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Enumeration of strings in Dyck paths: A bijective approach

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ABSTRACT

The statistics concerning the number of appearances of a string τ in Dyck paths as well as its appearances in odd and even level have been studied extensively by several authors using mostly algebraic methods. In this work a different, bijective approach is followed giving some known as well as some new results.

1. Introduction

A *Dyck path of semilength* n is a lattice path in the first quadrant, which begins at the origin $(0, 0)$, ends at $(2n, 0)$ and consists of steps $(1, 1)$ (called *rises*) and $(1, -1)$ (called *falls*). We can encode each rise by the letter u and each fall by d obtaining the encoding of a Dyck path by a so called *Dyck word*. For example, the encoding of the Dyck path of Fig. 1 is the Dyck word $\alpha = uuduuddududdduduudd$.

Throughout this paper we denote with $\mathcal{D}$ the set of all Dyck paths (or equivalently Dyck words). Furthermore, the subset of $\mathcal{D}$ that contains all the paths α of semilength $l(\alpha) = n$ is denoted by $\mathcal{D}_n$. We note that $\mathcal{D}_0$ consists only of the empty Dyck path, denoted by ϵ.

It is well-known that $|\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number (A000108 of [16]).

A word $\tau \in \{u, d\}^*$, called in this context *string*, *occurs* in a Dyck path α if $\alpha = \beta\tau\gamma$, where $\beta, \gamma \in \{u, d\}^*$. A string τ *occurs at height* j in a Dyck path if the minimum height of the points of τ in this occurrence is equal to j. For example, the Dyck path of Fig. 1 has three occurrences of the string udu; two at height 1 and one at height 0.

In this paper we deal with the statistics N_τ "number of occurrences of τ", E_τ "number of occurrences of τ at even height" and O_τ "number of occurrences of τ at odd height".

A wide range of articles dealing with occurrences of various strings appear frequently in the literature (e.g., see [2,8,11–15,18,19]). Recently, a systematic work concerning all strings of length up to four was given in [17]. There it has been proved that for every string τ of length up to three (except $\tau = ud$) there exist strings τ_1, τ_2 of length one more than the length of τ such that the statistics E_τ and O_τ are equidistributed with the statistics N_{τ_1} and N_{τ_2} respectively. These results have been proved algebraically, by identifying the corresponding generating functions. In this paper we give combinatorial proofs of these results as well as some new results for strings of length four.

Several bijections on Dyck paths appear in the literature (e.g., see [3–7,9,10]), usually introduced in order to show the equidistribution of statistics. Here two length-preserving bijections on $\mathcal{D}$, and some variations of them are presented. It is shown that some of the introduced bijections (depending on the string τ) send the statistic E_τ to the statistic N_{τ_1} and some send the statistic O_τ to the statistic N_{τ_2}, thus verifying the required equidistribution.

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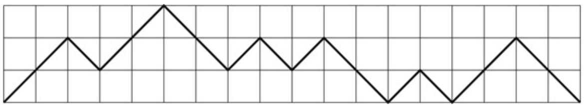

Fig. 1. The Dyck path $\alpha = uuduuddududduduudd$.

Fig. 2. The decomposition of a Dyck path into prime components.

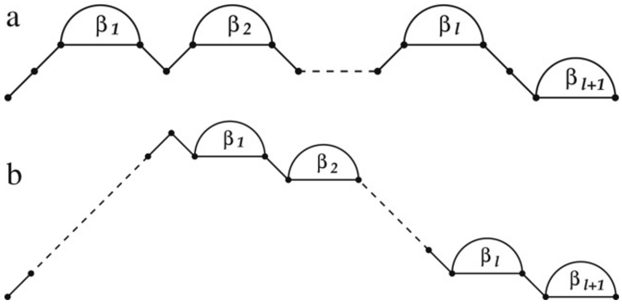

Fig. 3. Decompositions of $\mathcal{B}$.

It is well known that every non-empty Dyck path α can be decomposed uniquely in the form $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathcal{D}$. This is the so called *first return decomposition*.

For the construction of the required bijections we will consider some finer decompositions.

A Dyck path α which is the elevation of some $\beta \in \mathcal{D}$, i.e. $\alpha = \widehat{\beta} = u\beta d$, is called a *prime Dyck path*. We denote with $\widehat{\mathcal{D}}$ the set of all prime Dyck paths.

Using recursively the first return decomposition we obtain the decomposition of a Dyck path α into prime Dyck paths (usually called *prime components*), i.e. $\alpha = \widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}$, where $\beta_1, \beta_2, \ldots, \beta_l \in \mathcal{D}$; (see Fig. 2).

Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of all Dyck paths with length of the first ascent equal to (resp. greater than) one. In other words every path of $\mathcal{A}$ (resp. $\mathcal{B}$) starts with ud (resp. uu). The sets $\mathcal{A}$, $\mathcal{B}$ form a partition of the set of all non-empty Dyck paths.

Using the first prime component of a Dyck path $\alpha \in \mathcal{B}$ we obtain several types of decompositions of α.

Firstly, if we decompose β into prime components we obtain the following decomposition of α:

$$\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1}$$

where $\beta_1, \beta_2, \ldots, \beta_{l+1} \in \mathcal{D}$; (see Fig. 3(a)).

Next, by using the first ascent of β we obtain the following decomposition of α

$$\alpha = u^{l+1}d\beta_1 d\beta_2 d\cdots\beta_l d\beta_{l+1}$$

where $\beta_1, \beta_2, \ldots, \beta_{l+1} \in \mathcal{D}$; (see Fig. 3(b)).

The key to the construction of the required bijections is to send Dyck paths of the form of Fig. 3(a) to Dyck paths of either the form of Fig. 2 or the form of Fig. 3(b).

2. Main results

We start by giving two involutions, which are used in order to prove several equidistributions.

2.1. The involutions χ and θ

We define recursively two mappings $\chi, \theta : \mathcal{D} \to \mathcal{D}$ as follows:
$\chi(\epsilon) = \theta(\epsilon) = \epsilon$, $\chi(ud\gamma) = ud\chi(\gamma)$, $\chi(u\widehat{\beta}\delta d\gamma) = u\widehat{\chi(\gamma)}\theta(\delta)d\chi(\beta)$ and
$\theta(\widehat{\beta}\gamma) = \theta(\gamma)\widehat{\chi(\beta)}$; (see Fig. 4).
For example for the Dyck path α of Fig. 1 we obtain

$$\chi(\alpha) = uu\chi(uduudd)d\theta(uuddudud)d\chi(\epsilon) = uuuduudddududuuddd$$

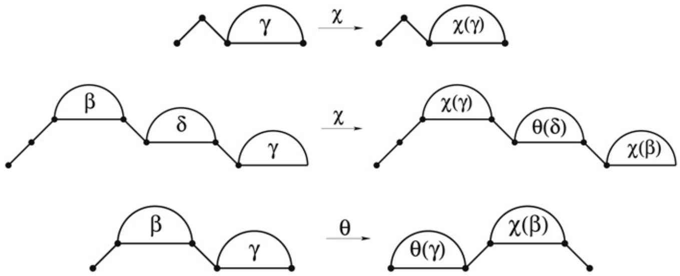

Fig. 4. The involutions χ and θ.

and

$$\theta(\alpha) = \theta(uduudd)u\chi(uduuddudud)d = uudduduuduuududdddd.$$

By iterating the recursion of the definition of θ we obtain

$$\theta(\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}) = \widehat{\chi(\beta_l)}\cdots\widehat{\chi(\beta_2)}\widehat{\chi(\beta_1)}.$$

Furthermore, using induction on path length it is shown simultaneously that the above two mappings are involutions. Indeed,

$$\chi^2(ud\gamma) = \chi(ud\chi(\gamma)) = ud\chi^2(\gamma) = ud\gamma,$$
$$\chi^2(u\widehat{\beta}\delta d\gamma) = \chi(u\widehat{\chi(\gamma)}\theta(\delta)d\chi(\beta)) = u\widehat{\chi^2(\beta)}\theta^2(\delta)d\chi^2(\gamma) = u\widehat{\beta}\delta d\gamma$$

and

$$\theta^2(\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}) = \theta(\widehat{\chi(\beta_l)}\cdots\widehat{\chi(\beta_2)}\widehat{\chi(\beta_1)}) = \widehat{\chi^2(\beta_1)}\widehat{\chi^2(\beta_2)}\cdots\widehat{\chi^2(\beta_l)} = \widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}.$$

These involutions have interesting properties: χ maps the sets $\mathcal{A}$, $\mathcal{B}$ to themselves and θ preserves the number of prime components and the length of each component. In addition, the following equalities can be proved inductively.

i. $E_{uuu}(\alpha) = E_{ddu}(\chi(\alpha))$, (A116424), $O_{uuu}(\alpha) = O_{ddu}(\theta(\alpha))$, (A114492).
ii. $E_{uuuu}(\alpha) = E_{dduu}(\chi(\alpha))$, $O_{uuuu}(\alpha) = O_{dduu}(\theta(\alpha))$.
iii. $E_{uuud}(\alpha) = E_{ddud}(\chi(\alpha))$, $O_{uuud}(\alpha) = O_{ddud}(\theta(\alpha))$.

We show only equalities ii since the proofs of i and iii are similar. For the first equality we restrict ourselves to the non-trivial case where $\alpha = u\widehat{\beta}\delta d\gamma$.

$$\begin{aligned} E_{uuuu}(\alpha) &= E_{uuuu}(\beta) + O_{uuuu}(\delta) + E_{uuuu}(\gamma) + [\beta \in \mathcal{B}] \\ &= E_{dduu}(\chi(\beta)) + O_{dduu}(\theta(\delta)) + E_{dduu}(\chi(\gamma)) + [\chi(\beta) \in \mathcal{B}] = E_{dduu}(\chi(\alpha)) \end{aligned}$$

where $[P]$ is the Iverson notation: $[P] = 1$ if P is true and $[P] = 0$ if P is false.
Furthermore, for $\alpha = \widehat{\beta}\gamma$ we have

$$O_{uuuu}(\widehat{\beta}\gamma) = E_{uuuu}(\beta) + O_{uuuu}(\gamma) = E_{dduu}(\chi(\beta)) + O_{dduu}(\theta(\gamma)) = O_{dduu}(\theta(\alpha)).$$

From the previous equalities we deduce that the statistics $E_\tau, E_{\tau'}$ as well as $O_\tau, O_{\tau'}$ are equidistributed when $(\tau, \tau') = (uuu, ddu)$, or $(uuuu, dduu)$, or $(uuud, ddud)$.

2.2. The bijection ϕ

We define recursively a mapping $\phi : \mathcal{D} \to \mathcal{D}$ as follows:
$\phi(\epsilon) = \epsilon$, $\phi(ud\gamma) = ud\phi(\gamma)$ and
for $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$, $\phi(\alpha) = u^{l+1}d\phi(\beta_1)d\phi(\beta_2)\cdots d\phi(\beta_l)d\phi(\beta_{l+1})$; (see Fig. 5).
For example for the Dyck path α of Fig. 1 we obtain

$$\phi(\alpha) = uuuuud\phi(\epsilon)d\phi(ud)d\phi(\epsilon)d\phi(\epsilon)d\phi(uduudd) = uuuuuddudddduduudd.$$

It is easy to check that ϕ is a bijection, it maps the sets $\mathcal{A}$, $\mathcal{B}$ to themselves, it preserves the length of the Dyck path, the number of prime components, the length of each component and it satisfies the product (concatenation) property

$$\phi(\alpha_1\alpha_2\cdots\alpha_k) = \phi(\alpha_1)\phi(\alpha_2)\cdots\phi(\alpha_k)$$

for every $\alpha_i \in \mathcal{D}$, $i \in [k]$.

The (non-empty) fixed points of ϕ are of the form $\prod_{i=1}^{k}\alpha_i$ where every α_i is a pyramid of height either 1 or 2. These paths are usually called *Fibonacci paths*; (see for example [1]).

Furthermore, ϕ can be used in order to show inductively the equidistribution of several statistics. In fact the following equalities are valid for every $\alpha \in \mathcal{D}$.

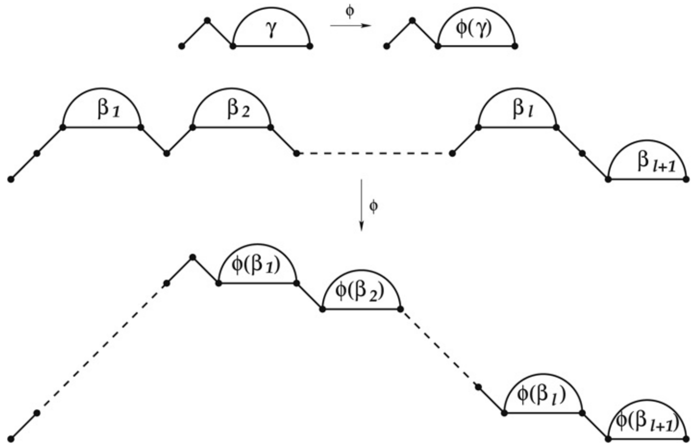

Fig. 5. The bijection ϕ.

1. $E_{uu}(\alpha) = N_{uud}(\phi(\alpha))$, (A091156).
2. $E_{ud}(\alpha) = \overline{N}_{dud}(\phi(\alpha))$, (A091867).
3. $O_{du}(\alpha) = N_{uuu}(\phi(\alpha))$, (A092107).
4. $E_{uuu}(\alpha) = N_{uudu}(\phi(\alpha))$, (A116424).
5. $E_{uud}(\alpha) = N_{uudd}(\phi(\alpha))$, (A098978).
6. $E_{uuuu}(\alpha) = N_{uuduu}(\phi(\alpha))$.
7. $E_{uudu}(\alpha) = N_{uuudd}(\phi(\alpha))$.
8. $E_{uuud}(\alpha) = N_{uudud}(\phi(\alpha))$.
9. $E_{uudd}(\alpha) = \overline{N}_{duudd}(\phi(\alpha))$.
10. $E_{udud}(\alpha) = \overline{N}_{dudud}(\phi(\alpha))$,

where $\overline{N}_{d\tau}(\alpha) = N_{d\tau}(\alpha) + [\alpha \text{ begins with } \tau]$.

Since the proofs are similar we show only equalities 3 and 8 for the non-trivial case where $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$.

$$\begin{aligned} O_{du}(\alpha) &= \sum_{i=1}^{l+1} O_{du}(\beta_i) + l - 1 = \sum_{i=1}^{l+1} N_{uuu}(\phi(\beta_i)) + l - 1 \\ &= N_{uuu}(\phi(\alpha)). \\ E_{uuud}(\alpha) &= \sum_{i=1}^{l+1} E_{uuud}(\beta_i) + [\beta_1 \in \mathcal{A}] = \sum_{i=1}^{l+1} N_{uudud}(\phi(\beta_i)) + [\phi(\beta_1) \in \mathcal{A}] \\ &= N_{uudud}(\phi(\alpha)). \end{aligned}$$

We remark that equalities 2, 9, 10 are special cases of the following result:

$$E_\tau(\alpha) = \overline{N}_{d\tau}(\phi(\alpha))$$

for every Fibonacci path τ.

For the proof we use the fact that ϕ satisfies the product property and τ is a fixed point of ϕ.

First for $\alpha = ud\gamma \in \mathcal{A}$ we have

$$\begin{aligned} E_\tau(\alpha) &= E_\tau(\gamma) + [\alpha \text{ begins with } \tau] \\ &= \overline{N}_{d\tau}(\phi(\gamma)) + [\phi(\alpha) \text{ begins with } \tau] \\ &= N_{d\tau}(\phi(\gamma)) + [\phi(\gamma) \text{ begins with } \tau] + [\phi(\alpha) \text{ begins with } \tau] \\ &= N_{d\tau}(\phi(\alpha)) + [\phi(\alpha) \text{ begins with } \tau] \\ &= \overline{N}_{d\tau}(\phi(\alpha)). \end{aligned}$$

Now, for $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$ we have

$$\begin{aligned} E_\tau(\alpha) &= \sum_{i=1}^{l+1} E_\tau(\beta_i) + [\alpha \text{ begins with } \tau] \\ &= \sum_{i=1}^{l+1} \overline{N}_{d\tau}(\phi(\beta_i)) + [\phi(\alpha) \text{ begins with } \tau] \end{aligned}$$

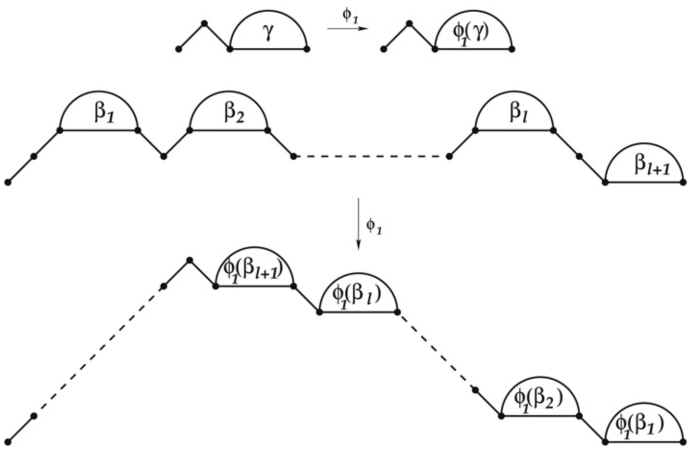

Fig. 6. The bijection ϕ_1.

$$\begin{aligned} &= \sum_{i=1}^{l+1} (N_{d\tau}(\phi(\beta_i)) + [\phi(\beta_i) \text{ begins with } \tau]) + [\phi(\alpha) \text{ begins with } \tau] \\ &= N_{d\tau}(\phi(\alpha)) + [\phi(\alpha) \text{ begins with } \tau] \\ &= \overline{N}_{d\tau}(\phi(\alpha)). \end{aligned}$$

There are two variations ϕ_1, ϕ_2 of ϕ obtained by changing the order of β_i's. The variation ϕ_1 is obtained by changing the order of $\phi(\beta_i)$'s, $i \in [l+1]$ in Fig. 3(b), placing $\phi(\beta_{l+1})$ first.

More precisely, we define recursively $\phi_1 : \mathcal{D} \to \mathcal{D}$ as follows:

$\phi_1(\epsilon) = \epsilon$, $\phi_1(ud\gamma) = ud\phi_1(\gamma)$ and

for $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$, $\phi_1(\alpha) = u^{l+1}d\phi_1(\beta_{l+1})d\phi_1(\beta_l)\cdots d\phi_1(\beta_2)d\phi_1(\beta_1)$; (see Fig. 6).

For example for the Dyck path α of Fig. 1 we obtain

$$\phi_1(\alpha) = uuuuud\phi_1(uduudd)d\phi_1(\epsilon)d\phi_1(\epsilon)d\phi_1(ud)d\phi_1(\epsilon) = uuuuududuudddddudd.$$

Using induction on path length it is shown that $\phi_1 = \phi \circ \chi$. Indeed,

$$(\phi \circ \chi)(ud\gamma) = \phi(ud\chi(\gamma)) = ud\phi(\chi(\gamma)) = ud\phi_1(\gamma) = \phi_1(ud\gamma)$$

and

$$\begin{aligned} (\phi \circ \chi)(u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1}) &= \phi(u\widehat{\chi(\beta_{l+1})}\theta(\widehat{\beta_2}\cdots\widehat{\beta_l})d\chi(\beta_1)) \\ &= \phi(u\widehat{\chi(\beta_{l+1})}\widehat{\chi(\beta_l)}\cdots\widehat{\chi(\beta_2)}d\chi(\beta_1)) \\ &= u^{l+1}d\phi(\chi(\beta_{l+1}))d\phi(\chi(\beta_l))\ldots d\phi(\chi(\beta_2))d\phi(\chi(\beta_1)) \\ &= u^{l+1}d\phi_1(\beta_{l+1})d\phi_1(\beta_l)\ldots d\phi_1(\beta_2)d\phi_1(\beta_1) \\ &= \phi_1(u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1}). \end{aligned}$$

Clearly, ϕ_1 is also a bijection, it maps the sets $\mathcal{A}$, $\mathcal{B}$ into themselves and it preserves the length of the Dyck path, although it does not preserve the number of prime components. Furthermore, ϕ_1 satisfies equalities 1, 2 and 3 of ϕ as well as the following equalities:

11. $E_{du}(\alpha) = N_{udu}(\phi_1(\alpha))$, (A091869).
12. $O_{uuu}(\alpha) = N_{dduu}(\phi_1(\alpha))$, (A114492).
13. $O_{uud}(\alpha) = N_{ddud}(\phi_1(\alpha))$, (A116424).
14. $E_{dud}(\alpha) = N_{udud}(\phi_1(\alpha))$, (A094507).
15. $E_{ddu}(\alpha) = N_{uudu}(\phi_1(\alpha))$, (A116424).
16. $E_{dudu}(\alpha) = N_{ududu}(\phi_1(\alpha))$.
17. $E_{dduu}(\alpha) = N_{uuduu}(\phi_1(\alpha))$.
18. $E_{ddud}(\alpha) = N_{uudud}(\phi_1(\alpha))$.
19. $O_{uudu}(\alpha) = N_{ddudu}(\phi_1(\alpha))$.

The proofs of the above equalities are in the same spirit as the proofs of equalities 1–10 and they are omitted.

Equalities 14, 16, 17 can be proved directly, or by using 1, 6 and 8 respectively, together with $\phi_1 = \phi \circ \chi$ and i, ii and iii.

We will show now a generalization of equalities 11 and 16, namely:

$E_{(du)^r}(\alpha) = N_{u(du)^r}(\phi_1(\alpha))$, for every $r \in \mathbb{N}^*$.

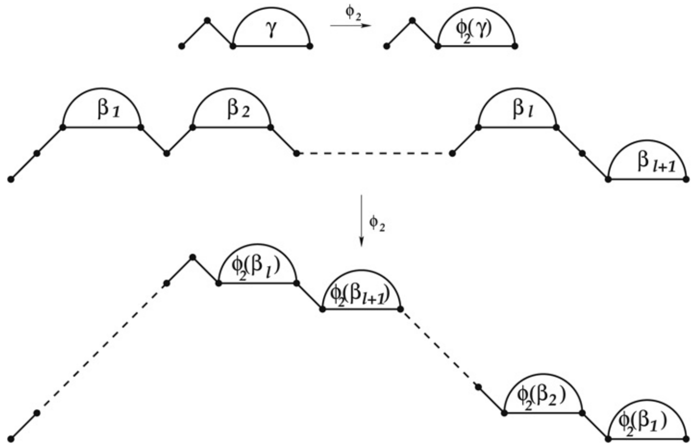

Fig. 7. The bijection ϕ_2.

Indeed, restricting again ourselves to the non-trivial case $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1}$ we have

$$\begin{aligned} E_{(du)^r}(\alpha) &= \sum_{i=1}^{l+1} E_{(du)^r}(\beta_i) + [\beta_{l+1} \text{ begins with } u(du)^{r-1}] \\ &= \sum_{i=1}^{l+1} N_{u(du)^r}(\phi_1(\beta_i)) + [\phi_1(\beta_{l+1}) \text{ begins with } u(du)^{r-1}] \\ &= N_{u(du)^r}(\phi_1(\alpha)). \end{aligned}$$

In addition, from i and 12 we can easily show the following equality:

20. $O_{ddu}(\alpha) = N_{dduu}((\phi_1 \circ \theta)(\alpha))$, (A114492).

We note that since ϕ_1 satisfies equalities 3 and 11, it sends the statistic N_{du} to the statistic N_{uxu} (*uxu* is either *uuu* or *udu*). Furthermore, if h is the reverse path involution (i.e., the mapping that flips every path with respect to a vertical axis) then $h \circ \phi_1$ is a *du* to *dxd* bijection. In [4] another *du* to *dxd* bijection has been constructed showing that the statistic N_{dxd} follows the Narayana distribution. The present *du* to *dxd* bijection has the advantage that sends also the statistics E_{du} and O_{du} to the statistics N_{dud} and N_{ddd} respectively.

The second variation $\phi_2 : \mathcal{D} \to \mathcal{D}$ is defined recursively as follows:
$\phi_2(\epsilon) = \epsilon$, $\phi_2(ud\gamma) = ud\phi_2(\gamma)$ and
for $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$, $\phi_2(\alpha) = u^{l+1}d\phi_2(\beta_l)d\phi_2(\beta_{l+1})\cdots d\phi_2(\beta_2)d\phi_2(\beta_1)$; (see Fig. 7).
For example, for the Dyck path α of Fig. 1 we obtain

$$\begin{aligned} \phi_2(\alpha) &= uuuud\phi_2(\epsilon)d\phi_2(uduudd)d\phi_2(\epsilon)d\phi_2(ud)d\phi_2(\epsilon) \\ &= uuuuudduduudddudd. \end{aligned}$$

Clearly, ϕ_2 is a bijection, it maps the sets $\mathcal{A}$, $\mathcal{B}$ to themselves and it preserves the length of the Dyck path, although it does not preserve the number of prime components.

Furthermore, it can be proved that ϕ_2 satisfies the following equality for every $\alpha \in \mathcal{D}$:

21. $E_{uddu}(\alpha) = N_{uuddu}(\phi_2(\alpha))$.

2.3. The bijection ψ

We define recursively a mapping $\psi : \mathcal{D} \to \mathcal{D}$ as follows:
$\psi(\epsilon) = \epsilon$, $\psi(ud\gamma) = \widehat{\psi(\gamma)}$ and
for $a = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$, $\psi(\alpha) = \widehat{\psi(\beta_1)}\widehat{\psi(\beta_2)}\cdots\widehat{\psi(\beta_l)}\widehat{\psi(\beta_{l+1})}$; (see Fig. 8).
For example, for the Dyck path α of Fig. 1 we obtain

$$\begin{aligned} \psi(\alpha) &= \widehat{\psi(\epsilon)}\widehat{\psi(ud)}\widehat{\psi(\epsilon)}\widehat{\psi(\epsilon)}u\psi(uduudd)d \\ &= \widehat{\epsilon}\widehat{ud}\widehat{\epsilon}\widehat{\epsilon}uu\psi(uudd)dd = uduuddududuuududdd. \end{aligned}$$

It is easy to check that the mapping ψ is a bijection, it maps the sets $\mathcal{A}$, $\mathcal{B}$ to the sets of prime and non-prime (non-empty) Dyck paths respectively and it preserves the length of the Dyck path. The non-empty fixed points of ψ are of the form $(ud)^r$, $r \in \mathbb{N}^*$.

Furthermore, ψ satisfies the following equalities:

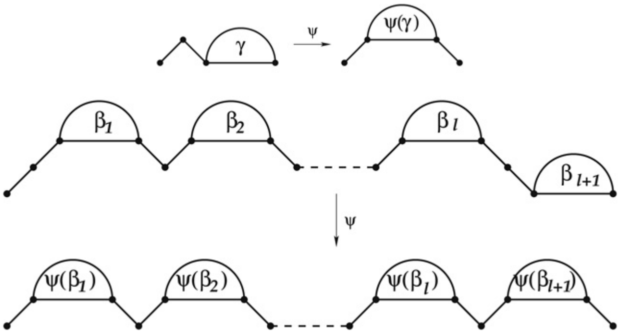

Fig. 8. The bijection ψ.

22. $O_{uu}(\alpha) = N_{ddu}(\psi(\alpha))$, (A091894).
23. $O_{ud}(\alpha) = N_{udu}(\psi(\alpha))$, (A091869).
24. $O_{dud}(\alpha) = N_{dudu}(\psi(\alpha))$, (A102405).
25. $O_{uudd}(\alpha) = N_{uuddu}(\psi(\alpha))$, (A114848).
26. $O_{udud}(\alpha) = N_{ududu}(\psi(\alpha))$.
27. $O_{ddud}(\alpha) = N_{ddudu}(\psi(\alpha))$.

Since the proofs are similar, we show only equalities 22 and 27. For the non-trivial case where $a = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$ we have

$$\begin{aligned} O_{uu}(\alpha) &= \sum_{i=1}^{l+1} O_{uu}(\beta_i) + \sum_{i=1}^{l}[\beta_i \neq \epsilon] = \sum_{i=1}^{l+1} N_{ddu}(\psi(\beta_i)) + \sum_{i=1}^{l}[\psi(\beta_i) \neq \epsilon] \\ &= N_{ddu}(\psi(\alpha)). \\ O_{ddud}(\alpha) &= \sum_{i=1}^{l+1} O_{ddud}(\beta_i) + \sum_{i=1}^{l-1}[\beta_i \neq \epsilon][\beta_{i+1} = \epsilon] \\ &= \sum_{i=1}^{l+1} N_{ddudu}(\psi(\beta_i)) + \sum_{i=1}^{l-1}[\psi(\beta_i) \neq \epsilon][\psi(\beta_{i+1}) = \epsilon] \\ &= N_{ddudu}(\psi(\alpha)). \end{aligned}$$

From iii and 27 we obtain the following result:

28. $O_{uuud}(\alpha) = N_{ddudu}((\psi \circ \theta)(\alpha))$.

Equalities 23, 25, 26 are special cases of the following result:

$$O_\tau(\alpha) = N_{\tau u}(\psi(\alpha))$$

for every Fibonacci path τ.

For the proof, we restrict ourselves to the non-trivial case $\alpha = u\widehat{\beta_1}\widehat{\beta_2}\cdots\widehat{\beta_l}d\beta_{l+1} \in \mathcal{B}$.

$$\begin{aligned} O_\tau(\alpha) &= \sum_{i=1}^{l+1} O_\tau(\beta_i) + \sum_{i=1}^{l-|\tau|}[\beta_i = \epsilon][\beta_{i+1} = \epsilon]\cdots[\beta_{i+|\tau|} = \epsilon] \\ &= \sum_{i=1}^{l+1} O_{\tau u}(\psi(\beta_i)) + \sum_{i=1}^{l-|\tau|}[\psi(\beta_i) = \epsilon][\psi(\beta_{i+1}) = \epsilon]\cdots[\psi(\beta_{i+|\tau|}) = \epsilon] \\ &= N_{\tau u}(\psi(\alpha)). \end{aligned}$$

The bijections so far do not cover the equidistributions of all strings of length 4. One of these cases concerns the statistic O_{uddu}. We will prove, using another bijection, that the statistic O_{uddu} is equidistributed with the statistic N_{uuduu}.

Indeed, since the parameters E_{dduu} and N_{uuduu} are equidistributed it is enough to prove the equidistribution of the parameters E_{dduu} and O_{uddu}. For this, we notice that for every $j \in \mathbb{N}$ there exists an involution ω_j of $\mathcal{D}$ (constructed in a similar way as ϕ_j of Section 3.1 in [17]) such that

(i) the number of *dduu*'s at height j in α is equal to the number of *uddu*'s at height $j+1$ in $\omega_j(\alpha)$ and

(ii) the number of *dduu*'s (resp. *uddu*'s) at height i (resp. $i+1$) in α is equal to the number of *dduu*'s (resp. *uddu*'s) in $\omega_j(\alpha)$ for $i \neq j$.

Furthermore, it is easy to check that the mapping $\omega : \mathcal{D} \to \mathcal{D}$ such that $\omega(\epsilon) = \epsilon$ and for $\alpha \neq \epsilon$, $\omega(\alpha) = \omega_{2\rho} \circ \omega_{2(\rho-1)} \circ \cdots \circ \omega_2 \circ \omega_0(\alpha)$ where $\rho = \left[\frac{h-1}{2}\right]$ (h is the height of the path α) is a bijection with $E_{dduu}(\alpha) = O_{uddu}(\omega(\alpha))$.

The remaining cases concern the statistics O_{uuuu} (or, its equidistributed statistic O_{dduu}), O_{dudu}, E_{dddu}, O_{dddu}.

By direct counting we have checked that even for small values of the semilength the first two of the above statistics do not have the same distribution as any N_τ (or $\overline{N}_\tau$) for every string of length 5. On the other hand, it can be proved using standard algebraic methods (see [17]) that the statistics E_{dddu} and O_{dddu} have the same distribution as N_{ddudu} and N_{dduuu} respectively. For the time being we cannot provide suitable mappings for the justification of the above results bijectively.

We close by giving tables that summarize all the above results.

τ	E_τ	O_τ
uu	N_{uud}	N_{duu}
ud	$\overline{N}_{dud}$	N_{udu}
du	N_{udu}	N_{uuu}

τ	E_τ	O_τ
uuu	N_{uudu}	N_{dduu}
uud	N_{uudd}	N_{uudu}
dud	N_{udud}	N_{dudu}
ddu	N_{uudu}	N_{dduu}

τ	E_τ	O_τ
$uuuu$	N_{uuduu}	
$uudu$	N_{uuudd}	N_{ddudu}
$dudu$	N_{ududu}	
$dduu$	N_{uuduu}	
$uuud$	N_{uudud}	N_{ddudu}
$uudd$	$\overline{N}_{duudd}$	N_{uuddu}
$udud$	$\overline{N}_{dudud}$	N_{ududu}
$ddud$	N_{uudud}	N_{ddudu}
$uddu$	N_{uuddu}	N_{uuduu}
$dddu$	N_{ddudu}	N_{dduuu}

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