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β-degree closures for graphs

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ABSTRACT

Bondy and Chvátal [J.A. Bondy, V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135] introduced a general and unified approach to a variety of graph-theoretic problems. They defined the k-closure $C_k(G)$, where k is a positive integer, of a graph G of order n as the graph obtained from G by recursively joining pairs of nonadjacent vertices a, b satisfying the condition $d(a) + d(b) \geq k$. For many properties P, they found a suitable k (depending on P and n) such that $C_k(G)$ has property P if and only if G does. In this paper we show that the condition $d(a) + d(b) \geq k$ can be replaced by a much better one: $d(a) + d(b) + |Q(G)| \geq k$, where $Q(G)$ is a well-defined subset of vertices nonadjacent to a, b.

1. Introduction

Bondy and Chvátal [6] observed that a graph $G = (V, E)$ of order n is Hamiltonian iff $G + ab$ is Hamiltonian where a, b are a pair of nonadjacent vertices satisfying the condition $d(a) + d(b) \geq n$. This observation motivated the introduction of the concept of the k-closure $C_k(G)$ of a graph G on n vertices. The graph $C_k(G)$ is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k. For a number of various properties of a graph G on n vertices, they showed that it is possible to find a suitable integer k, such that if G has property $P(k)$, so does $C_k(G)$. It was proved in [6] that seven of the classic sufficient degree conditions for Hamiltonicity guarantee that $C_n(G)$ is complete, yielding easier proofs of the corresponding results. Moreover they proved that it takes polynomial time to construct $C_n(G)$ and if $C_n(G)$ is complete then a polynomial algorithm is provided for obtaining a Hamiltonian cycle in the original graph. Although the Hamiltonicity Problem is NP-hard in general, it becomes polynomial if $C_n(G) = K_n$. The authors showed also that $C_k(G)$ is unique. The closure concept is becoming a major tool in Hamiltonian Graph Theory (see a recent survey by Broersma et al. [8]).

Preserving the uniqueness, Ainouche and Christofides (see [1,2]), Zhu et al. [10] and Broersma et al. [7] introduced other closure conditions stronger than that of Bondy and Chvátal. The condition of Zhu et al. is dominated by that given in [2]. Surprisingly the condition of Broersma et al. is dominated by that given in [1] (a simple proof is given in another paper in preparation).

In this paper we show that the condition $d(a) + d(b) \geq k$ used by Bondy and Chvátal to define the k-closure can be replaced by a more powerful one: $d(a) + d(b) + |Q| \geq k$, where, given k, Q is a well-defined subset of vertices nonadjacent to a, b and different from a and b. This new condition is similar to the condition $d(a) + d(b) + |R'| \geq k$ of Zhu et al., where R' is also a set of vertices nonadjacent to a, b. However one of the drawbacks of Zhu et al.'s condition is that R' has a specific definition for each property. Moreover for all properties considered here, $R' \subseteq Q$.

To state the different new conditions and to relate them to existing ones, we need some preliminary definitions and notations.

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2. Definitions and notations

We use Bondy and Murty [9] for terminology and notation not defined here and consider simple graphs only. Let $G = (V, E)$ be a graph of order $n \geq 3$. The set of neighbors of a vertex $v \in V$ is denoted $N_G(v)$ and $d_G(v) = |N_G(v)|$ is the degree of v. Paths and cycles in $G = (V, E)$ are considered as subgraphs and for simplicity we use the same notation to mean a subgraph, its vertex set or its edge set. If A is a subset of V, $G[A]$ will denote the subgraph induced by A. Let C be a cycle in G, in which a direction of traversing it is given. If a, b are vertices of a subset $R \subset V$ we let $R[a, b]$ denote the path with endpoints a, b and all internal vertices in R and if C (resp. P) is a cycle (resp. a path) with a chosen direction we let $C[a, b]$ (resp. $\pi[a, b]$) denote the sub-path of C (resp. π) from a to b in the chosen direction. We shall write $C(a, b)$ (resp. $\pi(a, b)$) if a, b or are excluded. Throughout, (a, b) is a pair of nonadjacent vertices and k is a positive integer. With (a, b) and k we associate.

$$\begin{aligned}
&\sigma_{ab}(G) := d_G(a) + d_G(b), \qquad \gamma_{ab}(G) := |N_G(a) \cup N_G(b)|, \qquad \lambda_{ab}(G) := |N_G(a) \cap N_G(b)| \\
&\Delta_{ab}(G) := \max\{d_G(x) | x \in T_{ab}(G)\}, \qquad \varpi_{ab}(G, x) := d_G(x) + \gamma_{ab}(G) + \varepsilon_{ab}(G) \\
&T_{ab}(G) := V \setminus (N_G(a) \cup N_G(b) \cup \{a, b\}), \qquad t := |T_{ab}(G)|, \qquad \overline{\alpha}_{ab}(G) = 2 + t \\
&Q_{ab}(G) := \{x \mid \varpi_{ab}(G, x) \geq k\}, \quad x \in T_{ab}(G) \neq \varnothing,
\end{aligned}$$

where $\varepsilon_{ab}(G)$ is a binary variable which takes the value 0 if and only if

$$\begin{cases} (\varepsilon.1) \quad G[T_{ab}(G)] = K_r \cup \overline{K_{t-r}}, \quad 1 \leq r \leq t \quad \text{and} \\ (\varepsilon.2) \quad \varpi_{ab}(G, x) = k - 1 \quad \forall x \in Q_{ab}(G). \end{cases} \tag{ε}$$

For simplicity of notation we may omit ab and/or G if no confusion can arise. In [2], we proved:

Theorem 1. *Let G be a 2-connected graph of order n and let $d_1^T \leq \cdots \leq d_t^T$ be the degree sequence (in G) of the vertices of the set $T \neq \varnothing$. If*

$$d_i^T \geq \overline{\alpha}_{ab} \quad \text{is true for all } i \text{ with } \max(1, \lambda_{ab} - 1) \leq i \leq t, \tag{1}$$

then G is Hamiltonian if and only if $G + ab$ is Hamiltonian.

In [3], we improved Theorem 1 as follows.

Theorem 2. *Let G be a 2-connected graph of order n and let $d_1^T \leq \cdots \leq d_t^T$ be the degree sequence (in G) of the vertices of the set $T \neq \varnothing$. If*

$$d_i^T + \varepsilon_{ab} \geq \overline{\alpha}_{ab} \quad \text{is true for all } i \text{ with } \max(1, \lambda_{ab} - 1) \leq i \leq t \tag{2}$$

then G is Hamiltonian if and only if $G + ab$ is Hamiltonian.

This new condition will be referred to as the "**β-cc**" for *β-closure condition*. This improved condition has two strong relaxations:

- a degree closure condition (**β-dcc**) involving the degree sum of (a, b), corresponding to the case $\max(1, \lambda_{ab}-1) = \lambda_{ab}-1$, which is the subject of this paper.
- a neighborhood closure condition (**β-ncc**), involving the neighborhood union of (a, b) and corresponding to the case $\max(1, \lambda_{ab} - 1) = 1$, which is the subject of another paper in preparation. For the particular case of the Hamiltonicity property, the "**β-ncc**" was used to obtain a large number of extensions of known sufficient conditions (see [4,5]).

Following Bondy and Chvátal [6] and recalling that $\sigma_{ab} = \gamma_{ab} + \lambda_{ab} = n - \overline{\alpha}_{ab} + \lambda_{ab}$, we define:

Definition 1. Let P be a property defined for all graphs G of order n and let k be an integer. Let a, b be two nonadjacent vertices satisfying the condition

$$P(k) : \sigma_{ab}(G) + |Q| \geq k \Leftrightarrow \overline{\alpha}_{ab}(G) \leq \lambda_{ab} + |Q| + n - k. \tag{*}$$

Then P is *k-degree stable* if whenever $G + ab$ has property P and $P(k)$ holds then G itself has property P. We denote by $dC_k(G)$ the associated k-degree closure.

The k-degree closure $dC_k(G)$ is then obtained from G by recursively joining pairs of nonadjacent vertices a, b for which $\sigma_{ab} + |Q| \geq k$ until no such pair remains. Obviously $dC_k(G)$ reduces to the Bondy–Chvátal's closure $C_k(G)$ if $Q = \varnothing$. Both closures are well defined. The Proposition below is an easy adaptation of Proposition 2.1 in [6].

Proposition 1. *If P is k-degree stable and $dC_k(G)$ has property P then G itself has property P.*

We would like to point out that the condition $d(a) + d(b) + |Q| \geq k$ can be checked in polynomial time. As in [6], our results lead to algorithms which construct the closure in polynomial time. Moreover, if the property P under study is easily verified in $dC_k(G)$ (in particular if $dC_k(G) = K_n$), then the property P is verified in the original graph G in polynomial time.

In this paper, we investigate the stability of a number of properties of graphs which remain in any super-graph of G (a graph obtained from G by addition of edges). Most of these properties are studied in [6,10]. We also provide new properties. Throughout (a, b) are a pair of nonadjacent vertices of a graph G satisfying the condition (*) for a given positive integer k. For each one of the considered properties P we fix k so that G has properties P whenever $G + ab$ does. Throughout, $S \subset V$ denotes a subset with s vertices.

3. Main results

Theorem 3. *The property of being Hamiltonian is n-degree stable.*

Proof. Suppose that a graph G satisfies the condition $P(n)$ for a given pair (a, b) of nonadjacent vertices. From (*) we get $\overline{\alpha}_{ab} \le \lambda_{ab} + |Q|$ or $|Q| \ge \overline{\alpha}_{ab} - \lambda_{ab}$. Since $\overline{\alpha}_{ab} = 2 + t$ and $Q \subseteq T$, we deduce that $\lambda_{ab} \ge 2$ or $\max\{1, \lambda_{ab} - 1\} = \lambda_{ab} - 1$. Moreover any $x \in Q$ satisfies the condition $d_i^T + \varepsilon_{ab} \ge \overline{\alpha}_{ab}$ by definition. Since $|\{i|\lambda_{ab} - 1 \le i \le t\}| = t + 1 - (\lambda_{ab} - 1) = \overline{\alpha}_{ab} - \lambda_{ab}$, it follows that $|Q| \ge |\{i|\lambda_{ab} - 1 \le i \le t\}|$. Thus G is Hamiltonian by Theorem 2. ■

Example 1. If G is the Petersen graph then $G' = G + K_1$ is Hamiltonian by Theorem 3 since for any pair (a, b) of nonadjacent vertices it is easy to check that $\sigma_{ab}(G') = 8$, $\varepsilon_{ab}(G') = 1$ and $\left|Q(G')\right| = 3$. Thus $dC_{11}(G') = K_{11}$.

The graph G is S-Hamiltonian, $s \le n - 3$, if it remains Hamiltonian whenever some or all vertices of S are removed. We simply say that it is s-Hamiltonian if we are only interested by the number s instead of the set of vertices.

Theorem 4. *The property of being S-Hamiltonian is $(n + s)$-degree stable.*

Proof. For some $W \subseteq S$, set $H := G - W$. Suppose now that $H + ab$ is Hamiltonian but H is not. Then

$$\begin{cases} \sigma_{ab} + |Q| \ge n + s & \text{by hypothesis} \\ \sigma_{ab}(H) + |Q(H)| < |H| = n - |W| & \text{by Theorem 3.} \end{cases} \tag{3}$$

Subtracting the two inequalities of (3) and noting that $\sigma_{ab} \le \sigma_{ab}(H) + 2\,|W \setminus T|$ we get $2\,|W \setminus T| + |Q| - |Q(H)| > s + |W|$. As $|Q| = |Q \cap V(H)| + |Q \cap W|$, $W \subseteq S$ and $Q \subseteq T$, we obtain

$$|Q \cap V(H)| > |Q(H)|\,. \tag{4}$$

By (4) there must exist $u \in (Q \cap V(H)) \setminus Q(H)$. Clearly $d(x) \le d_H(x) + |W|$ holds for all $x \in T$ and $\gamma_{ab} \le \gamma_{ab}(H) + |W \setminus T|$. If $x = u$, we have by hypothesis $\varpi(G, u) \ge n + s$. Therefore $\varpi(H, u) \ge n + s - |W| + |T \cap W| - \varepsilon_{ab} + \varepsilon_{ab}(H)$. As $W \subseteq S$ and $\varepsilon_{ab} - \varepsilon_{ab}(H) \le 1$ we obtain

$$\varepsilon_{ab} = 1, \qquad \varepsilon_{ab}(H) = 0, \qquad W = S, \qquad T \cap W = \varnothing.$$

So $T(G) = T(H)$ and hence $G[T] = H[T]$ since $T \cap W = \varnothing$. By (ε) we necessarily have $\Delta_{ab} + \gamma_{ab} \ge n + s$ while $\Delta_{ab}(H) + \gamma_{ab}(H) = |H| - 1$. Choose $z \in Q$ such that $d(z) = \Delta_{ab}$. Clearly $z \in T = T(H)$. Using the above inequalities, we easily get $d_H(z) + \gamma_{ab}(H) \ge |H|$. This is a contradiction to our assumption that $\Delta_{ab}(H) + \gamma_{ab}(H) = |H| - 1$. The proof of Theorem 4 is now complete. ■

We say that G is S-cyclable (S-traceable resp.) if it contains a cycle C (a path resp.) with all vertices of S.

Theorem 5. *The property "G is S-cyclable" is n-degree stable.*

Proof. Suppose that $(G + ab)$ contains a cycle C such that $S \subset V(C)$ but G does not. Then a, b are connected by a path $\pi := a_1 \dots a_p$ with $a = a_1$, $b = a_p$, $n \ge p \ge s$. Assume that π has a maximum length. If $|V(\pi)| = n$, then G is Hamiltonian by Theorem 3 and we are done. For the following we set $H := G[V(\pi)]$ and we assume that $R := V \setminus V(\pi) \neq \emptyset$. For simplicity we also denote $G[R]$ by R and $G[H]$ by H. Note that there exists no $[a, b]$-path with all internal vertices in R for otherwise G would contain a cycle C' such that $V(C) \subset V(C')$, a contradiction. In particular $N(a) \cap N(b) \subset V(\pi)$.

Denote by $a_{p_1}, a_{p_2}, \dots, a_{p_{\lambda_{ab}}}$, $p_1 < p_2 < \dots < p_{\lambda_{ab}}$, $\lambda_{ab} \ge 2$, the vertices of $N(a) \cap N(b)$. These vertices cannot be consecutive on π for otherwise a Hamiltonian cycle exists in H. For $i = 1, \dots, \lambda_{ab} - 1$, set $X_i := V\left(\pi(a_{r_i}, a_{s_i})\right)$, where $p_i \le r_i < s_i \le p_{i+1}$ are chosen so that:

(i) a_{s_i} is joined to a (resp. a_{r_i} is joined to b) by a path (possibly an edge) $R[a_{s_i}, a]$ say (resp. $R[a_{r_i}, b]$), whose internal vertices are all in R.

(ii) $X_i \subset T$.

(iii) For each i and among all possible choices of r_i, s_i we assume that $s_i - r_i$ is minimum. Clearly $R[a_{s_i}, a]$, $R[a_{r_i}, b]$ exist since $a_{p+1}a$, $a_{p_i}b$ satisfy (i).

As first consequences we have:

(iv) $X_i \neq \varnothing$ for otherwise the cycle $aa_2 \dots a_{r_i} R[a_{r_i}, b] b a_{p-1} a_{p-2} \dots a_{s_i} R[a_{s_i}, a]$ contains all vertices of S if $s_i = r_i + 1$.

(v) $R[a_{r_i}, b]$ and $R[a_{s_i}, a]$ are vertex disjoint for if $u \in V(R(a_{r_i}, b)) \cap V(R(a_{s_i}, a))$ then $V(\pi) \cup R[u, a] \cup R[u, b]$ induces a longer cycle than C, a contradiction,

(vi) there exists no path joining any $a_i \in X_i$ to either a or b with all internal vertices in R for otherwise we substitute this path to either $R[a_{r_i}, b]$ or to $R[a_{s_i}, a]$, in which case we contradict (iii).

Let us set $W_i := (V(R(a_{r_i}, b)) \cup V(R(a_{s_i}, a)))$ (possibly $W_i = \varnothing$), $W := \cup_{i=1}^{\lambda_{ab}-1} W_i$ and $X := \cup_{i=1}^{\lambda_{ab}-1} X_i$.

(vii) $N(a_j) \cap W = \varnothing$ holds whenever $a_j \in X$ for otherwise we contradict (iii). For simplicity we also set: $T_R := T \cap R$, $T_H := T \cap V(H)$, $Q_R := Q \cap R$ and $Q_H := Q \cap V(H)$.

(viii) $\forall a_i \in Q_H$, $N_R(a_i) \subseteq T_R$ and $N(v) \setminus R \subset H$ if $v \in N_R(a_i)$ as a consequence of (vi).

Claim 1. $\theta := |\{i | X_i \subset Q_H\}| > |T_R| - |Q \cap R|$.

Otherwise, suppose that $|T_H \setminus Q_H| = |T_H| - |Q_H| \geq \lambda_{ab} - 1 - \theta$. By hypothesis $\sigma_{ab} + |Q| \geq n \Leftrightarrow \overline{\alpha}_{ab} = 2 + |T_R| + |T_H| \leq \lambda_{ab} + |Q|$. Putting together these inequalities we get $\theta > |T_R| + |Q_H| - |Q| = |T_R| - |Q \cap R|$.

This result implies, since $Q_R \subseteq T_R$, that at least one subset X_i contains only vertices of Q_H and at least two if $|T_R| - |Q \cap R| > 0$.

Claim 2. For all $a_i \in Q_H$ we have

$$\begin{cases} (5.1) \quad d_H(a_i) + \gamma_{ab}(H) + \varepsilon_{ab}(H) \geq |H| - \varepsilon_{ab} + \varepsilon_{ab}(H) \\ (5.2) \quad \sigma_{ab}(H) + |Q_H| \geq |H| + |Q(H)| - |Q_H|. \end{cases} \tag{5}$$

By hypothesis we have $\sigma_{ab} + |Q| \geq n = |H| + |R|$ and $d(a_i) + \gamma_{ab} + \varepsilon_{ab} \geq n$. Combining with the obvious inequalities $d(a_i) \leq d_H(a_i) + |T_R|$, $\sigma_{ab} = \sigma_{ab}(H) + |R| - |T_R|$ and $\gamma_{ab} = \gamma_{ab}(H) + |R| - |T_R|$ we get (5).

Claim 3. $N_R(a_i) = T_R$ for all $a_i \in Q_H$.

By (5), $N_R(a_i) \subseteq T_R$. If $\varepsilon_{ab}(H) - \varepsilon_{ab} \geq 0$ we see from (5.1) that $a_i \in Q_H \Rightarrow a_i \in Q(H)$ and hence $|Q(H)| \geq |Q_H|$. By (5.2) and Theorem 3, H is Hamiltonian, a contradiction. So we are left with the case $\varepsilon_{ab}(H) - \varepsilon_{ab} < 0$ which implies $\varepsilon_{ab}(H) = 0$ and $\varepsilon_{ab} = 1$. By (5.1), $d_H(a_i) + \gamma_{ab}(H) \geq |H| - 1$. If $\Delta_{ab}(H) + \gamma_{ab}(H) \geq |H|$ then $\varepsilon_{ab}(H) = 1$ by (ε_2), a contradiction. Therefore

$$d_H(a_i) + \gamma_{ab}(H) = |H| - 1 \quad \text{is true for all } a_i \in Q_H. \tag{6}$$

On the other hand suppose $\Delta_{ab} + \gamma_{ab} \geq n$. Adding the inequalities $\Delta_{ab} \leq \Delta_{ab}(H) + |T_R|$ and $\gamma_{ab} = \gamma_{ab}(H) + |R| - |T_R|$ we get $\Delta_{ab}(H) + \gamma_{ab}(H) \geq |H|$, another contradiction. Therefore

$$d(a_i) + \gamma_{ab} = n - 1 \quad \text{is true for all } a_i \in Q_H. \tag{7}$$

Putting (viii), (6) and (7) together we get $d(a_i) = d_H(a_i) + |T_R|$ and hence the claim is proved.

Claim 4. $W = \varnothing$.

We have seen that there exists X_i such that $X_i \subset Q_H$. Choose $a_j \in X_i$. Then $N_R(a_j) = T_R$ and hence $W \subset N_R(a_j)$. We have now a contradiction to (vii) if $W \neq \varnothing$. From now on, we assume that $R\left[a_{r_i}, b\right]$ and $R\left[a_{s_i}, a\right]$ are respectively the edges $a_{r_i}b$ and $a_{s_i}a$.

Claim 5. $T_R \neq \varnothing$ and $|X_i| = 1$ for all $i \in [1, \lambda_{ab} - 1]$ satisfying the condition $X_i \subset Q_H$.

Otherwise suppose $T_R = \varnothing$. Now $G[T] = H[T]$ and $d(u) = d_H(u)$ holds for all $u \in T$. It is then obvious that $\varepsilon_{ab} = \varepsilon_{ab}(H)$, a contradiction. Since $T_R \neq \varnothing$, let u be any vertex of T_R. By Claim 3, $N_{X_1}(u) = X_1$ and hence we can insert u into two consecutive vertices π of X_1 to obtain a longer path than π in G if $|X_i| > 1$.

Claim 6. Last contradiction.

Since $T_R \neq \varnothing$, pick any vertex, v say, of T_R. Suppose first $Q \cap R = \varnothing$. Then by Claim 1, we have at least two indices, $h, e \in [1, \lambda_{ab} - 1]$ with $e < h$ such that $X_h \cup X_e \subset Q_H$. By Claim 3, $v \in N(a_h) \cap N(a_e)$. But now the cycle $aa_2 \ldots a_e v a_h a_{h+1} \ldots b a_{h-1} a_{h-2} \ldots a_{e+1} a$ is Hamiltonian in H. So we may assume $Q \cap R \neq \varnothing$. Then $d(v) + \gamma_{ab} \geq n - 1$ or equivalently $d(v) \geq \overline{\alpha}_{ab} - 1 = 1 + |T_H| + |T_R|$. By Claim 1, we may assume that $X_h \subset Q_H$ for some $h \in [1, \lambda_{ab} - 1]$ and hence $v \in N(a_h)$. Define

$$J(v) = \{a_i \in H | a_{i+1} \in N(v) \text{ if } i > h,\ a_{i-1} \in N(v) \text{ if } i < h\}.$$

Clearly $d(v) > 1 + |T_H| + |T_R| \Rightarrow d_H(v) \geq 2 + |T_H|$. It is easy to check that $J \cap \{a, b\} \neq \varnothing$. Suppose, without loss of generality that, for some $l > h$, $a_l \in N(a) \cup N(b)$ (note that $l \geq h + 2$ for otherwise we insert v into $a_h a_{h+1}$ to obtain a longer path than π). If $a_l \in N(a)$ then the cycle

$$aa_2 \ldots a_{h-1} b b a_{p-1} a_{p-2} \ldots a_{l+1} v a_h a_{h+1} \ldots a_{l-1} a$$

is Hamiltonian in H. If $a_l \in N(h)$ then the cycle

$$aa_2 \ldots a_h v a_l a_{l+1} \ldots a_{p-1} b \ldots a_{l-1} a_{l-2} \ldots a_{h+1} a$$

is Hamiltonian in H. The proof is now complete. ∎

The subgraph $G[S]$ is Hamiltonian if G is $V \setminus S$-Hamiltonian. Applying Theorem 4 we obtain:

Theorem 6. *The property "$G[S]$ is Hamiltonian" is $(2n - s)$-stable.*

A *caterpillar* is a particular tree which results in a path when its leaves are removed. The *spine* of the caterpillar is its longest path. The graph G is called *S-caterpillar spannable* if it has a spanning tree that is a caterpillar, whose leaves are the vertices of $S := \{x_1, \ldots, x_s\}$. Suppose that the spine is an $[x_1, x_2]$-path. Let G' be a graph obtained from G by adding a new vertex, v say, that is joined to x_1 and x_2. Then G is *S-caterpillar spannable if* G' is $(S - \{x_1, x_2\})$-Hamiltonian. Applying Theorem 4 to the graph G' we obtain:

Theorem 7. *Let $S \subset V(G)$ with s vertices, $2 \leq s < n$. Then the property "G is S-caterpillar spannable" is $(n+s-1)$-degree stable.*

A set $F \subset E$ of edges such that the components of the graph (V, F) are vertex disjoint paths is called F-cyclable (or $|F|$-*edge-Hamilton*) if there exists a cycle that contains F. It is F-traceable if there exists a path that contains F. Applying Theorem 3 to the graph obtained from G by subdividing each edge in F into two, we obtain:

Theorem 8. *The property "G is F-cyclable with $|F| \leq n-3$", is $n+|F|$-degree stable.*

A graph G is defined to be $|F|$-Hamilton-connected if for each pair (x, y) of vertices there is a Hamiltonian path with endpoints x, y that contains F. Now G must be $(F \cup xy)$-cyclable and using Theorem 8 we obtain:

Theorem 9. *The property "G is F-Hamilton-connected with $|F| \leq n-4$", is $n+|F|+1$-degree stable.*

Theorem 10. *Let n, s be positive integers with $s \leq \frac{n}{2}$. Then the property of containing sK_2 is $(2s-1)$-stable.*

Proof. If $G+ab$ contains an sK_2 but G does not, then there exists an $(s-1)$-matching $\{a_1b_1, \ldots, a_{s-1}b_{s-1}\}$ in G and an s-matching in $G+ab$. For $i \in [1, s-1]$ we set

$$\begin{cases} A := \{a_i\}, \quad B := \{b_i\}, \quad D := V \setminus (A \cup B \cup \{a, b\}) \\ M := \{a_ib_i | i \in [1, s-1]\}, \quad M_i = \{a_i, b_i\} \quad \text{and} \quad m_i := \left|N_{M_i}(a) \cup N_{M_i}(b)\right|. \end{cases}$$

We label the vertices of A, B so that $a_i \in N(a) \cup N(b)$ whenever $m_i \geq 1$. An M-augmenting path is a path with an even number of vertices, unsaturated endpoints in $D \cup \{a, b\}$ and whose edges are alternatively in $E-M$ and M. To avoid contradiction, we obviously assume that G contains no M-augmenting path. Moreover $D \cup \{a, b\}$ must be an independent set for otherwise an s-matching would exist in G. We shall assume $Q \neq \varnothing$, by Bondy and Chvátal's result [6].

To distinguish the all possible configurations we define the following independent sets: $J_0 := \{i | m_i = 0\}, J_{11} := \{i | m_i = 1 \text{ and } |N(a_i) \cap \{a, b\}| = 1\}, J_{12} := \{i | m_i = 1 \text{ and } |N(a_i) \cap \{a, b\}| = 2\}$ and $J_2 := \{i | m_i = 2\}$. If $j \in J_2$ then $d_{M_j}(a) + d_{M_j}(b) = 2$ and either $d_{M_j}(a) = 2$ or $d_{M_j}(b) = 2$ for if $aa_j, bb_j \in E$ then aa_jb_jb is an M-augmenting path. These sets form a partition of $J := J_0 \cup J_{11} \cup J_{12} \cup J_2$. We note that,

$$\begin{cases} s = |J| + 1, \quad \sigma_{ab} = |J_{11}| + 2(|J_{12}| + |J_2|), \\ \gamma_{ab} = |J_{11}| + |J_{12}| + 2\,|J_2| \quad \text{and} \quad \lambda_{ab} = |J_{12}|. \end{cases} \tag{8}$$

By hypothesis $\sigma_{ab} + |Q| \geq 2s-1$. Using (8) we obtain

$$|Q| > 2\,|J_0| + |J_{11}|. \tag{9}$$

At this point we split the proof into two cases.

Case 1. $Q \cap D \neq \varnothing$.

Choose $x \in Q \cap D$ such that $d(x) = \max\{d(y) | y \in Q \cap D\}$. Moreover we may assume without loss of generality that $d(x) \geq d(b_i)$ whenever $xa_i \in E$. If this is not the case, we exchange the labeling of x and b_i. We claim that $\forall i \in [1, s-1],\ d_{M_i}(x) + m_i = 2$. Indeed, if $d_{M_i}(x) + m_i \geq 3$, it is easy to check that G has an M-augmenting path with 4 vertices and a, x (or b, x) as extremities. Therefore $\sum_{i=1}^{s-1}\left(d_{M_i}(x) + m_i\right) = d(x) + \gamma_{ab} \leq 2(s-1)$ (recall that D is an independent set). As $x \in Q \cap D$, we must have $\varpi(G, x) \geq 2s-1$. This leads to the conclusion that $\varepsilon_{ab} = 1$ and $d_{M_i}(x) + m_i = 2$ as claimed. Moreover $xb_j \notin E$ whenever $j \in J \setminus J_0$ for if $xb_j \in E$ then aa_jb_jx (or ba_jb_jx) is an M-augmenting path. Similarly $xa_j \notin E$ whenever $j \in J_2$. It is then clear that

$$N(x) = \{M_i | i \in J_0\} \cup \left\{a_j | m_j = 1\right\}. \tag{10}$$

Obviously $d(x) = \Delta_{ab}$, by the choice of x. Since $\varpi(G, x) = 2(s-1)$ we deduce that $d(u) = \Delta_{ab}$ for any $u \in Q$. At this point we conclude that $G[T] \neq Kr \cup K_{t-r}$, $1 \leq r \leq t$ by (ε_1).

Sub-Case 1.1: $J_0 \neq \varnothing$.

We claim that $b_j \notin Q$ whenever $j \in J \setminus J_0$. This is obvious if $j \in J_2$ since then $a_j, b_j \in N(a) \cup N(b)$. If $j \in (J_{11} \cup J_{12})$ then $d(b_j) = d(x)$. Exchanging the role of x and b_j we see that, in particular, $b_jb_i \in E$ where $i \in J_0$. But then either $aa_jb_jb_ia_ix$ or $ba_jb_jb_ia_ix$ is an M-augmenting path. Therefore $Q \subset \{M_i | m_i = 0\} \cup D$. In fact $Q \cap D = \{x\}$ for if there exists $y \in (Q \cap D) \setminus \{x\}$ then, as for x, we have in particular $yb_i \in E$ with $i \in J_0$. But then xa_ib_iy is an M-augmenting path, a contradiction. Therefore $Q \subset \{M_i | m_i = 0\} \cup \{x\}$. By (9), we have more precisely $Q = \{M_i | m_i = 0\} \cup \{x\}$. Since each vertex of $\{M_i | m_i = 0\}$ can play the role of x, we conclude that $N(v) = \{M_i | i \in J_0\} \cup \left\{a_j | m_j = 1\right\}$ holds for any $v \in Q$. Furthermore it is easy to check that $T \setminus Q = \left\{b_j | j \in J \setminus J_0\right\} \cup (D \setminus \{x\})$ is an independent set. This implies that $G[T] = K_{|Q|} \cup \overline{K_{t-|Q|}}$. This is a contradiction.

Sub-Case 1.2: $J_0 = \varnothing$.

In this sub-case and by (10) we have $N(x) = \left\{a_j | m_j = 1\right\}$. It follows that $Q \subseteq \left\{b_j | j \in J_{11} \cup J_{12}\right\} \cup D$. We already know that $b_j \notin Q$ if $j \in J_2$. We claim that T is an independent set. We recall first that D is an independent set. Clearly $N(y) \subseteq \left\{a_j | m_j = 1\right\}$ for all $y \in D$ for if $b_j \in N(y)$ with $j \in J_{11} \cup J_{12}$ then we have an M-augmenting path with 4 vertices and extremities (a, y)

or (b, y). Finally assume $b_i b_j \in E$ with $j \in J_{11} \cup J_{12}$. Then $xa_j b_j b_i a_i a$ (or $xa_j b_j b_i a_i b$) is an M-augmenting path. So $G[T] = \overline{K_t}$ as claimed and a contradiction is again obtained.
Case 2.$Q \cap D = \varnothing$ (or $Q \subset A \cup B$).

By (9), $|Q| > |J_{11}|$ and hence there exists i, 1 say, such that $1 \in J_{12}$ and $b_1 \in Q$. As for the above sub-case we prove that T is an independent set. It is easy to see that an M-augmenting path with 4 vertices and extremities (a, y) or (b, y) exists if $b_i y \in E$ where $j \in J_{11} \cup J_{12} \cup J_2$ and $y \in D$. Suppose next $b_1 b_i \in E$, $1 \le i \le s-1$. Then either $aa_i b_i b_1 a_1 b$ or $ba_i b_i b_1 a_1 a$ is an M-augmenting path. This implies that $N(b_1) = \{a_j | m_j = 1\}$ since $b_1 \in Q$. Finally, suppose $b_i b_j \in E$, $1 < i < j \le s-1$. Assuming for instance $aa_i \in E$ then $ba_1 b_1 a_j b_j b_i a_i a$ is an M-augmenting path. From these observations we get $Q \subseteq \{b_i | i \in J_{11} \cup J_{12}\}$ and $N(v) = \{a_j | i \in J_{11} \cup J_{12}\}$ whenever $v \in Q$. Thus $d(v) + \gamma_{ab} = |J_{11}| + |J_{12}| + |J_{11}| + |J_{12}| + 2|J_2| = 2|J| = 2(s-1)$. This means that $\varepsilon_{ab} = 1$. On the other hand $\Delta_{ab} + \gamma_{ab} = 2(s-1)$ and $G[T]$ is an independent set. Therefore $\varepsilon_{ab} = 0$ by (ε). The proof is now complete. ■

Example 2. Consider the Petersen graph G. Obviously, it has a perfect matching. By Bondy–Chvátal's result, we have $C_k(G) = K_{10}$ only for $s \le 3$. By Theorem 3, $dC_k(G) = K_{10}$ for $s = 5$ since $\sigma_{ab} + |Q| = 6 + 3 \ge 9 = 2s - 1$ as each vertex of T is in Q since $d(x) + \gamma_{ab} \ge 8$ and $\varepsilon_{ab} = 1$.

As another example, consider the graph $G = C_6 = x_1 x_2 \ldots x_6 x_1$. Obviously this graph contains a 3-matching. For instance $d(x_1) + d(x_3) + |Q| = 5 \ge 2s - 1$ with $s = 3$ since $d(x_1) + d(x_3) = 4$ and $x_5 \in Q$ (it satisfies the condition $d(x_5) + \gamma_{x_1 x_3} = 2 + 3 \ge 2s - 1$). It is simple to check that $dC_5(G) = K_6$ while $C_5(G) = G$. We would like to point out that the $(2s-1)$-closures based on Bondy–Chvátal's condition and condition given in [10] are equal.

4. Corollaries

The following results can be easily derived as corollaries. Let G be a graph of order n, S be a subset of vertices and $s \le |S|$ be an integer. Then the property:

- G is "S-vertex Hamiltonian-connected" is $(n+s+1)$-degree stable (the graph is said to be S-vertex Hamiltonian-connected if it remains Hamiltonian-connected if s vertices of S or less vertices are removed);
- of being S-pancyclable is $(n + s - 3)$-degree stable with $3 \le s \le n$;
- of having $c(G) \ge s$ is n-degree stable, where $c(G)$ is the circumference of G;
- $\mu(G) \le p$ is $(n - p)$-stable, $p \ge 1$, where $\mu(G)$ is the number of paths that collectively contain the vertices of G;
- G is S-leaf-connected (G has a spanning tree whose leaves are the vertices of S) is $(n + s - 1)$-stable;
- of being s-edge-Hamiltonian is $(n + s)$-degree stable.

5. Open problem

We end the paper by posing the following open problem.

Problem 1. Let n, s be positive integers with $2 \le s < n$. Then the property of having an s-factor is $(n + 2s - 4)$-degree stable.

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