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Graphs of given degree and diameter obtained as abelian lifts of dipoles

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ABSTRACT

We derive an upper bound on the number of vertices in regular graphs of given degree and diameter arising as regular coverings of dipoles over abelian groups.

1. Introduction

The problem of finding the largest order $n_{d,k}$ of a graph of a given maximum degree d and a given diameter k has been known for nearly five decades as the *degree diameter problem*. We refer the reader to the recent survey article [1] for history and background. An obvious upper bound on $n_{d,k}$ is the Moore bound $M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$. Graphs of degree d, diameter k and order $M_{d,k}$ are called *Moore graphs*, where the equality $n_{d,k} = M_{d,k}$ holds trivially if $k = 1$ and $d \geq 1$ (complete graphs), or $k \geq 3$ and $d = 2$ (cycles). The only other cases where this equality holds are for $k = 2$ and $d = 3, 7$ and, possibly, 57 [2,3].

Some of the known examples of current largest graphs of given degree and diameter were found using regular coverings of dipoles, that is, graphs with exactly two vertices and a certain number of loops and multiple edges [5]. In particular, the McKay–Miller–Širáň graphs [4] can be constructed as regular covering constructions of dipoles [6] over abelian groups. Also, the Petersen and the Hoffman–Singleton graphs (the unique Moore graphs of order $M_{3,2}$ and $M_{7,2}$ respectively) can be constructed as regular covering constructions of dipoles with the voltage groups Z_5 and $Z_5 \times Z_5$, respectively. Regular covering constructions will be briefly explained in Section 2.

Let $D_{d,k}$ denote the largest order of a regular graph of degree d and diameter k, which regularly covers a dipole over an abelian group. In particular $D_{l,c,k}$ is the largest order of a regular covering of a dipole with l loops and c multiple edges over an abelian group (note that $d = 2l + c$). It was shown in [8,9] respectively that when d is arbitrarily large then $D_{d,2} < 0.93M_{d,2}$ and $D_{d,3} < 0.6M_{d,3}$.

We present explicit upper bounds $U_{d,k}$ and $U_{l,c,k}$ on the quantities $D_{d,k}$ and $D_{l,c,k}$ respectively in Section 3. A special case where additional constraints on the selection of group elements are applied is in Section 4. In Section 5 we give an asymptotic upper bound for $U_{d,k}$ for $k \geq 2$ and arbitrarily large values of d, and thus showing $D_{d,k}$ to be much smaller than $M_{d,k}$ for large values of k. We also present some approximate values for $U_{d,k}$, $k \in \{2, \ldots, 20\}$, in tabular form for both the general and special cases, confirming and extending the results of [8,9].

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2. Voltage assignments, lifts, dipoles and bouquets

Let Γ be a finite, undirected graph, possibly with loops and multiple edges. To facilitate the description of voltages, we will think of the (undirected) edges of Γ as pairs of oppositely directed edges, called *darts*. The number of elements in the set D of all darts of Γ is therefore twice the number of all edges of Γ. If e is a dart, then e^{-1} will denote the dart reverse to e. Let G be a finite group. A mapping $\alpha : D \to G$ will be called a *voltage assignment* if $\alpha(e^{-1}) = (\alpha(e))^{-1}$, for any dart $e \in D$. Thus, a voltage assignment sends a pair of mutually reverse darts onto a pair of mutually inverse elements of the group. The pair (Γ, α) is the *voltage graph*, which determines the *lift* Γ^α of Γ as follows. Let V be the vertex set of Γ. The vertex set and the dart set of the lift are $V^\alpha = V \times G$ and $D^\alpha = D \times G$. In the lift, (e, g) is a dart from the vertex (u, g) to the vertex (v, h) if and only if e is a dart from u to v in Γ and $h = g\alpha(e)$. The lift itself is considered to be undirected, since (e, g) and $(e^{-1}, g\alpha(e))$ form a pair of mutually reverse darts and therefore give rise to an undirected edge of Γ^α.

The projection $\pi : \Gamma^\alpha \to \Gamma$ given by $\pi(e, g) = e$ and $\pi(v, g) = v$ is, topologically, a regular covering of Γ by Γ^α. This is why Γ is often called a (regular) *quotient*. For any vertex v and any dart e of the quotient, the sets $\pi^{-1}(v)$ and $\pi^{-1}(e)$ are called *fibers* above v and e. For any fixed $h \in G$ the mapping $(e, g) \mapsto (e, hg)$ determines an automorphism of the lift Γ^α. This way, the voltage group G acts regularly (that is, transitively and freely) on each fiber as a group of automorphisms of the lift.

A sequence $e_1e_2 \dots e_t$ of darts in Γ such that the terminal vertex of e_i coincides with the initial vertex of $e_{i+1} (1 \le i < t)$ is a walk in Γ of length t. If $W = e_1e_2 \dots e_t$ is a walk in Γ then we set $\alpha(W) = \alpha(e_1)\alpha(e_2)\dots\alpha(e_t)$. Examining walks in Γ we are able to determine the diameter of Γ^α as stated in the following lemma [4].

Lemma 1. *Let α be a voltage assignment on a graph Γ in a group G. Then,* $\mathrm{diam}(\Gamma^\alpha) \le k$ *if and only if for each ordered pair of vertices u, v (possibly, $u = v$) of Γ and for each $g \in G$ there exists a $u \to v$ walk of length at most k of net voltage g.*

We denote the dipole with l loops attached to each of its two adjacent vertices u and v, and c multiple edges connecting u to v, as $\delta_{l,c}$. Denote the set of all voltages assigned to the $u \to v$ darts of $\delta_{l,c}$ as C and the set of their inverses (or voltages assigned to the oppositely directed darts) as $\overline{C}$. In what follows we never assign the same voltage to distinct $u \to v$ darts and therefore we will always have $c = |C| = |\overline{C}|$. We denote the set of voltages assigned to the loops attached to u as L_u, and that for the loops attached to v as L_v, with the set $L = L_u \cup L_v$. Again, we will assume that all voltages in both L_u and L_v are distinct and hence $l = |L_u| = |L_v| = \frac{L}{2}$.

A bouquet is the graph composed of a single vertex with a collection of loops and *semi-edges* (edges with just one end-vertex and with the other end dangling) attached to it. Note that all Cayley graphs C_G over a finite group G and a generating set from G are isomorphic to lifts of bouquets, where the generators are the voltages assigned to the loops and semi-edges. However, in this paper we do not consider semi-edges. We denote the bouquet with l loops as β_l. We will use the fact that β_l is a subgraph of $\delta_{l,c}$.

3. Explicit upper bounds

We define the following three sums:

$$S_{l,k} = \sum_{i=0}^{l} 2^i \binom{l}{i} \binom{k}{i}$$

$$U^u_{l,c,k} = S_{l,k} + S_{2l,k-2} - S_{l,k-2} + \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} S_{2l,k-2j} \sum_{i=0} \binom{c}{i} \binom{j-1}{i-1} \binom{c+j-i-1}{j}$$

$$U^v_{l,c,k} = \sum_{j=1}^{\lceil \frac{k}{2} \rceil} S_{2l,k-2j+1} \sum_{i=0} \binom{c}{i} \binom{j-1}{i-1} \binom{c+j-i-2}{j-1}.$$

Our first proposition now follows:

Proposition 1. *Let α be a voltage assignment on $\delta_{l,c}$ in an abelian group G such that the lift $\delta^\alpha_{l,c}$ has diameter k. Then the number of vertices of $\delta^\alpha_{l,c}$ is at most $U_{l,c,k}$, where*

$$D_{l,c,k} \le U_{l,c,k} = 2\min\{U^u_{l,c,k}, U^v_{l,c,k}\}.$$

Proof. Assume that $\delta^\alpha_{l,c}$ has diameter k. By Lemma 1, the number of vertices in the fiber above u cannot exceed the number of distinct voltages on the $u \to u$ walks in $\delta_{l,c}$ of length at most k. On the other hand, the number of vertices in $\delta^\alpha_{l,c}$ cannot exceed two times the number of distinct voltages on $u \to v$ walks in $\delta_{l,c}$ of length at most k. We proceed with the following three claims in order to justify the expressions for $S_{l,k}$, $U^u_{l,c,k}$ and $U^v_{l,c,k}$.

Claim 1. *The number of distinct voltages on the $u \to u$ walks in $\delta_{l,c}$ of length at most k such that all the voltages collected in the walks are from L_u is bounded above by $S_{l,k}$.*

It is straightforward to show that the preimage of the vertex u and the set L_u in $\delta^{\alpha}_{l,c}$ is a Cayley subgraph C_G of $\delta^{\alpha}_{l,c}$, as β_l is a subgraph of $\delta_{l,c}$. The upper bound on the order of the vertex set of C_G of diameter k (with the generators set L_u) is given by $S_{l,k}$ [7].

Claim 2. *The number of distinct voltages on the $u \to u$ walks in $\delta_{l,c}$ of length at most k is bounded above by $U^u_{l,c,k}$.*

From Claim 1 we see that the number of distinct voltages on the walks of length at most k with voltages from L_u is bounded above by $S_{l,k}$. The next crucial observation is that the parity of the number of voltages collected from $C \cup \overline{C}$ in any given $\delta_{l,c}$ walk will determine the fiber in which the terminal vertex lies; any $u \to v$ walk will have an odd number of voltages collected from $C \cup \overline{C}$ (where if $j \geq 1$ voltages were collected from C then $j-1$ voltages were collected from $\overline{C}$) and any $u \to u$ walk will have an even number of voltages collected from $C \cup \overline{C}$ (with $j \geq 0$ voltages collected from C and j from $\overline{C}$).

If the sum of the set of voltages collected from $C \cup \overline{C}$ in the group G is zero, then only the voltages collected from L in the $u \to u$ walk will determine the voltage of the walk. Suppose, without loss of generality, that a $u \to u$ walk in $\delta_{l,c}$ collects $g \in C$ and $-g \in \overline{C}$ where all other voltages in the walk were collected from L; then from Claim 1 there are at most $S_{2l,k-2}$ distinct voltages on such a walk of length k, as the voltage of the walk is independent from the ordering of the voltages collected from L in the walk (as G is abelian). From $S_{2l,k-2}$ we need to subtract $S_{l,k-2}$ voltages that were already counted in $S_{l,k}$.

Suppose that the sum of the set of voltages collected from $C \cup \overline{C}$ in the group G is non-zero. Denote the number of the voltages collected from C as j; thus, there are exactly j voltages collected from $\overline{C}$. Therefore, we need to sum over all possible selections of $i \in \{1, \dots, j\}$ voltages from C to collect words of length j followed by words of length j with voltages collected from $\overline{C}$, where no voltage g from C that was collected will have its inverse $-g$ collected from $\overline{C}$ in the same walk, as this will create over-counting of walks already considered for smaller values of j, thus allowing for at most $\min\{c-i, j\}$ voltages to be selected from $\overline{C}$. Hence, the summation below is the upper bound on the number of distinct voltages on walks of length $2j$ collected from $C \cup \overline{C}$. We can simplify this summation using *Vandermonde's identity*:

$$\sum_{i=1} \binom{c}{i}\binom{j-1}{i-1} \sum_{t=1}^{\min\{c-i,j\}} \binom{c-i}{t}\binom{j-1}{t-1} = \sum_{i=0} \binom{c}{i}\binom{j-1}{i-1}\binom{c+j-i-1}{j}.$$

At most $k-2j$ voltages from L, $j \in \{1 \dots \lfloor \frac{k}{2} \rfloor\}$, can be collected in $u \to u$ walks as well, in at most $S_{2l,k-2j}$ ways. We thus multiply the sum above by $S_{2l,k-2j}$ and by summing for j we get the final summation:

$$\sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} S_{2l,k-2j} \sum_{i=0} \binom{c}{i}\binom{j-1}{i-1}\binom{c+j-i-1}{j}.$$

Adding all these up we get $U^u_{l,c,k}$.

Claim 3. *The number of distinct voltages on the $u \to v$ walks in $\delta_{l,c}$ of length at most k is bounded above by $U^v_{l,c,k}$.*

When a $\delta_{l,c}$ walk starts at u and terminates at v then the number of voltages collected from C is always one greater than the number of the voltages collected from $\overline{C}$ and therefore no walks are collected exclusively from L. Using a similar reasoning to Claim 2 we get the summation in $U^v_{l,c,k}$. □

The exact expression for $U_{d,k}$ is a trivial consequence of Proposition 1:

Proposition 2. *Let α be a voltage assignment on $\delta_{l,c}$ in an abelian group G such that the lift $\delta^{\alpha}_{l,c}$ has diameter k and $2l+c=d$. Then the number of vertices of $\delta^{\alpha}_{l,c}$ over all possible values of c and l is at most $U_{d,k}$, where*

$$D_{d,k} \leq U_{d,k} = \max\{U_{l,c,k} | 2l + c = d\}.$$

4. A special case

In [5], random voltages from a finite group G were assigned to a variety of quotients in order to find large regular graphs of given degree d and diameter k. When given any two adjacent vertices u and v in a quotient Γ, and the set C of voltages on the $u \to v$ darts, assignments satisfying the condition "$g \in C$ implies $-g \notin C$" for all $g \in G$ seem to do much better in the generation of successful degree diameter record graphs. We are motivated by this observation to investigate the following special case: Consider any set H of distinct elements of G such that $-h \notin H$ whenever $h \in H$, with the corresponding set $\overline{H}$ such that $-h \in \overline{H}$ for all $h \in H$. Let $C' = \{0\} \cup H \cup \overline{H}$, and suppose that C' is the set of the voltages on the $u \to v$ darts in $\delta_{l,2c+1}$, where $|H| = c$. Denote any voltage assignment that assigns a set C' to the $u \to v$ darts in $\delta_{l,2c+1}$ as α'. Define $D'_{d,k}$ to be the largest order of the vertex set of a regular graph of degree d and diameter k for any such α'. In particular we define $D'_{l,2c+1,k}$ to be the largest such order when $d = 2l + 2c + 1$. We define the following sums:

$$U'^{u}_{l,2c+1,k} = S_{l,k} + S_{2l,k-2} - S_{l,k-2} + \sum_{j=2, j \text{ is even}}^{k} S_{2l,k-j} \sum_{i=0}^{\min\{j,c\}} 2^i \binom{c}{i} \left(\binom{j-1}{i-1} + \binom{j-2}{i-1} \right)$$

$$U'^{v}_{l,2c+1,k} = (2c+1)S_{2l,k-1} + \sum_{j=3, j \text{ is odd}}^{k} S_{2l,k-j} \sum_{i=0}^{\min\{j,c\}} 2^i \binom{c}{i} \left(\binom{j-1}{i-1} + \binom{j-2}{i-1} \right).$$

Proposition 3. *Let α' be the voltage assignment presented above on $\delta_{l,2c+1}$ in an abelian group G such that the lift $\delta^{\alpha'}_{l,2c+1}$ has diameter k. Then the number of vertices of $\delta^{\alpha'}_{l,2c+1}$ is at most $U'_{l,2c+1,k}$, where*

$$D'_{l,2c+1,k} \le U'_{l,2c+1,k} = 2\min\{U'^{u}_{l,2c+1,k}, U'^{v}_{l,2c+1,k}\}.$$

Proof. Using Lemma 1 and a similar argument to the proof of Proposition 1 it will suffice to show that the expressions for $U'^{v}_{l,2c+1,k}$ and $U'^{u}_{l,2c+1,k}$ are constructed correctly. When looking at the $u \to u$ walks in $\delta_{l,2c+1}$, the number of voltages j collected from C' is always even. It is easy to see that for each even value of j we can have at most $\sum_{i=0} 2^i \binom{c}{i}\binom{j-1}{i-1}$ distinct voltages of walks of length j collected from $H \cup \overline{H}$. As such walks can collect zero from C' we need to consider all walks that collect zero exactly once, where any walks that collect zero more than once will have the same voltage as a shorter walk, and collecting zero once will correspond to a voltage that can be collected for an odd value of j. There are at most $\sum_{i=0} 2^i \binom{c}{i}\binom{j-1}{i-1}$ distinct voltages of walks of length j collected from C' where zero is collected exactly once. We therefore get $U'^{u}_{l,2c+1,k}$. Similarly, allowing j to be odd, as in the case for $u \to v$ walks, we get $U'^{v}_{l,2c+1,k}$. $\square$

The following proposition regarding an exact expression for $U'_{d,k}$, the upper bound on $D'_{d,k}$, is a direct consequence of Proposition 3 and is given without a proof.

Proposition 4. *Let α' be the voltage assignment presented above on $\delta_{l,2c+1}$ in an abelian group G such that the lift $\delta^{\alpha'}_{l,2c+1}$ has diameter k and $2l + 2c + 1 = d$. Then the number of vertices of $\delta^{\alpha'}_{l,2c+1}$ over all possible values of c and l is at most $U'_{d,k}$, where*

$$D'_{d,k} \le U'_{d,k} = \max\{U'_{l,2c+1,k} | 2l + 2c + 1 = d\}.$$

5. Asymptotic upper bound and approximate values

When k is fixed, $a_k d^k$ is the dominant term in both $U_{d,k}$ and $M_{d,k}$. Using numerical computations we were able to construct Table 1 for a_k when $k \in \{2, \dots, 20\}$ and d is not too large (a range of values for $d > 1000$ were tested). We note here that the values for a_2 in $U_{d,2}$ and a_3 in $U_{d,3}$ agree with those found in [8,9].

However, when bigger values of d and k are considered, numerical computations are insufficient. Therefore, we are interested in the asymptotic size of a_k in $U_{d,k}$ for large values of d and k in order to find an asymptotic upper bound for $D_{d,k}$ in terms of d^k. In Table 1, we can see that $\frac{2^{k-1}}{k!}$ seems to be a good upper bound on the asymptotic size of a_k in $U_{d,k}$. This observation is good computational evidence pointing to the following theorem and our main result. For simplicity, we will assume that $2l > c$ and both l and c grow larger with d.

Theorem 1. *Suppose that $2l > c$. For every r_1, $0 < r_1 < 1$, there exists a positive integer r_2 such that for every $d > r_2$ and any $c, l > dr_1$ we have*

$$2U^{v}_{l,c,k} < \frac{2^{k-1}}{k!} d^k.$$

Proof. The following five auxiliary computations and notation will be used in the proof (the details include standard techniques in asymptotic calculations with binomial coefficients and are left to the reader):

1. Suppose that $c < 2l$. Using binomial expansion we get

$$(2c + 4l)^k - (-2c + 4l)^k = 2\sum_{j=1} \binom{k}{2j-1} (2c)^{2j-1}(4l)^{k-2j+1} \le (2c + 4l)^k = (2d)^k.$$

2. Using induction on $j > 1$ yields $\binom{2j-1}{j-1} < 4^{j-1}$.

3. Define $E_{c,j}$ to be the following sum:

$$E_{c,j} = \sum_{i=0} \binom{c}{i} \binom{j-1}{i-1} \binom{c+j-i-2}{j-1}.$$

Thus, we can write $U^{v}_{l,c,k} = \sum_{j=1}^{\lceil \frac{k}{2} \rceil} S_{2l,k-2j+1} E_{c,j}$.

4. Using limit calculations we can show

$$\lim_{l\to\infty} \frac{S_{2l,k-2j+1}(k-2j+1)!}{(4l)^{k-2j+1}} = 1.$$

5. Using limit calculations we can show

$$\lim_{c\to\infty} \frac{E_{c,j}(j-1)!j!}{c^{2j-1}} = 1.$$

Now, let us calculate an upper bound on the following limit involving $U^{v}_{l,c,k}$ and $(2d)^k$:

$$\begin{aligned}
\lim_{d\to\infty} \frac{U^{v}_{l,c,k}}{(2d)^k} &= \lim_{d\to\infty} \frac{1}{(2d)^k} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} S_{2l,k-2j+1} E_{c,j} \\
&= \lim_{d\to\infty} \frac{1}{(2d)^k} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} \frac{S_{2l,k-2j+1}(k-2j+1)!}{(4l)^{k-2j+1}} \frac{E_{c,j}(j-1)!j!}{c^{2j-1}} \frac{(4l)^{k-2j+1}}{(k-2j+1)!} \frac{c^{2j-1}}{(j-1)!j!} \\
&= \lim_{d\to\infty} \frac{1}{(2d)^k} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} \frac{(4l)^{k-2j+1}}{(k-2j+1)!} \frac{c^{2j-1}}{(j-1)!j!} \frac{(2j-1)!}{(2j-1)!} \\
&= \lim_{d\to\infty} \frac{1}{(2d)^k} \frac{1}{k!} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} \binom{k}{2j-1} (4l)^{k-2j+1} \binom{2j-1}{j-1} c^{2j-1} \\
&< \lim_{d\to\infty} \frac{1}{(2d)^k} \frac{1}{k!} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} \binom{k}{2j-1} (4l)^{k-2j+1} 4^{j-1} c^{2j-1} \\
&= \lim_{d\to\infty} \frac{1}{(2d)^k} \frac{1}{2k!} \sum_{j=1}^{\lceil \frac{k}{2} \rceil} \binom{k}{2j-1} (4l)^{k-2j+1} (2c)^{2j-1} \\
&\leq \lim_{d\to\infty} \frac{1}{(2d)^k} \frac{1}{2k!} \frac{(2d)^k}{2} = \frac{1}{4k!}.
\end{aligned}$$

Note that for $k = 2$ we get $\binom{2j-1}{j-1} = 4^{j-1}$ as $j = 1$; however, in this case $(2c + 4l)^2 - (-2c + 4l)^2 < (2d)^2$. We therefore get that $\lim_{d\to\infty} \frac{U^{v}_{l,c,k}}{(2d)^k} < \frac{1}{4k!}$ which implies that for all values of $d > r_2$ for some r_2, the following inequality holds:

$$2U^{v}_{l,c,k} < \frac{(2d)^k}{2k!} = \frac{2^{k-1}}{k!} d^k. \quad \square$$

Our main result, regarding a general upper bound on $D_{d,k}$, with the given conditions that $2l > c$ and that both c and l grow large as $d \to \infty$, is a direct consequence of Theorem 1 and Proposition 2. We therefore present Theorem 2 without a proof.

Theorem 2. *Suppose that $2l > c$. For every r_1, $0 < r_1 < 1$, there exists a positive integer r_2 such that for every $d > r_2$ and any $c, l > dr_1$ we have*

$$D_{d,k} < \frac{2^{k-1}}{k!} d^k.$$

Table 1
Approximate values for a_k in $U_{d,k}$ and $U'_{d,k}$

k	$\frac{2^{k-1}}{k!}$	a_k in $U_{d,k}$	$a_k$in $U'_{d,k}$
2	1	0.932	0.669
3	0.6667	0.599	0.311
4	0.3334	0.289	0.136
5	0.1334	0.113	0.05
6	0.04445	0.0371	0.0159
7	0.01269	0.0105	0.00441
8	3.17×10^{-3}	2.6×10^{-3}	1.07×10^{-3}
9	7×10^{-4}	5.7×10^{-4}	2.35×10^{-4}
10	1.41×10^{-4}	1.14×10^{-4}	4.66×10^{-5}
11	2.56×10^{-5}	2.07×10^{-5}	8.4×10^{-6}
12	4.27×10^{-6}	3.44×10^{-6}	1.39×10^{-6}
13	6.57×10^{-7}	5.28×10^{-7}	2.12×10^{-7}
14	9.39×10^{-8}	7.53×10^{-8}	3.025×10^{-8}
15	1.25×10^{-8}	1.002×10^{-8}	4.019×10^{-9}
16	1.56×10^{-9}	1.25×10^{-9}	5.009×10^{-10}
17	1.84×10^{-10}	1.47×10^{-10}	5.87×10^{-11}
18	2.04×10^{-11}	1.63×10^{-11}	6.52×10^{-12}
19	2.15×10^{-12}	1.71×10^{-12}	6.85×10^{-13}
20	2.15×10^{-13}	1.71×10^{-13}	6.84×10^{-14}

6. Remarks

Theorem 1 can be regarded as an illustration of the technique required to determine the asymptotic order of $U_{d,k}$. We would like to note that the proof of Theorem 1 can be modified in order to allow any values of c and l to be considered. This claim is based on the fact that using induction on $j \geq 5$ it is easy to see that $\binom{2j-1}{j-1} < \frac{4^{j-1}}{2}$ and also $(2c+4l)^k - (-2c+4l)^k \leq 2(2d)^k$. Therefore, we need only to carefully consider the first few terms in $\sum_{j=1}^{\lceil \frac{k}{2} \rceil} S_{2l,k-2j+1} E_{c,j}$ to show that $U_{d,k} < \frac{2^{k-1}}{k!} d^k$ always holds (other than for $k = 2$ and $d = 3$ or 7). However, a full presentation of all the details will be too long and thus is not included here.

Improving the Moore bound in the degree diameter problem seems to be very hard in general. However, restricting the problem to special families of graphs seems more promising. We have shown that $U_{d,k}$ is an upper bound on $D_{d,k}$, and furthermore that for any fixed value of $d > 2$ the limit $\lim_{k\to\infty} \frac{M_{d,k}}{U_{d,k}}$ diverges quickly. This is obviously a notable improvement to the general upper bound $M_{d,k}$.

Proposition 4 and the entries in Table 1 suggests that any additional restrictions on the construction of $\alpha : G \to \delta_{c,l}$ will decrease the upper bound on the order of $\delta^{\alpha}_{c,l}$.

Additional work is necessary in order to find a lower bound for the order of the vertex set of a regular covering of a dipole over an abelian voltage group for $k \geq 3$. The case where $k = 2$ was already investigated in depth in [4,6].

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