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Degree conditions for graphs to be λ_3-optimal and super-λ_3 ☆

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ABSTRACT

For a positive integer m, an edge-cut S of a connected graph G is an m-restricted edge-cut if each component of $G-S$ contains at least m vertices. The m-restricted edge connectivity of G, denoted by $\lambda_m(G)$, is defined as the minimum cardinality of all m-restricted edge-cuts. Let $\xi_m(G) := \min\{|\partial(X)| : X \subseteq V(G), |X| = m, \text{ and } G[X] \text{ is connected}\}$, where $\partial(X)$ denotes the set of edges of G each having exactly one endpoint in X. A graph G is said to be λ_m-optimal if $\lambda_m(G) = \xi_m(G)$, and super-λ_m if every minimum m-restricted edge-cut isolates a component of size exactly m.

In this paper, firstly, we give some relations among λ_3-optimal, λ_i-optimal and super-λ_i for $i = 1, 2$. Then we present degree conditions for arbitrary, triangle-free and bipartite graphs to be λ_3-optimal and super-λ_3, respectively; moreover, we give some examples which prove that our results are the best possible.

1. Introduction and notations

Let G be a connected undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $n(G)$ denote the order of G, $d_G(u, v)$ the *distance* between vertices u and v in G, and $g(G)$ the girth of G. For a vertex $v \in V(G)$, $N_G(v)$ denotes the set of vertices adjacent to v in G, $N_G[v] := N_G(v) \cup \{v\}$. Then $d(v) = |N_G(v)|$ is the *degree* of v in G, and $\delta(G)$ is the *minimum degree* of G. If $X \subseteq V(G)$, then $G[X]$ denotes the subgraph of G induced by X, and $\bar{X} = V(G) \setminus X$. For disjoint sets X and Y of vertices of G, $[X, Y]$ denotes the set of edges of G with one endpoint in X and the other one in Y. Put $\partial(X) := [X, \bar{X}]$. We denote $N_{G[X]}(v)$ by $N_X(v)$, the complete graph with order n by K_n, and the complete bipartite graph with bipartite sets of cardinalities m and n by $K_{m,n}$. A (p, r)-*barbell* ($p \geq 3, r \leq p$) [19] is a graph G obtained by joining two copies of the complete graph K_p with pr additional edges such that $d(v) = p + r - 1$ for each vertex $v \in V(G)$.

It is well known that the underlying topology of an interconnection network is usually modeled by a graph G with vertices and edges representing the nodes and links, respectively. An edge-cut S of a connected graph G is called *a restricted edge-cut* if $G - S$ contains no isolated vertex. The minimum cardinality of all restricted edge-cuts, denoted by $\lambda'(G)$, is called the *restricted edge connectivity* of G. Edge connectivity $\lambda(G)$ and restricted edge connectivity $\lambda'(G)$ have been used to measure the reliability of a network. In order to more accurately measure the reliability, the parameter $\lambda_m(G)$ received much attention. Under some reasonable conditions, Wang and Li [18] showed that for two regular graphs G_1 and G_2 with $\lambda(G_1) = \lambda(G_2) = \lambda$ and $\lambda'(G_1) = \lambda'(G_2) = \lambda'$, and $m_\lambda(G_1) = m_\lambda(G_2)$ and $m_{\lambda'}(G_1) = m_{\lambda'}(G_2)$, G_1 is more reliable than G_2 if $\lambda_3(G_1) > \lambda_3(G_2)$ or $\lambda_3(G_1) = \lambda_3(G_2) = \lambda_3$ and $m_{\lambda_3}(G_1) < m_{\lambda_3}(G_2)$, where $m_i(G)$ denotes the number of disconnecting edge sets of size i in graph G. So graphs with maximal 3-restricted edge connectivity $\lambda_3(G)$ (namely λ_3-optimal graphs) and the fewest minimum 3-restricted edge-cuts (super-λ_3 graphs have these two properties) have higher reliability.

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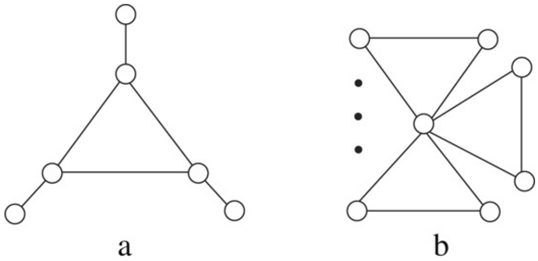

Fig. 1. (a) The 3-leg spider graph, and (b) the friendship graph.

The m-restricted edge connectivity $\lambda_m(G)$ was defined by Fábrega and Fiol [4,5] as follows:

Definition 1.1. An edge set S of a connected graph G is called an *m-restricted edge-cut* if $G - S$ is disconnected and each component of $G - S$ contains at least m vertices. The *m-restricted edge connectivity* of G, denoted by $\lambda_m(G)$, is the minimum cardinality of all m-restricted edge-cuts of G.

Balbuena et al. [1] improved the results contained in [4,5], and more recently Bonsma et al. [2] and Meng and Ji [11] have obtained very interesting results concerning the existence of m-restricted edge-cuts. Also see the survey by Hellwig and Volkmann [9].

Note that $\lambda_1(G) = \lambda(G)$ and $\lambda_2(G)$ is just the usual *restricted edge connectivity* $\lambda'(G)$. An m-restricted edge-cut S in G is called a *λ_m-cut*, if $|S| = \lambda_m(G)$, and *trivial* if S isolates a component of size exactly m. Obviously, for any λ_m-cut S, the graph $G - S$ has exactly two components.

For a connected graph G, let

$$\xi_m(G) := \min\{|\partial(X)| : X \subseteq V(G), |X| = m, \text{ and } G[X] \text{ is connected}\}.$$

Note that $\xi_1(G) = \delta(G)$ and $\xi_2(G)$ is just the *minimum edge-degree* $\xi(G)$ of G. A connected graph G is *λ_m-connected* if $\lambda_m(G)$ exists. Clearly, if G is λ_m-connected for $m \geq 2$, then G is also λ_{m-1}-connected and $\lambda_{m-1}(G) \leq \lambda_m(G)$. In 1988, Esfahanian and Hakimi [3] showed that every connected graph G of order $n(G) \geq 4$, except a star $K_{1,n-1}$, $\lambda_2(G)$ exists and satisfies $\lambda_2(G) \leq \xi_2(G)$. Bonsma, Ueffing and Volkman [2], Wang and Li [22] characterized λ_3-connected graphs as follows.

Theorem 1.2 (*[2,22]*). (a) *A connected graph G of order $n(G) \geq 6$ is λ_3-connected if and only if G is not isomorphic to the 3-leg spider graph (Fig. 1(a)) or any subgraph of the friendship graph (Fig. 1(b)).*

(b) *If G is λ_3-connected, then $\lambda_3(G) \leq \xi_3(G)$.*

For $m \geq 4$, Bonsma et al. [2] pointed out that the inequality $\lambda_m(G) \leq \xi_m(G)$ is no longer true in general, Ou characterized graphs of order at least $3m - 2$ that contain m-restricted edge-cuts [12] and showed that a λ_4-connected graph G with order at least 11 has the property $\lambda_4(G) \leq \xi_4(G)$ [13], and Zhang and Yuan [24] showed that for $m \leq \delta(G) + 1$, every connected graph G with order at least $2(\delta(G) + 1)$ except the graph $G^*_{n,t}$ is λ_m-connected and $\lambda_m(G) \leq \xi_m(G)$, where $G^*_{n,t}$ is obtained from n copies of K_t by adding a new vertex u that is adjacent to every vertex of them. To maximize $\lambda_m(G)$ and minimize the number of λ_m-cuts of G, the following definition was proposed in [11,23,25].

Definition 1.3. For a positive integer m, a λ_m-connected graph G with $\lambda_m(G) \leq \xi_m(G)$ is said to be *optimally m-restricted edge connected*, for short *λ_m-optimal*, if $\lambda_m(G) = \xi_m(G)$, and *super-m-restricted edge connected*, for short *super-λ_m*, if every λ_m-cut of G is trivial.

Note that λ_1-optimal is just *maximally edge-connected* and λ_2-optimal is the *λ'-optimal*; super-λ_1 is just the *super-edge connected* and super-λ_2 is the *super-λ'*.

For the λ_3-optimal and super-λ_3 graphs, Bonsma et al. [2] showed that the complete bipartite graph $K_{r,s}$ with $r, s \geq 2$ and $r + s \geq 6$ is λ_3-optimal, Ou and Zhang characterized the 3-restricted edge connectivity of vertex transitive graphs with girth four [14] and that of 3-regular and 4-regular vertex transitive graphs with girth three [15], Zhang and Meng [23] studied the λ_3-optimal vertex transitive graphs, Wang [19] presented Ore type sufficient conditions for graphs with diameter 2 to be λ_3-optimal and super-λ_3, Zhang and Yuan [25] gave degree conditions for graphs with diameter 2 to be λ_m-optimal, and Zhang [26] gave sufficient conditions expressed in terms of $\xi_m(G)$ for graphs to be λ_m-optimal, $m = 2, 3$. For more information on m-restricted edge connectivity of graphs, please refer to [6,7,13,16,20,24].

In this paper, we study the index λ_3 of graphs and present degree conditions for arbitrary, triangle-free, and bipartite graphs to be λ_3-optimal and super-λ_3, respectively; moreover, we give some examples which prove that our results are the best possible.

Now we discuss some relations between λ_m-optimal and super-λ_m for $m \leq 3$. A super-λ_m graph is also λ_m-optimal, but the converse is not true, and a λ_3-optimal graph is not always λ_2-optimal. Hellwig and Volkman [8] gave the following proposition about the relations between λ_2-optimal, λ_1-optimal and super-λ_1.

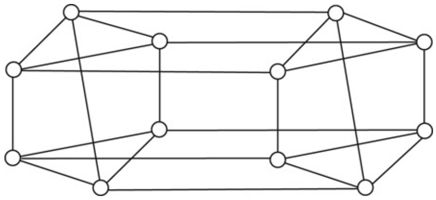

Fig. 2. A λ_3-optimal but not super-λ_2 graph with $\delta(G) = 4$.

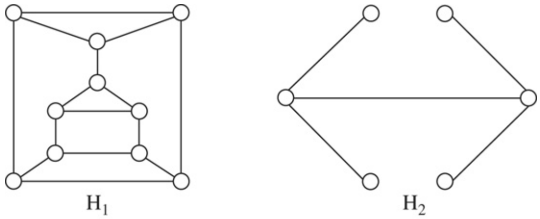

Fig. 3. Two λ_3-optimal but not λ_2-optimal graphs.

Proposition 1.4 ([8]). (a) *If G is λ_2-optimal, then G is also λ_1-optimal.*(b) *If G is λ_2-optimal and $\delta(G) \geq 3$, then G is super-λ_1.*

We give relations below between λ_3-optimal, λ_i-optimal and super-λ_i for $i = 1, 2$.

Proposition 1.5. *Let G be a λ_3-optimal graph.*
(a) *If $\delta(G) \geq 4$, then G is λ_i-optimal for $i = 1, 2$ and super-λ_1; if $\delta(G) > 4$, then G is super-λ_i for $i = 1, 2$.*
(b) *Assume that G is triangle-free. If $\delta(G) \geq 2$, then G is λ_i-optimal for $i = 1, 2$; if $\delta(G) > 2$, then G is super-λ_i for $i = 1, 2$.*

Proof. Since G is λ_3-optimal,

$$\begin{aligned}\lambda_3(G) &= \xi_3(G)\\ &= \min\{|\partial(X)| : X \subset V(G), |X| = 3, \text{ and } G[X] \text{ is connected}\}\\ &= \min\{\min\{d(x) + d(y) + d(z) - 6 : G[\{x, y, z\}] \text{ is a triangle}\},\\ &\quad \min\{d(x) + d(y) + d(z) - 4 : G[\{x, y, z\}] \text{ is a path}\}\}\\ &\geq \begin{cases}\xi_2(G) + \delta(G) - 4, & \text{if } G \text{ contains a triangle;}\\ \xi_2(G) + \delta(G) - 2, & \text{if } G \text{ is triangle-free.}\end{cases}\end{aligned}$$

Hence, $\lambda_3(G) \geq \xi_2(G)$ if $\delta(G) \geq 4$ and $\lambda_3(G) > \xi_2(G)$ if $\delta(G) > 4$; and when G is triangle-free, $\lambda_3(G) \geq \xi_2(G)$ if $\delta(G) \geq 2$ and $\lambda_3(G) > \xi_2(G)$ if $\delta(G) > 2$. Since $\lambda_2(G) \leq \lambda_3(G)$, $\lambda_3(G) \geq \xi_2(G)$ implies $\lambda_2(G) = \xi_2(G)$ and $\lambda_3(G) > \xi_2(G)$ implies that each λ_2-cut is trivial, so by Proposition 1.4, both statements (a) and (b) hold. □

Remark 1. From the proof of Proposition 1.5, we know that a λ_3-optimal graph G is super-λ_2 if $\xi_3(G) > \xi_2(G)$ and λ_2-optimal if $\xi_3(G) \geq \xi_2(G)$. A λ_3-optimal graph G is not always super-λ_2 if $\xi_3(G) = \xi_2(G)$ or λ_2-optimal if $\xi_3(G) < \xi_2(G)$. In Fig. 2, we give an example of a graph with $\delta(G) = 4$, $\xi_3(G) = \xi_2(G) = 6$, and $\lambda_3(G) = 6$. So G is λ_3-optimal but not super-λ_2. The cycle $C_n (n \geq 6)$, a λ_3-optimal triangle-free graph with $\delta(C_n) = 2$ and $\xi_3(C_n) = \xi_2(C_n) = 2$, is not super-$\lambda_2$. In Fig. 3, $\lambda_3(H_1) = \xi_3(H_1) = 3$, $\lambda_3(H_2) = \xi_3(H_2) = 1$, but $\lambda_2(H_1) = 3 < 4 = \xi_2(H_1)$, $\lambda_2(H_2) = 1 < 2 = \xi_2(H_2)$. So H_1 and H_2 are λ_3-optimal but not λ_2-optimal.

We next present degree conditions for arbitrary, triangle-free, and bipartite graphs to be λ_3-optimal and super-λ_3, respectively.

2. Conditions for arbitrary graphs

Lemma 2.1. *Let G be a λ_3-connected graph. Then:*
(a) *G is λ_3-optimal if and only if either G is non-λ_4-connected, or G is λ_4-connected and $\lambda_4(G) \geq \xi_3(G)$.*
(b) *G is super-λ_3 if and only if either G is non-λ_4-connected, or G is λ_4-connected and $\lambda_4(G) > \xi_3(G)$.*

Proof. Since G is λ_3-optimal, then $\lambda_3(G) = \xi_3(G)$. Thus to prove the necessity observe that if G is λ_4-connected, then by $\lambda_4(G) \geq \lambda_3(G)$, we have $\lambda_4(G) \geq \xi_3(G)$. If G is super-λ_3, then $\lambda_4(G) > \lambda_3(G)$ and we have $\lambda_4(G) > \xi_3(G)$.

To prove the sufficiency note that if G is non-λ_4-connected, then each λ_3-cut of G is trivial, and G is λ_3-optimal and super-λ_3. Let G be λ_4-connected with $\lambda_4(G) \geq \xi_3(G)$. If $\lambda_4(G) > \lambda_3(G)$, then each λ_3-cut of G is trivial and G is thus super-λ_3. Otherwise $\lambda_3(G) = \lambda_4(G) \geq \xi_3(G)$. Then $\lambda_3(G) = \xi_3(G)$, and G is λ_3-optimal. □

In the following, we first list some degree conditions for graphs to be λ_m-optimal and super-λ_m for $m = 1, 2, 3$, then present sufficient conditions for arbitrary graphs to be λ_3-optimal and super-λ_3.

Theorem 2.2. *Let G be a connected graph.*

(a) [10] *If $d(u) + d(v) \geq n(G) - 1$ for all pairs u, v of nonadjacent vertices, then G is λ_1-optimal.*

(b) [10] *If $d(u)+d(v) \geq n(G)$ for all pairs u, v of nonadjacent vertices, and G is different from $K_{n(G)/2} \times K_2$, then G is super-λ_1.*

(c) [20] *If $n(G) \geq 4$ and $d(u) + d(v) \geq n(G) + 1$ for all pairs u, v of nonadjacent vertices, then G is λ_2-optimal.*

(d) [8] *Let G be a λ_2-connected graph such that $\delta(G) \geq \lfloor n(G)/2 \rfloor - 1$. If for each triangle T of G there exists at least one vertex $w \in V(T)$ such that $d(w) \geq \lfloor n(G)/2 \rfloor + 1$, then G is λ_2-optimal.*

(e) [21] *If G is not a $(p, 2)$-barbell and $d(u) + d(v) \geq n(G) + 2$ for all pairs u, v of nonadjacent vertices, then G is super-λ_2.*

(f) [19] *If $n(G) \geq 6$ and $d(u) + d(v) \geq n(G) + 3$ for all pairs u, v of nonadjacent vertices, then G is λ_3-optimal.*

(g) [19] *If G is not $(p, 3)$-barbell $(p \geq 4)$, $n(G) \geq 6$, and $d(u) + d(v) \geq n(G) + 3$ for all pairs u, v of nonadjacent vertices, then G is super-λ_3.*

Theorem 2.3. *Let G be a connected graph with $n(G) \geq 6$. Then G is λ_3-optimal if the following three conditions hold:*

(a) *$d(x) + d(y) \geq 2\lfloor n(G)/2 \rfloor - 5$ for each pair $x, y \in V(G)$ with $d_G(x, y) = 3$,*

(b) *$d(x) + d(y) \geq 2\lfloor n(G)/2 \rfloor - 1$ for each pair $x, y \in V(G)$ with $d_G(x, y) = 2$, and*

(c) *for each subgraph K_4 of G, there exists at least one vertex $v \in K_4$ with $d(v) \geq \lfloor n(G)/2 \rfloor + 2$.*

Proof. From Condition (b) and $n(G) \geq 6$, it follows that $d(x) + d(y) \geq 5$ for each pair $x, y \in V(G)$ with $d_G(x, y) = 2$. Hence G cannot be isomorphic to the 3-leg spider graph or a subgraph of the friendship graph. From Theorem 1.2 we know that G is λ_3-connected. So by Lemma 2.1 (a), it suffices to show that $\lambda_4(G) \geq \xi_3(G)$. Let $\partial(X)$ be any λ_4-cut of G with $|X| \leq |\bar{X}|$. This implies $4 \leq |X| \leq \lfloor n(G)/2 \rfloor$. Choose three vertices u, v and w in X such that $G[\{u, v, w\}]$ is connected and satisfies

$$|\partial(\{u, v, w\})| = \min\{|\partial(A)| : A \subset X, |A| = 3, \text{ and } G[A] \text{ is connected}\}.$$

Case 1. $G[\{u, v, w\}]$ is a path. Assume that $uw \notin E(G)$, by the choice of u, v and w, we have

$$d(a) \geq d(w) \text{ and } 2 \leq d_G(a, w) \leq 3, \quad \text{for each } a \in N_X(u) \setminus N_X[\{v, w\}]; \tag{1}$$

$$d(b) \geq d(u) \text{ and } d_G(b, u) = 2, \quad \text{for each } b \in N_X(v) \setminus N_X[\{u, w\}]; \tag{2}$$

$$d(c) \geq d(u) \text{ and } 2 \leq d_G(c, u) \leq 3, \quad \text{for each } c \in N_X(w) \setminus N_X[\{u, v\}]; \tag{3}$$

$$d(d) \geq d(w) + 2 \text{ and } d_G(d, w) = 2, \quad \text{for each } d \in (N_X(u) \cap N_X(v)) \setminus N_X[w]; \tag{4}$$

$$d(e) \geq d(v) \text{ and } d_G(e, v) = 2, \quad \text{for each } e \in (N_X(u) \cap N_X(w)) \setminus N_X[v]; \tag{5}$$

$$d(f) \geq d(u) + 2 \text{ and } d_G(f, u) = 2, \quad \text{for each } f \in (N_X(v) \cap N_X(w)) \setminus N_X[u]; \tag{6}$$

and each vertex $g \in N_X(u) \cap N_X(v) \cap N_X(w)$ satisfies that

$$d(g) \geq d(u) + 2, \qquad d(g) \geq d(w) + 2, \quad \text{and} \quad d_G(u, w) = 2. \tag{7}$$

For each vertex $a \in N_X(u) \setminus N_X[\{v, w\}]$, according to (1), $|X| \leq \lfloor n(G)/2 \rfloor$, and Conditions (a) and (b), we obtain

$$\begin{aligned} |N_{\bar{X}}(a)| &= d(a) - |N_X(a)| \\ &\geq \frac{1}{2}(d(a) + d(w)) - (|X| - 3) \\ &\geq \lfloor n(G)/2 \rfloor - \frac{5}{2} - (\lfloor n(G)/2 \rfloor - 3) \\ &= \frac{1}{2}. \end{aligned}$$

Since $|N_{\bar{X}}(a)|$ is an integer, it follows that $|N_{\bar{X}}(a)| \geq 1$. Similarly, we can deduce that

$$|N_{\bar{X}}(b)| \geq 3, \quad \text{for each } b \in N_X(v) \setminus N_X[\{u, w\}];$$
$$|N_{\bar{X}}(c)| \geq 1, \quad \text{for each } c \in N_X(w) \setminus N_X[\{u, v\}];$$
$$|N_{\bar{X}}(d)| \geq 3, \quad \text{for each } d \in (N_X(u) \cap N_X(v)) \setminus N_X[w];$$
$$|N_{\bar{X}}(e)| \geq 2, \quad \text{for each } e \in (N_X(u) \cap N_X(w)) \setminus N_X[v];$$
$$|N_{\bar{X}}(f)| \geq 3, \quad \text{for each } f \in (N_X(v) \cap N_X(w)) \setminus N_X[u];$$
$$|N_{\bar{X}}(g)| \geq 3, \quad \text{for each } g \in N_X(u) \cap N_X(v) \cap N_X(w).$$

Case 2. $G[\{u, v, w\}]$ is a triangle. By the choice of vertices u, v and w, we have

$$d(a) \geq d(w) - 2 \text{ and } d_G(a, w) = 2, \quad \text{for each } a \in N_X(u) \setminus N_X[\{v, w\}]; \tag{8}$$

$$d(b) \geq d(u) - 2 \text{ and } d_G(b, u) = 2, \quad \text{for each } b \in N_X(v) \setminus N_X[\{u, w\}]; \tag{9}$$

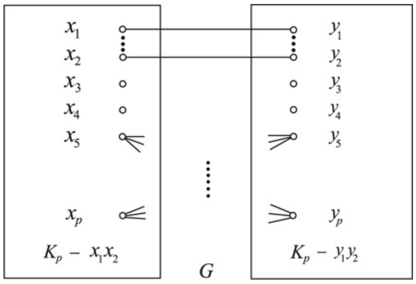

Fig. 4. A non-λ_3-optimal graph not satisfying Condition (b) of Theorem 2.3.

$$d(c) \geq d(v) - 2 \text{ and } d_G(c, v) = 2, \quad \text{for each } c \in N_X(w) \setminus N_X[\{u, v\}]; \tag{10}$$

$$d(d) \geq d(w) \text{ and } d_G(d, w) = 2, \quad \text{for each } d \in (N_X(u) \cap N_X(v)) \setminus N_X[w]; \tag{11}$$

$$d(e) \geq d(v) \text{ and } d_G(e, v) = 2, \quad \text{for each } e \in (N_X(u) \cap N_X(w)) \setminus N_X[v]; \tag{12}$$

$$d(f) \geq d(u) \text{ and } d_G(f, u) = 2, \quad \text{for each } f \in (N_X(v) \cap N_X(w)) \setminus N_X[u]; \tag{13}$$

and each vertex $g \in N_X(u) \cap N_X(v) \cap N_X(w)$ satisfies that

$$d(g) \geq \max\{d(u), d(v), d(w)\} \quad \text{and} \quad G[\{u, v, w, g\}] \text{ is a } K_4. \tag{14}$$

For each vertex $a \in N_X(u) \setminus N_X[\{v, w\}]$, according to (8), $|X| \leq \lfloor n(G)/2 \rfloor$ and Condition (b), we obtain

$$\begin{aligned} |N_{\bar{X}}(a)| &= d(a) - |N_X(a)| \\ &\geq \frac{1}{2}(d(a) + d(w) - 2) - (|X| - 3) \\ &\geq \lfloor n(G)/2 \rfloor - \frac{3}{2} - (\lfloor n(G)/2 \rfloor - 3) \\ &= \frac{3}{2}. \end{aligned}$$

So $|N_{\bar{X}}(a)| \geq 2$. Similarly, we can deduce that

$$\begin{aligned} &|N_{\bar{X}}(b)| \geq 2, \quad \text{for each } b \in N_X(v) \setminus N_X[\{u, w\}]; \\ &|N_{\bar{X}}(c)| \geq 2, \quad \text{for each } c \in N_X(w) \setminus N_X[\{u, v\}]; \\ &|N_{\bar{X}}(d)| \geq 2, \quad \text{for each } d \in (N_X(u) \cap N_X(v)) \setminus N_X[w]; \\ &|N_{\bar{X}}(e)| \geq 2, \quad \text{for each } e \in (N_X(u) \cap N_X(w)) \setminus N_X[v]; \\ &|N_{\bar{X}}(f)| \geq 2, \quad \text{for each } f \in (N_X(v) \cap N_X(w)) \setminus N_X[u]; \\ &|N_{\bar{X}}(g)| \geq 3, \quad \text{for each } g \in N_X(u) \cap N_X(v) \cap N_X(w). \end{aligned}$$

Hence, in both two cases, we have

$$\begin{aligned} \lambda_4(G) &= |\partial(X)| = |[\{u, v, w\}, \bar{X}]| + |[X \setminus \{u, v, w\}, \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |[N_X(u) \setminus N_X[\{v, w\}], \bar{X}]| + |[N_X(v) \setminus N_X[\{u, w\}], \bar{X}]| \\ &\quad + |[N_X(w) \setminus N_X[\{u, v\}], \bar{X}]| + |[(N_X(u) \cap N_X(v)) \setminus N_X[w], \bar{X}]| + |[(N_X(u) \cap N_X(w)) \setminus N_X[v], \bar{X}]| \\ &\quad + |[(N_X(v) \cap N_X(w)) \setminus N_X[u], \bar{X}]| + |[N_X(u) \cap N_X(v) \cap N_X(w), \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |N_X(u) \setminus N_X[\{v, w\}]| + 2|N_X(v) \setminus N_X[\{u, w\}]| \\ &\quad + |N_X(w) \setminus N_X[\{u, v\}]| + 2|(N_X(u) \cap N_X(v)) \setminus N_X[w]| \\ &\quad + 2|(N_X(u) \cap N_X(w)) \setminus N_X[v]| + 2|(N_X(v) \cap N_X(w)) \setminus N_X[u]| + 3|N_X(u) \cap N_X(v) \cap N_X(w)| \\ &\geq |\partial(\{u, v, w\})| \geq \xi_3(G). \quad \square \end{aligned}$$

Remark 2. The following examples illustrate that Conditions (b) and (c) in Theorem 2.3 cannot be weakened.

Example 1. Let H_i, $i = 1, 2$ be two copies of K_p, $p \geq 7$ with $V(H_1) = \{x_1, x_2, \ldots, x_p\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$. The graph G is defined as the disjoint union of $H_1 - x_1x_2$ and $H_2 - y_1y_2$ together with additional x_1y_1, x_2y_2 and $3p - 12$ edges between $\{x_5, x_6, \ldots, x_p\}$ and $\{y_5, y_6, \ldots, y_p\}$ such that $d(x_i) = d(y_i) = p + 2$ for $i = 5, 6, \ldots, p$ (Fig. 4). Then, $n(G) = 2p$, $d(x_i) = d(y_i) = p - 1$ for $i = 1, 2, 3, 4$; $d(x_j) = d(y_j) = p + 2$ for $j = 5, 6, \ldots, p$. Clearly, G satisfies Conditions (a)

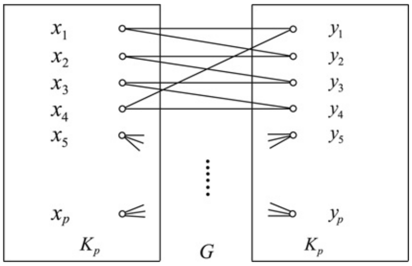

Fig. 5. A non-λ_3-optimal graph not satisfying Condition (c) of Theorem 2.3.

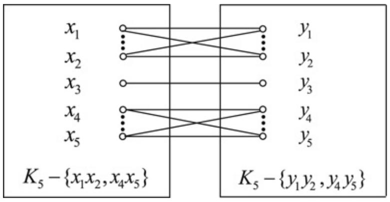

Fig. 6. A non-super-λ_3 graph G not satisfying Condition (b) of Theorem 2.5.

and (c) but not (b) of Theorem 2.3 as $d_G(x_1, x_2) = 2$ and $d(x_1) + d(x_2) = 2p - 2 < 2p - 1 = 2\lfloor n(G)/2\rfloor - 1$. However, $\xi_3(G) = 3p - 9$, and since the set S of edges joining $H_1 - x_1x_2$ and $H_2 - y_1y_2$ is a 3-restricted edge-cut and $|S| = 3p - 10$, then $\lambda_3(G) \le 3p - 10$. Thus, G is non-λ_3-optimal.

Example 2. Let H_i, $i = 1, 2$ be two copies of K_p, $p \ge 7$ with $V(H_1) = \{x_1, x_2, \ldots, x_p\}$ and $V(H_2) = \{y_1, y_2, \ldots, y_p\}$. The graph G is defined as the disjoint union of H_1 and H_2 by adding 8 edges x_4y_1, x_4y_4, x_iy_i and x_iy_{i+1} for $i = 1, 2, 3$, and $3p - 12$ edges between $\{x_5, x_6, \ldots, x_p\}$ and $\{y_5, y_6, \ldots, y_p\}$ such that $d(x_j) = d(y_j) = p + 2$ for $j = 5, 6, \ldots, p$ (Fig. 5). Then, $n(G) = 2p$ and G satisfies Conditions (a) and (b) but not (c) of Theorem 2.3 as $G[\{x_1, x_2, x_3, x_4\}]$ is a K_4 and $d(x_i) = p + 1 < p + 2 = \lfloor n(G)/2\rfloor + 2$ for $i = 1, 2, 3, 4$. However, $\xi_3(G) = 3p - 3$ and $\lambda_3(G) \le 3p - 4$ (since the set S of edges that join H_1 and H_2 is a 3-restricted edge-cut and $|S| = 3p - 4$). Hence, G is not λ_3-optimal.

Corollary 2.4. *Let G be a connected K_4-free graph with $n(G) \ge 6$. Then G is λ_3-optimal if the following two conditions hold:*
(a) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor - 5$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 3$, *and*
(b) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor - 1$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 2$.

Similarly to the proof of Theorem 2.3, by Lemma 2.1 (b) we can obtain the following theorem.

Theorem 2.5. *Let G be a connected graph with $n(G) \ge 6$. Then G is super-λ_3 if the following three conditions hold:*
(a) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor - 3$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 3$,
(b) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor + 1$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 2$, *and*
(c) *for each subgraph* K_4 *of* G, *there exists at least one vertex* $v \in K_4$ *with* $d(v) \ge \lfloor n(G)/2\rfloor + 3$.

Remark 3. (1) The example depicted in Fig. 6 shows that Condition (b) in Theorem 2.5 cannot be weakened. In Fig. 6, $n(G) = 10$, $d(v) = 5$ for $v \in V(G)$, and G fulfills Conditions (a) and (c) but not (b) of Theorem 2.5. Furthermore, by Theorem 2.3, $\lambda_3(G) = \xi_3(G) = 9$, and the edge set $S = \{x_iy_i, x_1y_2, x_2y_1, x_4y_5, x_5y_4 : i = 1, 2, \ldots, 5\}$ is a nontrivial λ_3-cut of G, so G is non-super-λ_3.

(2) $(p, 3)$-barbell ($p \ge 4$) is any graph G obtained by joining two copies of the complete graph K_p with $3p$ additional edges such that $d(v) = p+2$ for each vertex $v \in V(G)$. We see that $(p, 3)$-barbell satisfies Conditions (a) and (b) but not Condition (c) of Theorem 2.5. By Theorem 2.3, $(p, 3)$-barbell is λ_3-optimal. Also, the set of $3p$ edges joining two copies of the complete graph K_p is a nontrivial λ_3-cut. So it is not super-λ_3 and thus Condition (c) of Theorem 2.5 cannot be weakened.

Corollary 2.6. *Let G be a connected K_4-free graph with $n(G) \ge 6$. Then G is super-λ_3 if the following two conditions hold:*
(a) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor - 3$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 3$, *and*
(b) $d(x) + d(y) \ge 2\lfloor n(G)/2\rfloor + 1$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 2$.

3. Conditions for triangle-free graphs

Hellwig and Volkmann [8] gave the following result about the λ_2-optimality of triangle-free graphs:

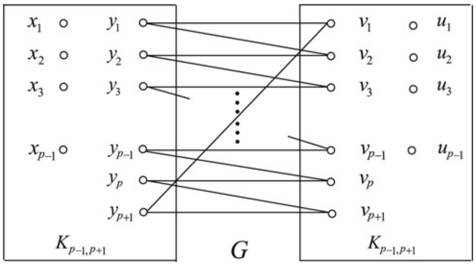

Fig. 7. A non-λ_3-optimal triangle-free graph with $d(v) = p + 1$.

Theorem 3.1 ([8]). *Let G be a λ_2-connected triangle-free graph. If $d(x) \geq \lfloor (n(G) + 2)/4 \rfloor + 1$ for all vertices x in G with at most one exception, then G is λ_2-optimal.*

Inspired by the ideas in [8], we present the following two theorems.

Theorem 3.2. *Let G be a connected triangle-free graph with $n(G) \geq 6$. If $d(x) \geq \lfloor (n(G) + 2)/4 \rfloor + 2$ for all vertices x in $V(G)$ with at most one exception, then G is λ_3-optimal.*

Proof. Since $n(G) \geq 6$, then $d(x) \geq \lfloor (n(G) + 2)/4 \rfloor + 2 \geq 4$ for all vertices x in $V(G)$ with at most one exception. Hence G cannot be isomorphic to the 3-leg spider graph or a subgraph of the friendship graph. By Theorem 1.2, G is λ_3-connected. It now suffices to prove that $\lambda_4(G) \geq \xi_3(G)$ by Lemma 2.1 (a). Let $\partial(X)$ be any λ_4-cut of G with $|X| \leq |\bar{X}|$. This implies $4 \leq |X| \leq \lfloor n(G)/2 \rfloor$. Choose one vertex v in X such that $d(v) = \min\{d(x) : x \in X\}$ and let $u, w \in X$ such that $G[\{u, v, w\}]$ is connected. Using Turán's [17] bound $2|E(G)| \leq n(G)^2/2$ for triangle-free graphs G, we have

$$\begin{aligned}
\lambda_4(G) &= |\partial(X)| = \sum_{x \in X} d(x) - 2|E(G[X])| \\
&\geq d(u) + d(v) + d(w) - 4 + 4 + \sum_{x \in X \setminus \{u,v,w\}} d(x) - \frac{|X|^2}{2} \\
&\geq \xi_3(G) + (|X| - 3)(\lfloor (n(G) + 2)/4 \rfloor + 2) - \frac{1}{2}(|X|^2 - 8) \\
&= \xi_3(G) + \frac{1}{2}(|X| - 3)(2\lfloor (n(G) + 2)/4 \rfloor - |X| + 1) - \frac{1}{2} \\
&\geq \xi_3(G) + \frac{1}{2}(2\lfloor (n(G) + 2)/4 \rfloor - \lfloor n(G)/2 \rfloor + 1) - \frac{1}{2} \\
&\geq \xi_3(G). \quad \square
\end{aligned}$$

In the proof above, when $n(G) \geq 10$, we have
if $|X| = 4$, then

$$\lambda_4(G) \geq \xi_3(G) + \lfloor (n(G) + 2)/4 \rfloor - 2 > \xi_3(G);$$

if $|X| \geq 5$, then

$$\lambda_4(G) \geq \xi_3(G) + 2\lfloor (n(G) + 2)/4 \rfloor - \lfloor n(G)/2 \rfloor + 1 - \frac{1}{2} > \xi_3(G).$$

By Lemma 2.1 (b), G is super-λ_3. So we have the following theorem.

Theorem 3.3. *Let G be a connected triangle-free graph with $n(G) \geq 10$. If $d(x) \geq \lfloor (n(G) + 2)/4 \rfloor + 2$ for all vertices x in $V(G)$ with at most one exception, then G is super-λ_3.*

Remark 4. The example depicted in Fig. 7 ($p \geq 4$) shows that the results of Theorems 3.2 and 3.3 are the best possible. In Fig. 7, G is a bipartite graph with $n(G) = 4p$, and $d(v) = p + 1 < p + 2 = \lfloor (n(G) + 2)/4 \rfloor + 2$ for all $v \in V(G)$. However, $\xi_3(G) = 3p - 1$ and $\lambda_3(G) \leq |\{y_i v_i, y_i v_{i+1}, y_{p+1} v_{p+1}, y_{p+1} v_1 : i = 1, 2, \ldots, p\}| = 2p + 2$, so G is non-λ_3-optimal.

4. Conditions for bipartite graphs

In regard to the λ_3-optimality of bipartite graphs, also inspired by the ideas of Hellwig and Volkmann in [8], we obtain the following result.

Theorem 4.1. *Let G be a connected bipartite graph with $n(G) \geq 6$. If:*
(a) $d(x) + d(y) \geq 2\lfloor (n(G)+2)/4 \rfloor - 1$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 3$ *and*
(b) $d(x) + d(y) \geq 2\lfloor (n(G)+2)/4 \rfloor + 3$ *for each pair* $x, y \in V(G)$ *with* $d_G(x, y) = 2$
hold, then G is λ_3*-optimal.*

Proof. From $n(G) \geq 6$ and Condition (b), it follows that $d(x)+d(y) \geq 2\lfloor (n(G)+2)/4 \rfloor + 3 \geq 7$ for each pair $x, y \in V(G)$ with $d_G(x, y) = 2$. So G cannot be isomorphic to the 3-leg spider graph or a subgraph of the friendship graph. By Theorem 1.2, G is λ_3-connected. By Lemma 2.1 (a), it suffices to show that $\lambda_4(G) \geq \xi_3(G)$. Let (A, B) be the bipartition of G and $\partial(X)$ any λ_4-cut of G with $|X| \leq |\bar{X}|$. This implies $4 \leq |X| \leq \lfloor n(G)/2 \rfloor$. Set $X' := X \cap A$ and $X'' := X \cap B$. We assume, without loss of generality, that $|X'| \leq |X''|$. It follows that $|X'| \leq \lfloor n(G)/4 \rfloor$. Choose three vertices u, v, and w in X such that $G[\{u, v, w\}]$ is connected and satisfies that

$$|\partial(\{u, v, w\})| = \min\{|\partial(H)| : H \subseteq X, |H| = 3, \text{ and } G[H] \text{ is connected}\}$$

and X' contains as more as possible vertices of $\{u, v, w\}$. Since G is bipartite, $G[\{u, v, w\}]$ is a path. We assume that $uw \notin E(G)$. By the choice of u, v and w, we have

$$d(a) \geq d(w) \text{ and } d_G(a, w) = 3, \quad \text{for each } a \in N_X(u) \setminus N_X(w); \tag{15}$$

$$d(b) \geq d(u) \text{ and } d_G(b, u) = 3, \quad \text{for each } b \in N_X(w) \setminus N_X(u); \tag{16}$$

$$d(c) \geq d(u) \text{ and } d_G(c, u) = 2, \quad \text{for each } c \in N_X(v) \setminus \{u, w\}; \tag{17}$$

$$d(f) \geq d(v) \text{ and } d_G(f, v) = 2, \quad \text{for each } f \in (N_X(u) \cap N_X(w)) \setminus \{v\}. \tag{18}$$

Case 1. $|X'| = \lfloor n(G)/4 \rfloor$. It follows that $\lfloor n(G)/4 \rfloor \leq |X''| \leq \lfloor n(G)/4 \rfloor + 1$ from $|X| \leq \lfloor n(G)/2 \rfloor$.

Subcase 1.1. $|X''| = \lfloor n(G)/4 \rfloor$. We assume, without loss of generality, that $u, w \in X'$ and $v \in X''$. According to (15) and Condition (a), we obtain

$$\begin{aligned} |N_{\bar{X}}(a)| &= d(a) - |N_X(a)| \\ &\geq \frac{1}{2}(d(a) + d(w)) - (|X'| - 1) \\ &\geq \lfloor (n(G)+2)/4 \rfloor - \frac{1}{2} - (\lfloor n(G)/4 \rfloor - 1) \\ &\geq \frac{1}{2}, \end{aligned}$$

for each $a \in N_X(u) \setminus N_X(w)$. So $|N_{\bar{X}}(a)| \geq 1$. Similarly, we have

$$|N_{\bar{X}}(b)| \geq 1, \quad \text{for each } b \in N_X(w) \setminus N_X(u);$$
$$|N_{\bar{X}}(c)| \geq 2, \quad \text{for each } c \in N_X(v) \setminus \{u, w\};$$
$$|N_{\bar{X}}(f)| \geq 2, \quad \text{for each } f \in (N_X(u) \cap N_X(w)) \setminus \{v\}.$$

Hence,

$$\begin{aligned} \lambda_4(G) &= |\partial(X)| = |[\{u, v, w\}, \bar{X}]| + |[X \setminus \{u, v, w\}, \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |[N_X(u) \setminus N_X(w), \bar{X}]| + |[N_X(w) \setminus N_X(u), \bar{X}]| \\ &\quad + |[N_X(v) \setminus \{u, w\}, \bar{X}]| + |[(N_X(u) \cap N_X(w)) \setminus \{v\}, \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |N_X(u) \setminus N_X(w)| + |N_X(w) \setminus N_X(u)| \\ &\quad + 2|N_X(v) \setminus \{u, w\}| + 2|(N_X(u) \cap N_X(w)) \setminus \{v\}| \\ &\geq |\partial(\{u, v, w\})| \geq \xi_3(G). \end{aligned}$$

Subcase 1.2. $|X''| = \lfloor n(G)/4 \rfloor + 1$. Then $n(G) \equiv 2$ or $3 \pmod 4$ by $\lfloor n(G)/4 \rfloor + \lfloor n(G)/4 \rfloor + 1 = |X'| + |X''| = |X| \leq \lfloor n(G)/2 \rfloor$. This implies that $\lfloor (n(G)+2)/4 \rfloor = \lfloor n(G)/4 \rfloor + 1$, hence, Conditions (a) and (b) are equivalent to the following (a)′ and (b)′, respectively.
(a)′ $d(x) + d(y) \geq 2\lfloor n(G)/4 \rfloor + 1$ for each pair $x, y \in V(G)$ such that $d_G(x, y) = 3$;
(b)′ $d(x) + d(y) \geq 2\lfloor n(G)/4 \rfloor + 5$ for each pair $x, y \in V(G)$ such that $d_G(x, y) = 2$.
By a similar proof of Subcase 1.1, we can obtain the desired result.

Case 2. $|X'| \leq \lfloor n(G)/4 \rfloor - 1$.

Subcase 2.1 $u, w \in X'$ and $v \in X''$. According to (15)–(18) and Conditions (a) and (b), by a similar reasoning of Subcase 1.1, we have

$$|N_{\bar{X}}(a)| \geq 2, \quad \text{for each } a \in N_X(u) \setminus N_X(w);$$
$$|N_{\bar{X}}(b)| \geq 2, \quad \text{for each } b \in N_X(w) \setminus N_X(u);$$
$$|N_{\bar{X}}(f)| \geq 3, \quad \text{for each } f \in (N_X(u) \cap N_X(w)) \setminus \{v\}.$$

For each $c \in N_X(v) \setminus \{u, w\}$, if $|N_{\bar{X}}(c)| \geq 1$, as in Subcase 1.1, we can obtain $\lambda_4(G) \geq \xi_3(G)$. Otherwise, there exists one vertex $c_0 \in N_X(v) \setminus \{u, w\}$ such that $|N_{\bar{X}}(c_0)| = 0$, and by (17) and Condition (b), we have

$$|N_X(c_0)| = d(c_0) \geq \frac{1}{2}(d(u) + d(c_0)) \geq \lfloor (n(G)+2)/4 \rfloor + \frac{3}{2}. \tag{19}$$

Choose one vertex x in $N_X(c_0) \setminus \{v\}$ such that

$$d(x) = \min\{d(y) : y \in N_X(c_0) \setminus \{v\}\}.$$

For each $y \in N_X(c_0) \setminus (N_X(\{u, w\}) \cup \{x\})$, $d(y) \geq d(x)$ and $d_G(x, y) = 2$, hence

$$\begin{aligned} |N_{\bar{X}}(y)| &\geq \frac{1}{2}(d(x) + d(y)) - |X'| \\ &\geq \lfloor (n(G)+2)/4 \rfloor + \frac{3}{2} - \lfloor n(G)/4 \rfloor + 1 \\ &\geq \frac{5}{2}. \end{aligned}$$

From (19) and $|N_X(v) \setminus \{u, w\}| \leq |X'| - 2 \leq \lfloor n(G)/4 \rfloor - 3$, we have

$$|N_X(c_0) \setminus \{x, v\}| \geq \lfloor (n(G)+2)/4 \rfloor - \frac{1}{2} > |N_X(v) \setminus \{u, w\}|. \tag{20}$$

Then

$$\begin{aligned} \lambda_4(G) &= |\partial(X)| = |[\{u, v, w\}, \bar{X}]| + |[X \setminus \{u, v, w\}, \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |[N_X(u) \setminus N_X(w), \bar{X}]| + |[N_X(w) \setminus N_X(u), \bar{X}]| \\ &\quad + |[(N_X(u) \cap N_X(w)) \setminus \{v\}, \bar{X}]| + |[N_X(c_0) \setminus (N_X(\{u, w\}) \cup \{x\}), \bar{X}]| \\ &\geq |[\{u, v, w\}, \bar{X}]| + 2|N_X(u) \setminus N_X(w)| + 2|N_X(w) \setminus N_X(u)| \\ &\quad + 3|(N_X(u) \cap N_X(w)) \setminus \{v\}| + 3|N_X(c_0) \setminus (N_X(\{u, w\}) \cup \{x\})| \\ &\geq |[\{u, v, w\}, \bar{X}]| + |N_X(u) \setminus N_X(w)| + |N_X(w) \setminus N_X(u)| + 2|(N_X(u) \cap N_X(w)) \setminus \{v\}| + |N_X(c_0) \setminus \{x, v\}| \\ &> |\partial(\{u, v, w\})| \geq \xi_3(G). \end{aligned}$$

Subcase 2.2. $u, w \in X''$ and $v \in X'$. According to (17), each vertex $c \in N_X(v) \setminus \{u, w\}$ satisfies

$$\begin{aligned} |N_{\bar{X}}(c)| &\geq \frac{1}{2}(d(u) + d(c)) - |X'| \\ &\geq \lfloor (n(G)+2)/4 \rfloor + \frac{3}{2} - \lfloor n(G)/4 \rfloor + 1 \\ &\geq \frac{5}{2}. \end{aligned}$$

If each vertex $x \in (N_X(u) \setminus N_X(w)) \cup (N_X(w) \setminus N_X(u))$ has at least one neighbor in $\bar{X}$ and each vertex $f \in (N_X(u) \cap N_X(w)) \setminus \{v\}$ has at least two neighbors in $\bar{X}$, as in Subcase 1.1, we can deduce that $\lambda_4(G) \geq \xi_3(G)$.

Otherwise, if there exists one vertex $x \in (N_X(u) \setminus N_X(w)) \cup (N_X(w) \setminus N_X(u))$ such that $|N_{\bar{X}}(x)| = 0$. By the choice of the vertices u, v and w, either $d(x) > d(w)$ and $d_G(x, w) = 3$ or $d(x) > d(u)$ and $d_G(x, u) = 3$. According to Condition (a),

$$|N_X(x) \setminus \{u, w\}| = d(x) - 1 > \lfloor (n(G)+2)/4 \rfloor - \frac{3}{2}. \tag{21}$$

If there exists one vertex $x \in N_X(u) \cap N_X(w)$ such that $|N_{\bar{X}}(x)| \leq 1$, then from (18) it follows that

$$|N_X(x) \setminus \{u, w\}| \geq \frac{1}{2}(d(v) + d(x)) - 3 \geq \lfloor (n(G)+2)/4 \rfloor - \frac{3}{2}. \tag{22}$$

Choose one vertex z_0 in $N_X(x) \setminus \{u, w\}$ such that

$$d(z_0) = \min\{d(z) : z \in N_X(x) \setminus \{u, w\}\}.$$

Then, each vertex $z \in N_X(x) \setminus (N_X(v) \cup \{z_0\})$ satisfies that $d(z) \geq d(z_0)$ and $d_G(z, z_0) = 2$. By Condition (b),

$$|N_{\bar{X}}(z)| \geq \lfloor (n(G)+2)/4 \rfloor + \frac{3}{2} - (\lfloor n(G)/4 \rfloor - 1) \geq \frac{5}{2}.$$

According to (21), (22) and $|N_X(\{u, w\}) \setminus \{v\}| \leq |X'| - 1 \leq \lfloor n(G)/4 \rfloor - 2$, we have

$$|N_X(x) \setminus \{u, w, z_0\}| \geq \lfloor (n(G)+2)/4 \rfloor - 2 \geq |N_X(\{u, w\}) \setminus \{v\}|. \tag{23}$$

Hence,

$$\begin{aligned}\lambda_4(G) &= |\partial(X)| = |[\{u,v,w\},\bar{X}]| + |[X \setminus \{u,v,w\},\bar{X}]| \\ &\geq |[\{u,v,w\},\bar{X}]| + |[N_X(v) \setminus \{u,w\},\bar{X}]| + |[N_X(x) \setminus (N_X(v) \cup \{z_0\}),\bar{X}]| \\ &\geq |[\{u,v,w\},\bar{X}]| + 3|N_X(v) \setminus \{u,w\}| + 3|N_X(x) \setminus (N_X(v) \cup \{z_0\})| \\ &\geq |[\{u,v,w\},\bar{X}]| + |N_X(v) \setminus \{u,w\}| + 2|N_X(x) \setminus \{u,w,z_0\}| \\ &\geq |[\{u,v,w\},\bar{X}]| + |N_X(v) \setminus \{u,w\}| + 2|N_X(\{u,w\}) \setminus \{v\}| \\ &\geq |\partial(\{u,v,w\})| \geq \xi_3(G). \quad \square\end{aligned}$$

Remark 5. It is easy to test that the bipartite graph G depicted in Fig. 7 satisfies Condition (a) but not Condition (b) of Theorem 4.1, and by Remark 4, G is not λ_3-optimal. Hence Condition (b) of Theorem 4.1 cannot be weakened.

Similarly to the proof of Theorem 4.1, by Lemma 2.1 (b) we can show the following theorem.

Theorem 4.2. *Let G be a connected bipartite graph with $n(G) \geq 6$. If:*
(a) $d(x) + d(y) \geq 2\lfloor (n(G)+2)/4 \rfloor + 1$ *for each pair* $x, y \in V(G)$ *with* $d_G(x,y) = 3$ *and*
(b) $d(x) + d(y) \geq 2\lfloor (n(G)+2)/4 \rfloor + 5$ *for each pair* $x, y \in V(G)$ *with* $d_G(x,y) = 2$
hold, then G is super-λ_3.

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