Discrete Mathematics 309 (2009) 3219–3232

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Image partition regularity near zero

Dibyendu De [a], Neil Hindman [b,*]

[a] School of Mathematics, University of Witwatersrand, Private Bag 3, Wits 2050, South Africa

[b] Department of Mathematics, Howard University, Washington, DC 20059, USA

ARTICLE INFO

Article history:
Received 5 July 2007
Received in revised form 9 August 2008
Accepted 11 September 2008
Available online 9 October 2008

Keywords:
Ramsey Theory
Image partition regular matrix
Central Sets Theorem

ABSTRACT

Many of the classical results of Ramsey Theory are naturally stated in terms of *image partition regularity* of matrices. Many characterizations are known of image partition regularity over $\mathbb{N}$ and other subsemigroups of $(\mathbb{R}, +)$. We study several notions of *image partition regularity near zero* for both finite and infinite matrices, and establish relationships which must hold among these notions.

1. Introduction

One of the earliest results of Ramsey Theory is Schur's Theorem [17] which says that whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many classes (or *finitely colored*) there exist x and y such that x, y, and $x+y$ are contained in one cell of the partition (or are *monochromatic*). This theorem can be viewed as saying that the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is *kernel partition regular* over $\mathbb{N}$.

Definition 1.1. Let S be a subsemigroup of $(\mathbb{R}, +)$. Let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *kernel partition regular over* S (abbreviated KPR/S) if and only if, whenever S is finitely colored there exists monochromatic $\vec{x} \in S^v$ such that $A\vec{x} = \overline{0}$.

The terminology is due to Walter Deuber and refers to the fact that the vector $\vec{x}$ is in the kernel of the linear transformation defined by $\vec{y} \mapsto A\vec{y}$.

Schur's Theorem may also be viewed as saying that the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is *image partition regular* over $\mathbb{N}$.

Definition 1.2. Let S be a subsemigroup of $(\mathbb{R}, +)$, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *image partition regular over* S (abbreviated IPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic.

* Corresponding author.
E-mail addresses: dibyendude@gmail.com (D. De), nhindman@aol.com (N. Hindman).
URL: http://members.aol.com/nhindman/ (N. Hindman).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.09.023

Another of the earliest results of Ramsey Theory is van der Waerden's Theorem [19] which says that whenever $\mathbb{N}$ is finitely colored there must exist arbitrarily long arithmetic progressions. The length five version of van der Waerden's Theorem is clearly equivalent to the statement that the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

is image partition regular. On the other hand while one can write matrices whose kernel partition regularity imply any of the instances of van der Waerden's Theorem, it is impossible to write a kernel partition regular matrix such that any element of the kernel has entries constituting a nontrivial length five arithmetic progression (or any other length greater than two). See [7, Theorem 2.6].

In 1933 Rado [15] characterized those finite matrices that are kernel partition regular over $\mathbb{N}$ and later, in [16] those that are kernel partition regular over other subsets of $\mathbb{R}$. It was not until 1993 that characterizations of finite matrices that are image partition regular over $\mathbb{N}$ were obtained in [8]. (See [7, Theorem 4.8] for a list of 17 known equivalences to IPR/$\mathbb{N}$.)

While there are several partial results, nothing near a characterization of either kernel or image partition regularity of infinite matrices has been obtained. (See [7, Section 6] for a summary of some of what is known about partition regularity of infinite matrices.)

In [9], a paper primarily concerned with algebraic results in the Stone–Čech compactification of various semigroups of $(\mathbb{R}, +)$ with the discrete topology, a few results about image partition regularity near zero were obtained. In this paper we are investigating this subject in greater detail.

Definition 1.3. Let S be a subsemigroup of $(\mathbb{R}, +)$ with $0 \in c\ell S$, let $u, v \in \mathbb{N}$, and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *image partition regular over S near zero* (abbreviated IPR/S_0) if and only if, whenever $S \setminus \{0\}$ is finitely colored and $\delta > 0$, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic and lie in the interval $(-\delta, \delta)$.

In Section 2 we shall investigate those finite matrices which are IPR/S_0 for arbitrary dense subsemigroups of $(\mathbb{R}, +)$ and of $((0, \infty), +)$, and determine the precise relationships among these notions for the semigroups $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{D}$, $\mathbb{D}^+$, $\mathbb{R}$, and $\mathbb{R}^+$, where $S^+ = \{x \in S : s > 0\}$ and $\mathbb{D}$ is the set of dyadic rationals.

Definitions 1.2 and 1.3 have obvious generalizations to $\omega \times \omega$ matrices with finitely many nonzero entries in each row, where $\omega = \mathbb{N} \cup \{0\}$ is the first infinite cardinal. There is also a new notion which makes sense only if the matrix is infinite which we present in Definition 3.1. In Section 3 we investigate the relationships among these notions for the same semigroups and almost succeed in determining the precise relationships that hold among them.

Central sets in an arbitrary semigroup are known to have substantial combinatorial structure, and there is a natural extension of this notion to *central near zero* which was introduced in [9]. Both of these notions involve the algebraic structure of the Stone–Čech compactification of a discrete semigroup. Since Sections 2 and 3 do not require any knowledge of this structure, we postpone a description of it until Section 4, where we will derive a new version of the Central Sets Theorem near zero and get some combinatorial consequences thereof.

In Section 5 we establish that *Milliken–Taylor* matrices (which we will define there) are image partition regular near zero in the strong sense introduced in Section 3.

2. Finite matrices

We show in this section that there are precisely two distinct notions of image partition regularity of S near zero, depending on whether S is dense in $(0, \infty)$ or in $\mathbb{R}$.

Lemma 2.1. *Let $u, v \in \mathbb{N}$ let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ such that A is* IPR/$\mathbb{N}$*, and let S be a dense subsemigroup of* $((0, \infty), +)$*. Then A is* IPR/S_0.

Proof. Let $r \in \mathbb{N}$, let $S = \bigcup_{i=1}^{r} C_i$, and let $\delta > 0$. By a standard compactness argument (see [12, Section 5.5] or [6, Section 1.5]) pick $k \in \mathbb{N}$ such that whenever $\{1, 2, \ldots, k\} = \bigcup_{i=1}^{r} D_i$, there exist $\vec{x} \in \{1, 2, \ldots, k\}^v$ and $i \in \{1, 2, \ldots, r\}$ such that $A\vec{x} \in (D_i)^u$. Pick $z \in S \cap (0, \frac{\delta}{k})$. For $i \in \{1, 2, \ldots, r\}$ let $D_i = \{t \in \{1, 2, \ldots, k\} : tz \in C_i\}$. Pick $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \{1, 2, \ldots, k\}^v$ such that $A\vec{x} \in (D_i)^u$ and let $\vec{y} = z\vec{x}$. Then $A\vec{y} \in (C_i \cap (0, \delta))^u$. □

Lemma 2.2. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ such that A is* IPR/$\mathbb{Z}$*, and let S be a dense subsemigroup of* $(\mathbb{R}, +)$*. Then A is* IPR/S_0.

Proof. This is essentially identical to the previous proof. Given $r \in \mathbb{N}$, pick $k \in \mathbb{N}$ such that whenever $\{-k, -k+1, \ldots, k-1, k\} = \bigcup_{i=1}^{r} D_i$, there exist $\vec{x} \in \{-k, -k+1, \ldots, k-1, k\}^v$ and $i \in \{1, 2, \ldots, r\}$ such that $A\vec{x} \in (D_i)^u$. □

Theorem 2.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let S be a dense subsemigroup of $((0, \infty), +)$. The following statements are equivalent.*
(a) *A is* IPR/$\mathbb{N}$.
(b) *A is* IPR/S_0.
(c) *A is* IPR/S.
(d) *A is* IPR/$\mathbb{R}^+$.

Proof. That (a) implies (b) is Lemma 2.1. Trivially (b) implies (c) and (c) implies (d). That (d) implies (a) follows from [13, Theorem 2.4(I)]. □

Theorem 2.4. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and let S be a dense subsemigroup of $(\mathbb{R}, +)$. The following statements are equivalent.*
(a) *A is* IPR/$\mathbb{Z}$.
(b) *A is* IPR/S_0.
(c) *A is* IPR/S.
(d) *A is* IPR/$\mathbb{R}$.

Proof. That (a) implies (b) is Lemma 2.2. Trivially (b) implies (c) and (c) implies (d). That (d) implies (a) follows from [13, Theorem 2.4(II)]. □

Lemma 2.5. *Let $A = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$ and for $i \in \{0, 1, 2, 3\}$ let*

$$C_i = \bigcup_{t=0}^{\infty} \left[\left(\frac{2}{3}\right)^{4t+i+1}, \left(\frac{2}{3}\right)^{4t+i} \right).$$

Then there do not exist $i \in \{0, 1, 2, 3\}$ and $\vec{x} \in (\mathbb{R}^+)^2$ such that $A\vec{x} \in (C_i)^2$. Thus A is not IPR/$\mathbb{R}_0^+$. *On the other hand A is* IPR/$\mathbb{Z}$.

Proof. Suppose we have such i and $\vec{x}$ and pick $t \in \omega$ such that

$$\left(\frac{2}{3}\right)^{4t+i+1} \leq x_1 + 2x_2 < \left(\frac{2}{3}\right)^{4t+i}.$$

Then

$$\left(\frac{2}{3}\right)^{4t+i} = \frac{3}{2}\left(\frac{2}{3}\right)^{4t+i+1} \leq \frac{3}{2}x_1 + 3x_2 < 3x_1 + 3x_2 < 3x_1 + 6x_2 < 3\left(\frac{2}{3}\right)^{4t+i} < \left(\frac{2}{3}\right)^{4t+i-3}$$

so $3x_1 + 3x_2 \notin C_i$, a contradiction.

On the other hand

$$A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$

so A is IPR/$\mathbb{Z}$. □

Theorem 2.6. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The seven statements in* (I) *below are equivalent and are strictly stronger than the seven equivalent statements in* (II).

(I) (a) *A is* IPR/$\mathbb{N}$.
(b) *A is* IPR/$\mathbb{D}^+$.
(c) *A is* IPR/$\mathbb{Q}^+$.
(d) *A is* IPR/$\mathbb{R}^+$.
(e) *A is* IPR/$\mathbb{D}_0^+$.
(f) *A is* IPR/$\mathbb{Q}_0^+$.
(g) *A is* IPR/$\mathbb{R}_0^+$.

(II) (a) *A is* IPR/$\mathbb{Z}$.
(b) *A is* IPR/$\mathbb{D}$.
(c) *A is* IPR/$\mathbb{Q}$.
(d) *A is* IPR/$\mathbb{R}$.
(e) *A is* IPR/$\mathbb{D}_0$.
(f) *A is* IPR/$\mathbb{Q}_0$.
(g) *A is* IPR/$\mathbb{R}_0$.

Proof. The equivalences in (I) and (II) follow from Theorems 2.3 and 2.4. To see that the statements in (I) are strictly stronger than those in (II), let

$$A = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}.$$

By Lemma 2.5, A is not IPR/$\mathbb{R}_0^+$ and is IPR/$\mathbb{Z}$. □

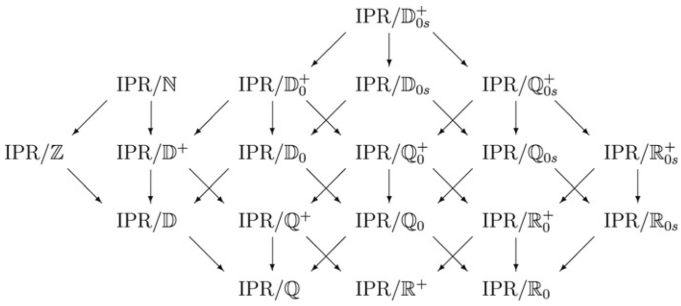

Fig. 1. Diagram of implications.

3. Infinite matrices

We shall see in this section that the situation with respect to infinite matrices is substantially different from that with respect to finite matrices. Recall that $\omega = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$ is the first infinite ordinal (and also the first infinite cardinal).

The notions defined in Definitions 1.2 and 1.3 both have obvious interpretations where u and v are both replaced by ω. In addition there is the following notion which only makes sense for infinite matrices.

Definition 3.1. Let S be a subsemigroup of $(\mathbb{R}, +)$ with $0 \in c\ell S$, and let A be an $\omega \times \omega$ matrix with entries from $\mathbb{Q}$ and finitely many nonzero entries in each row. Then A is *image partition regular over S near zero in the strong sense* (abbreviated IPR/S_{0s}) if and only if, whenever $S \setminus \{0\}$ is finitely colored and $\delta > 0$, there exists $\vec{x} \in S^\omega$ such that $\lim_{n\to\infty} x_n = 0$ and the entries of $A\vec{x}$ are monochromatic and lie in the interval $(-\delta, \delta)$.

Consider now the diagram of implications in Fig. 1.

All of the implications in the diagram hold trivially. We shall show in the remainder of the section that most of the missing implications do not hold in general. If we had an example of a matrix which is IPR/$\mathbb{N}$ but not IPR/$\mathbb{R}_0$, we would know that the only implications that hold in general are those diagrammed and those that follow from them by transitivity.

Lemma 3.2. *Let A be an $\omega \times \omega$ matrix having all possible rows with a single* 1 *and a single* 2*, and all other entries equal to* 0*. Then A is* IPR/$\mathbb{D}_0^+$ *and is* IPR/$\mathbb{N}$*, but is not* IPR/$\mathbb{R}_{0s}$.

Proof. Since constant vectors produce constant solutions, we have immediately that A is IPR/$\mathbb{D}_0^+$ and is IPR/$\mathbb{N}$. We show that A is not IPR/$\mathbb{R}_{0s}$.

For $x \in (0, 1)$ choose $I(x) \subseteq \mathbb{N}$ such that $x = \sum_{t\in I(x)} 2^{-t}$ and if there is a finite $F \subseteq \mathbb{N}$ such that $x = \sum_{t\in F} 2^{-t}$, then $I(x) = F$. (That is, choose the terminating binary expansion of x if it has one.) For $x \in (0, 1)$, define $\varphi(x) = \min I(x)$. Let

$$C_0 = \{x \in (-1, 1) \setminus \{0\} : \varphi(|x|) \text{ is even}\} \quad\text{and}\quad C_1 = \{x \in (-1, 1) \setminus \{0\} : \varphi(|x|) \text{ is odd}\} \cup (\mathbb{R} \setminus (-1, 1))\,.$$

Suppose that we have $i \in \{0, 1\}$ and a sequence $\langle x_n\rangle_{n=0}^\infty$ in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$ and all entries of $A\vec{x}$ are in C_i. If all but finitely many terms of $\langle x_n\rangle_{n=0}^\infty$ are negative, replace $\vec{x}$ by $-\vec{x}$. We can thus assume that infinitely many terms of $\langle x_n\rangle_{n=0}^\infty$ are positive. Pick j such that $0 < x_j < 1$. Pick $k > \varphi(x_j)$ such that $k \notin I(x_j)$. (Such a k exists by the second requirement in the definition of $I(x)$.) Pick l such that $x_l > 0$ and $\varphi(x_l) > k + 1$. When the sum $x_j + 2x_l$ is computed there is no carrying past position k. When the sum $2x_j + x_l$ is computed there is no carrying past position $k - 1$. Thus $\varphi(x_j + 2x_l) = \varphi(x_j)$ and $\varphi(2x_j + x_l) = \varphi(x_j) - 1$. This contradiction completes the proof. □

Lemma 3.3. *Let*

$$A = \begin{pmatrix} 3 & 3 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 3 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 2 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 3 & 3 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 2 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is IPR/$\mathbb{D}_{0s}$ *but is not* IPR/$\mathbb{R}^+$.

Proof. By Lemma 2.5, A is not IPR/$\mathbb{R}^+$. To see that A is IPR/$\mathbb{D}_{0s}$ let $r \in \mathbb{N}$, let $\mathbb{D} \setminus \{0\} = \bigcup_{i=1}^r C_i$, let $\delta > 0$, and pick $i \in \{1, 2, \ldots, r\}$ such that $0 \in c\ell\, C_i \cap 3\mathbb{D}$, and pick a sequence $\langle y_n \rangle_{n=0}^\infty$ in $C_i \cap 3\mathbb{D} \cap (-\delta, \delta)$ which converges to 0. For $n < \omega$ let $x_{2n} = -\frac{1}{3} y_n$ and let $x_{2n+1} = \frac{2}{3} y_n$. Since $y_n \in 3\mathbb{D}$, x_{2n} and x_{2n+1} are in $\mathbb{D}$. Then

$$A\vec{x} = \begin{pmatrix} y_0 \\ y_0 \\ y_1 \\ y_1 \\ \vdots \end{pmatrix}. \quad \square$$

We need some preliminary results in order to prove Lemma 3.6. We are grateful to Fred Galvin for supplying us with the proof of the following theorem which was stated without proof as [5, Theorem 9(3)]. According to Galvin this proof is "a straightforward generalization of the Erdős–Rado proof of the partition relation $\omega_1 \to (\omega + 1)_k^r$ which is stated in [3, page 472, line 6]."

For a set X and a cardinal κ we let $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$.

Theorem 3.4 (*Galvin*). *Let $(P, <)$ be a partially ordered set with the property that whenever P is colored with countably many colors, there is a monochromatic subset of order type ω. Let $r \in \mathbb{N}$. If the set of length r chains in P is finitely colored, there exists a chain in P of order type $\omega + 1$ all of whose length r subchains are monochromatic.*

Proof. Notice that the $r = 1$ case follows immediately from the $r = 2$ case. (If $k \in \mathbb{N}$ and $\gamma : P \to \{1, 2, \ldots, k\}$, define ψ taking the 2-element chains in P to $\{1, 2, \ldots, k\}$ so that if $x, y \in P$ and $x < y$, then $\psi(\{x, y\}) = \gamma(y)$. If X is a subset of P of order type $\omega + 1$ such that ψ is constant on the set of 2-element chains, and $z = \min X$, then $X \setminus \{z\}$ is a subset of P of order type $\omega + 1$ on which γ is constant.) Thus we shall assume that $r \geq 2$.

Let $\mathcal{C}$ be the set of r-element chains in P, let $k \in \mathbb{N}$, and let $\psi : \mathcal{C} \to \{1, 2, \ldots, k\}$.

Call a subset X of P *end-homogeneous* if and only if X is a chain in P and whenever $y_1, y_2, \ldots y_{r+1} \in X$ and $y_1 < y_2 < \cdots < y_{r+1}$, one has

$$\psi(\{y_1, y_2, \ldots, y_{r-1}, y_r\}) = \psi(\{y_1, y_2, \ldots, y_{r-1}, y_{r+1}\}).$$

We claim that it suffices to show that there is an end-homogeneous subset X of P such that the order type of X is $\omega + 1$. So assume we have such X, let $u = \max X$, and let $Y = X \setminus \{u\}$. Pick by Ramsey's Theorem an infinite subset Y' of Y and $i \in \{1, 2, \ldots, k\}$ such that for all $B \in [Y']^r$, $\psi(B) = i$. Then $Y' \cup \{u\}$ has order type $\omega + 1$ and whenever $B \in [Y' \cup \{u\}]^r$, $\psi(B) = i$. (If $u \in B$, pick $z \in Y'$ with $z > \max B \setminus \{u\}$. Then $\psi(B) = \psi((B \setminus \{u\}) \cup \{z\}) = i$.)

So suppose that there is no end-homogeneous subset of P with order type $\omega + 1$. Fix a well ordering W of P and for nonempty $A \subseteq P$ write $\min_W(A)$ for the smallest element of A with respect to this well ordering. Given $u \in P$ and $X \subseteq P$ write $X < u$ if and only if for all $x \in X$, $x < u$. Given $u \in P$ and $X \subseteq P$ such that $X \cup \{u\}$ is end-homogeneous and $X < u$, let

$$S(X, u) = \{y \in P : X < y < u \text{ and } X \cup \{y, u\} \text{ is end-homogeneous}\}.$$

Observe that for any $u \in P$, $S(\emptyset, u) = \{y \in P : y < u\}$.

We claim that for each $u \in P$ for which $S(\emptyset, u) \neq \emptyset$, there exist $n(u) < \omega$ and $x_1(u), x_2(u), \ldots, x_{n(u)}(u) \in P$ such that:

(1) $x_1(u) = \min_W (S(\emptyset, u))$,
(2) for $i \in \{2, 3, \ldots, n(u)\}$, $x_i(u) = \min_W (S(\{x_1(u), x_2(u), \ldots, x_{i-1}(u)\}, u))$, and
(3) $S(\{x_1(u), x_2(u), \ldots, x_{n(u)}(u)\}, u) = \emptyset$.

To see this, note that otherwise one may inductively define a sequence $\langle x_n \rangle_{n=1}^\infty$ by $x_1 = \min_W (S(\emptyset, u))$ and for $n \in \mathbb{N}$, $x_{n+1} = \min_W (S(\{x_1, x_2, \ldots, x_n\}, u))$. Then $\{x_n : n \in \mathbb{N}\} \cup \{u\}$ is an end-homogeneous subset of P of order type $\omega + 1$.

Given $u \in P$ such that $S(\emptyset, u) \neq \emptyset$, let $X(u) = \{x_1(u), x_2(u), \ldots, x_{n(u)}(u)\}$. Define an equivalence relation $\sim$ on P by $u \sim v$ if and only if either $S(\emptyset, u) = S(\emptyset, v) = \emptyset$ or:

(a) $n(u) = n(v)$ and
(b) whenever $1 \leq i_1 < i_2 < \cdots < i_{r-1} \leq n(u)$, $\psi(\{x_{i_1}(u), x_{i_2}(u), \ldots, x_{i_{r-1}}(u), u\}) = \psi(\{x_{i_1}(v), x_{i_2}(v), \ldots, x_{i_{r-1}}(v), v\})$.

There are only countably many equivalence classes mod $\sim$, so by the hypothesis on P we may choose an increasing sequence $\langle u_i \rangle_{i=1}^\infty$ in P such that $u_i \sim u_j$ for all $i, j \in \mathbb{N}$. There do not exist two comparable elements of the equivalence class determined by $S(\emptyset, u) = \emptyset$, so we may pick n such that $n = n(u_i)$ for all $i \in \mathbb{N}$.

We show now by induction on $j \in \{1, 2, \ldots, n\}$ that there are some $l(j) \in \mathbb{N}$ and $z_j \in P$ such that for all $i \in \mathbb{N}$, if $i \geq l(j)$, then $x_j(u_i) = z_j$. Assume first that $j = 1$. Then for each $i \in \mathbb{N}$, $x_1(u_i) = \min_W (S(\emptyset, u_i))$ and $S(\emptyset, u_i) \subseteq S(\emptyset, u_{i+1})$ so $x_1(u_{i+1}) \leq_W x_1(u_i)$. Since W is a well ordering, the sequence $\langle x_1(u_i) \rangle_{i=1}^\infty$ is eventually constant as required.

Now assume that $j \in \{2, 3, \ldots, n\}$, $m \in \mathbb{N}$, and $z_1, z_2, \ldots, z_{j-1} \in P$ such that for all $i \geq m$ and all $t \in \{1, 2, \ldots, j-1\}$, $x_t(u_i) = z_t$. Then given $i \in \mathbb{N}$ with $i \geq m$,

$$\begin{aligned} S(\{x_1(u_i), x_2(u_i), \ldots, x_{j-1}(u_i)\}, u_i) &= S(\{z_1, z_2, \ldots, z_{j-1}\}, u_i) \\ &\subseteq S(\{z_1, z_2, \ldots, z_{j-1}\}, u_{i+1}) = S(\{x_1(u_{i+1}), x_2(u_{i+1}), \ldots, x_{j-1}(u_{i+1})\}, u_{i+1}) \end{aligned}$$

so that $x_j(u_{i+1}) \le_W x_j(u_i)$. (The inclusion uses the fact that $u_i \sim u_{i+1}$.) Thus the sequence $\langle x_j(u_i)\rangle_{i=1}^\infty$ is eventually constant.

We therefore have some i such that $X(u_i) = X(u_{i+1})$. But then

$$u_i \in S(\{x_1(u_{i+1}), x_2(u_{i+1}), \ldots, x_{n(u_{i+1})}(u_{i+1})\}, u_{i+1}),$$

a contradiction. □

Galvin also provided the proof of the following corollary.

Corollary 3.5. *Let S be an uncountable subset of $\mathbb{R}$, let $k \in \mathbb{N}$, and let $\varphi : [S]^2 \to \{1, 2, \ldots, k\}$. There exists an increasing sequence $\langle y_n\rangle_{n<\omega+1}$ such that φ is constant on $\{\{y_k, y_l\} : k < l < \omega + 1\}$ and $y_\omega = \lim_{n\to\infty} y_n$.*

Proof. We first show that S satisfies the hypothesis of Theorem 3.4. That is whenever S is colored with countably many colors, there is a monochromatic subset of order type ω. Since whenever S is colored with countably many colors there must exist an uncountable monochromatic subset, it suffices to show that S contains a subset of order type ω. Trivially any nonempty subset which does not have a largest element contains a subset of order type ω. So if S contains no subset of order type ω, then every nonempty subset has a largest element. But this means that $-S$ is well ordered, while $\mathbb{R}$ trivially does not contain any uncountable well ordered subset. (One could pick a rational between any element of such a subset and its successor.)

We may presume that S is bounded since it must contain an uncountable bounded subset. Define $\psi : [S]^3 \to \{1, 2\}$ as follows. Given $x < y < z$ in S, let $\psi(\{x, y, z\}) = 1$ if $\varphi(\{x, y\}) = \varphi(\{x, z\})$ and $y-x > z-y$ and let $\psi(\{x, y, z\}) = 2$ otherwise. Pick by Theorem 3.4 a set $B \subseteq S$ of order type $\omega + 1$ such that ψ is constant on $[B]^3$. We claim that the constant value is 1. So suppose instead it is 2. By Ramsey's Theorem pick $C \in [B]^\omega$ such that φ is constant on $[C]^2$. We can choose an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in C. Given any n we have that $\varphi(\{x_n, x_{n+1}\}) = \varphi(\{x_n, x_{n+2}\})$ so it must be that $x_{n+1} - x_n \le x_{n+2} - x_{n+1}$. Since S is bounded, this is impossible.

Consequently the constant value of ψ is 1. Let $z = \max B$. By the pigeon hole principle, we may presume that φ is constant on $\{\{x, z\} : x \in B \setminus \{z\}\}$. Therefore, φ is constant on $[B]^2$. Again choose an increasing sequence $\langle x_n\rangle_{n=1}^\infty$ in B. Since $\{x_n : n \in \mathbb{N}\}$ is bounded, there must exist arbitrarily small values of $x_{n+1} - x_n$, and thus $z - x_{n+1}$ must be arbitrarily small since $z - x_{n+1} < x_{n+1} - x_n$. □

Lemma 3.6. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & -1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & -1 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is IPR/$\mathbb{R}^+_{0s}$ *but A is not* IPR/$\mathbb{Q}$.

Proof. It is shown in the proof of [13, Theorem 2.6] that A is not IPR/$\mathbb{Q}$. To see that A is IPR/$\mathbb{R}^+_{0s}$, let $k \in \mathbb{N}$, let $\delta > 0$ and let $\tau : \mathbb{R}^+ \to \{1, 2, \ldots, k\}$. Note that for $\vec{x} \in \mathbb{R}^\omega$, the entries of $A\vec{x}$ are

$$\{x_n : n < \omega\} \cup \{x_n - x_{n+1} : n < \omega\}.$$

Define

$$\varphi : [\mathbb{R}]^2 \to \{1, 2, \ldots, k\}$$

by $\varphi(\{x, y\}) = \tau(|x - y|)$. Pick by Corollary 3.5 an increasing sequence $\langle y_n\rangle_{n<\omega+1}$ in $(0, \delta)$ such that φ is constant on $\{\{y_k, y_l\} : k < l < \omega + 1\}$ and $y_\omega = \lim_{n\to\infty} y_n$.

For each $n < \omega$, let $x_n = y_\omega - y_n$. Then $\lim_{n\to\infty} x_n = 0$ and τ is constant on the entries of $A\vec{x}$. □

Lemma 3.7. *Let*

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \ldots \\ 1/3 & 0 & -1 & 0 & 0 & \ldots \\ 1/5 & 0 & 0 & -1 & 0 & \ldots \\ 1/7 & 0 & 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is IPR/$\mathbb{Q}^+_{0s}$ *but is not* IPR/$\mathbb{D}$.

Proof. To see that A is not IPR/$\mathbb{D}$ we show that there is no $\vec{x} \in \mathbb{D}^\omega$ such that $\vec{y} = A\vec{x} \in \mathbb{D}^\omega$. Indeed, suppose one has such $\vec{x}$ and pick $n \in \mathbb{N}$ such that $x_0/(2n+1) \notin \mathbb{D}$. Then $y_n = x_0/(2n+1) - x_{n+1} \notin \mathbb{D}$.

To see that A is IPR/$\mathbb{Q}^+_{0s}$ let $r \in \mathbb{N}$, let $\delta > 0$, let $(0, \infty) \cap \mathbb{Q} = \bigcup_{i=1}^r C_i$, pick $i \in \{1, 2, \ldots, r\}$ such that $0 \in c\ell C_i$, and pick a sequence $\langle y_n \rangle_{n=0}^\infty$ in C_i which converges to 0. We may also assume that for each n, $y_n < 1/(2n+1)$ and $y_n < \delta$. Let $x_0 = 1$ and for $n \in \mathbb{N}$, let $x_n = 1/(2n-1) - y_{n-1}$. Then $A\vec{x} = \vec{y} \in (C_i)^\omega$. □

Lemma 3.8. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1/2 & -1 & 0 & 0 & \ldots \\ 1/4 & 0 & -1 & 0 & \ldots \\ 1/8 & 0 & 0 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is IPR/$\mathbb{D}^+_{0s}$ *but is not* IPR/$\mathbb{Z}$.

Proof. To see that A is IPR/$\mathbb{D}^+_{0s}$, let $\delta > 0$ be given and let $(0, \infty) \cap \mathbb{D} = \bigcup_{i=1}^r C_i$. Pick i such that $0 \in c\ell C_i$ and choose a sequence $\langle y_n \rangle_{n=0}^\infty \in C_i$ such that for each $n \in \mathbb{N}$, $y_n < y_0/2^n$. Let $x_0 = y_0$ and for $n \in \mathbb{N}$, let $x_n = y_0/2^n - y_n$. Then $A\vec{x} = \vec{y}$.

To see that A is not IPR/$\mathbb{Z}$, suppose one has $\vec{x} \in \mathbb{Z}^\omega$ such that all entries of $A\vec{x}$ are in $\mathbb{Z} \setminus \{0\}$. Pick $n \in \mathbb{N}$ such that $x_0/2^n \notin \mathbb{Z}$. Then $x_0/2^n - x_n \notin \mathbb{Z}$. □

Lemma 3.9. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & \ldots \\ 4 & 0 & 1 & 0 & \ldots \\ 8 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is IPR/$\mathbb{N}$ *but is not* IPR/$\mathbb{R}^+_0$.

Proof. To see that A is IPR/$\mathbb{N}$, let $\mathbb{N}$ be finitely colored and pick a monochromatic sequence $\langle y_n \rangle_{n=0}^\infty$ such that for each $n \in \mathbb{N}$, $y_n > 2^n y_0$. Let $x_0 = y_0$ and for each $n \in \mathbb{N}$, let $x_n = y_n - 2^n y_0$. Then $A\vec{x} = \vec{y}$.

Now suppose one has $\vec{x} \in (\mathbb{R}^+)^\omega$ such that $\vec{y} = A\vec{x} \in ((0, 1))^\omega$. Then $x_0 = y_0 > 0$. Pick $k \in \mathbb{N}$ such that $2^k x_0 > 1$. Then $y_k = 2^k x_0 + x_k > 1$, a contradiction. □

Now consider the table in Fig. 2. In this table, the entry in row S and column T is labeled as follows. If the fact that any matrix which is IPR/S is also IPR/T follows from the implications in Fig. 1, then a "+" is entered. An entry of "n.k" means that an example of a matrix which is IPR/S but is not IPR/T is given in Lemma n.k. (Only one lemma is cited when multiple lemmas provide examples.) If we cannot determine whether every matrix which is IPR/S is also IPR/T, a "?" is entered.

If we knew that there is a matrix which is IPR/$\mathbb{N}$ but is not IPR/$\mathbb{R}_0$ we would know that none of the missing implications in Fig. 1 are valid.

Question 3.10. *Is there an $\omega \times \omega$ matrix with rational entries which is* IPR/$\mathbb{N}$ *but is not* IPR/$\mathbb{R}_0$.

Lemma 3.11. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \ldots \\ 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \ldots \\ 8 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then A is not IPR/$\mathbb{R}_0$.

	$\mathbb{D}_{0s}^+$	$\mathbb{N}$	$\mathbb{D}_0^+$	$\mathbb{D}_{0s}$	$\mathbb{Q}_{0s}^+$	$\mathbb{Z}$	$\mathbb{D}^+$	$\mathbb{D}_0$	$\mathbb{Q}_0^+$	$\mathbb{Q}_{0s}$	$\mathbb{R}_{0s}^+$	$\mathbb{D}$	$\mathbb{Q}^+$	$\mathbb{Q}_0$	$\mathbb{R}_0^+$	$\mathbb{R}_{0s}$	$\mathbb{Q}$	$\mathbb{R}^+$	$\mathbb{R}_0$
$\mathbb{D}_{0s}^+$	+	3.8	+	+	+	3.8	+	+	+	+	+	+	+	+	+	+	+	+	+
$\mathbb{N}$	3.2	+	3.9	3.2	3.2	+	+	?	3.9	3.2	3.2	+	+	?	3.9	3.2	+	+	?
$\mathbb{D}_0^+$	3.2	3.8	+	3.2	3.2	3.8	+	+	+	3.2	3.2	+	+	+	+	3.2	+	+	+
$\mathbb{D}_{0s}$	3.3	3.3	3.3	+	3.3	3.8	3.3	+	3.3	+	3.3	+	3.3	+	3.3	+	+	3.3	+
$\mathbb{Q}_{0s}^+$	3.7	3.7	3.7	3.7	+	3.7	3.7	3.7	+	+	+	3.7	+	+	+	+	+	+	+
$\mathbb{Z}$	2.5	2.5	2.5	3.2	2.5	+	2.5	?	2.5	3.2	2.5	+	2.5	?	2.5	3.2	+	2.5	?
$\mathbb{D}^+$	3.2	3.8	3.9	3.2	3.2	3.8	+	?	3.9	3.2	3.2	+	+	?	3.9	3.2	+	+	?
$\mathbb{D}_0$	3.2	3.3	3.3	3.2	3.2	3.8	3.3	+	3.3	3.2	3.2	+	3.3	+	3.3	3.2	+	3.3	+
$\mathbb{Q}_0^+$	3.2	3.7	3.7	3.2	3.2	3.7	3.7	3.7	+	3.2	3.2	3.7	+	+	+	3.2	+	+	+
$\mathbb{Q}_{0s}$	3.3	3.3	3.3	3.7	3.3	3.7	3.3	3.7	3.3	+	3.3	3.7	3.3	+	3.3	+	+	3.3	+
$\mathbb{R}_{0s}^+$	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	+	3.6	3.6	3.6	+	+	3.6	+	+
$\mathbb{D}$	2.5	3.3	2.5	3.2	2.5	3.8	3.3	?	2.5	3.2	2.5	+	3.3	?	2.5	3.2	+	3.3	?
$\mathbb{Q}^+$	3.2	3.7	3.7	3.2	3.2	3.7	3.7	3.7	3.9	3.2	3.2	3.7	+	?	3.9	3.2	+	+	?
$\mathbb{Q}_0$	3.2	3.3	3.3	3.2	3.2	3.7	3.3	3.7	3.3	3.2	3.2	3.7	3.3	+	3.3	3.2	+	3.3	+
$\mathbb{R}_0^+$	3.2	3.6	3.6	3.2	3.2	3.6	3.6	3.6	3.6	3.2	3.2	3.6	3.6	3.6	+	3.2	3.6	+	+
$\mathbb{R}_{0s}$	3.3	3.3	3.3	3.6	3.3	3.6	3.3	3.6	3.3	3.6	3.3	3.6	3.3	3.6	3.3	+	3.6	3.3	+
$\mathbb{Q}$	2.5	3.3	2.5	3.2	2.5	3.7	3.3	3.7	2.5	3.2	2.5	3.7	3.3	?	2.5	3.2	+	3.3	?
$\mathbb{R}^+$	3.2	3.6	3.6	3.2	3.2	3.6	3.6	3.6	3.6	3.2	3.2	3.6	3.6	3.6	3.9	3.2	3.6	+	?
$\mathbb{R}_0$	3.2	3.3	3.3	3.2	3.2	3.6	3.3	3.6	3.3	3.2	3.2	3.6	3.3	3.6	3.3	3.2	3.6	3.3	+

Fig. 2. Table of implications.

Proof. Let $C_1 = (0, \infty)$ and let $C_2 = (-\infty, 0)$. Suppose one has $i \in \{1, 2\}$ and $\vec{x} \in \mathbb{R}^\omega$ such that $A\vec{x} \in (C_i \cap (-1, 1))^\omega$. We may assume without loss of generality that $i = 1$. Then $x_0 > 0$. Pick $k \in \mathbb{N}$ such that $2^k x_0 > 1$. Then $\sum_{t=2^{k-1}}^{2^k-1} x_t > 0$ so pick some $t \in \{2^{k-1}, 2^{k-1}+1, \ldots, 2^k - 1\}$ such that $x_t > 0$. Then $2^k x_0 + x_t$ is an entry of $A\vec{x}$ which is bigger than 1. □

Question 3.12. *Is the matrix A of Lemma 3.11 IPR/$\mathbb{N}$?*

Of course an affirmative answer to Question 3.12 would provide an affirmative answer to Question 3.10.

4. Central sets near zero

Central subsets of a semigroup are intimately related with structures that are partition regular over that semigroup. In this section we will deal with sets that are *central near zero* and show that similar relationships hold with respect to partition regularity near zero. In order to do this, we need to discuss the algebra of the Stone–Čech compactification of a discrete semigroup.

If S is a discrete space, we take the points of the Stone–Čech compactification βS of S to be the ultrafilters on S, identifying the principal ultrafilters with the points of S (and thus pretending that $S \subseteq \beta S$). Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The sets $\{\overline{A} : A \subseteq S\}$ form a basis for the open sets of S as well as a basis for the closed sets of S.

Given a discrete semigroup $(S, +)$ the operation extends to βS making $(\beta S, +)$ a right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, we have that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

Note that, even if S is commutative, βS is not likely to be commutative. In particular, in the cases with which we are concerned, namely dense subsemigroups of $(\mathbb{R}, +)$ or $((0, \infty), +)$, $(\beta S, +)$ is not commutative.

A subset I of a semigroup $(T, +)$ is a right ideal provided $I \neq \emptyset$ and $I + T \subseteq I$, a left ideal provided $I \neq \emptyset$ and $T + I \subseteq I$, and a two sided ideal provided it is both a left and right ideal. Any compact right topological semigroup $(T, +)$ has a smallest two sided ideal $K(T)$ which is the union of all minimal right ideals and is the union of all minimal left ideals. If L is a minimal left ideal and R is a minimal right ideal, then $L \cap R$ is a group. In particular $K(T)$ has idempotents. An idempotent in T is *minimal* if and only if it is a member of $K(T)$. See [12] for an introduction to the algebraic structure of βS, as well as any unfamiliar algebraic statements encountered here.

Central sets were introduced by Furstenberg in [4] and were defined in terms of topological dynamics. The following algebraic definition is simpler.

Definition 4.1. Let $(S, +)$ be a discrete semigroup. A set $C \subseteq S$ is *central* if and only if there is an idempotent $p \in \overline{C} \cap K(\beta S)$.

Central sets have substantial combinatorial properties which are consequences of the *Central Sets Theorem*. The original Central Sets Theorem [4, Proposition 8.21] applied to $(\mathbb{N}, +)$. See [12, Part III] for a more general version and a presentation of many of these combinatorial properties.

We have been considering semigroups which are dense in $(\mathbb{R}, +)$ or $((0, \infty), +)$. Here, of course, "dense" means with respect to the usual topology on $\mathbb{R}$. When passing to the Stone–Čech compactification of such a semigroup S, we deal with S_d, which is the set S with the discrete topology.

Definition 4.2. Let S be a dense subsemigroup of $(\mathbb{R}, +)$ or of $((0, \infty), +)$. Then $0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\ ((0, \epsilon) \cap S \in p)\}$. If S is a dense subsemigroup of $(\mathbb{R}, +)$, then $0^-(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\ ((-\epsilon, 0) \cap S \in p)\}$.

It was shown in [9, Lemma 2.5] that $0^+(S)$ is a subsemigroup of $(\beta S_d, +)$. Also, it was noted that $0^+(S) \cap K(\beta S_d) = \emptyset$, so one does not obtain any information about $K\left(0^+(S)\right)$ based on knowledge of $K(\beta S_d)$. But as a compact right topological semigroup, $K\left(0^+(S)\right)$ does exist, and has idempotents.

Definition 4.3. Let S be a dense subsemigroup of $(\mathbb{R}, +)$ or of $((0, \infty), +)$. A set $C \subseteq S$ is *central near zero* if and only if there is an idempotent $p \in \overline{C} \cap K\left(0^+(S)\right)$.

In [1] a new stronger version of the Central Sets Theorem for arbitrary semigroups was proved. In Theorem 4.6 we shall show that analogues of this theorem hold for dense subsemigroups of $((0, \infty), +)$ and for dense subsemigroups of $(\mathbb{R}, +)$.

Definition 4.4. Let S be a dense subsemigroup of $(\mathbb{R}, +)$ or of $((0, \infty), +)$. A set $C \subseteq S$ is *piecewise syndetic near zero* if and only $\overline{C} \cap K\left(0^+(S)\right) \neq \emptyset$.

Notice that any set which is central near zero is also piecewise syndetic near zero. In [9] a mildly complicated elementary characterization of sets central near zero was given.

Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X.

Lemma 4.5. *Let S be a dense subsemigroup of $((0, \infty), +)$, let $l \in \mathbb{N}$, and let $C \subseteq S$ be piecewise syndetic near zero. If there is a dense subsemigroup T of $(\mathbb{R}, +)$ such that $S = T \cap (0, \infty)$, then for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in $T \cup \{0\}$ such that $\lim_{n\to\infty} y_{i,n} = 0$. Otherwise, for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in $S \cup \{0\}$ such that $\lim_{n\to\infty} y_{i,n} = 0$. For each $m \in \mathbb{N}$ there exist $a \in S \cap (0, 1/m)$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > m$ and for each $i \in \{1, 2, \ldots, l\}$, $a + \sum_{t\in H} y_{i,t} \in C \cap (0, 1/m)$.*

Proof. Let $Y = \times_{i=1}^{l} 0^+(S)$ and let $Z = \times_{i=1}^{l} \beta S_d$. By [12, Theorem 2.22], Y and Z are right topological semigroups, and if $\vec{x} \in \times_{i=1}^{l} S$, then $\lambda_{\vec{x}} : Z \to Z$ is continuous.

For $k \in \mathbb{N}$ let $I_k = \{(a + \sum_{t\in H} y_{1,t}, a + \sum_{t\in H} y_{2,t}, \ldots, a + \sum_{t\in H} y_{l,t}) : a \in S, H \in \mathcal{P}_f(\mathbb{N}),\ \min H > k$, and $(\forall i \in \{1, 2, \ldots, l\})\left(a + \sum_{t\in H} y_{i,t} \in S \cap (0, 1/k)\right)\}$ and let $E_k = I_k \cup \{(a, a, \ldots, a) : a \in S \cap (0, 1/k)\}$. Let $E = \bigcap_{k=1}^{\infty} c\ell_Z E_k$ and let $I = \bigcap_{k=1}^{\infty} c\ell_Z I_k$.

Since $0^+(S) = \bigcap_{k=1}^{\infty}\left(\beta S_d \cap \overline{(0, 1/k)}\right)$ and each $E_k \subseteq S \cap (0, 1/k)$ we have that $E \subseteq Y$. Trivially $I \subseteq E$. We claim that E is a subsemigroup of Y and I is an ideal of E. To see that $I \neq \emptyset$, it suffices to let $k \in \mathbb{N}$ and show that $I_k \neq \emptyset$. So let $k \in \mathbb{N}$ be given. Pick $n > k$ such that for each $i \in \{1, 2, \ldots, l\}$, $|y_{i,n}| < \frac{1}{3k}$, and pick $a \in S \cap (\frac{1}{3k}, \frac{2}{3k})$. Then $(a + y_{1,n}, a + y_{2,n}, \ldots, a + y_{l,n}) \in I_k$.

Now let $\vec{p}, \vec{q} \in E$. We show that $\vec{p} + \vec{q} \in E$ and if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{p} + \vec{q} \in I$. Let U be an open neighborhood of $\vec{p} + \vec{q}$ and let $k \in \mathbb{N}$. Since $\rho_{\vec{q}}$ is continuous, pick a neighborhood V of $\vec{p}$ such that $V + \vec{q} \subseteq U$. Pick $\vec{x} \in E_{2k} \cap V$ with $\vec{x} \in I_{2k}$ if $\vec{p} \in I$. If $\vec{x} \in I_{2k}$, pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > 2k$ and $a + \sum_{t\in H} y_{i,t} \in S \cap (0, 1/(2k))$ for each $i \in \{1, 2, \ldots, l\}$. In this case, let $j = \max H$. If $\vec{x} \notin I_{2k}$ pick $a \in S \cap (0, 1/(2k))$ such that $\vec{x} = (a, a, \ldots, a)$ and let $j = 2k$.

Since $\vec{x} + \vec{q} \in U$ and $\lambda_{\vec{x}}$ is continuous, pick a neighborhood W of $\vec{q}$ such that $\vec{x} + W \subseteq U$. Pick $\vec{y} \in E_j \cap W$ with $\vec{y} \in I_j$ if $\vec{q} \in I$. Then $\vec{x} + \vec{y} \in U \cap E_k$ and if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{x} + \vec{y} \in U \cap I_k$.

By [12, Theorem 2.23] $K(Y) = \times_{i=1}^{l} K\left(0^+(S)\right)$. Since C is piecewise syndetic near zero, pick $p \in K\left(0^+(S)\right) \cap \overline{C}$. Then $\overline{p} = (p, p, \ldots, p) \in K(Y)$. We claim that $\overline{p} \in E$. To see this, let $k \in \mathbb{N}$, let U be a neighborhood of $\overline{p}$ in Z, and for $i \in \{1, 2, \ldots, l\}$ pick $A_i \in p$ such that $\times_{i=1}^{l} \overline{A_i} \subseteq U$. Pick $a \in (0, 1/k) \cap \bigcap_{i=1}^{l} A_i$. Then $(a, a, \ldots, a) \in U \cap E_k$. Thus $\overline{p} \in E \cap K(Y)$ so by [12, Theorem 1.65], $K(E) = E \cap K(Y)$. Since I is an ideal of E, $K(E) \subseteq I$ and consequently $\overline{p} \in I$.

Now let $m \in \mathbb{N}$ be given. Then $\overline{p} \in c\ell_Z I_m$ so $\times_{i=1}^{l} \overline{C} \cap I_m \neq \emptyset$. □

The original Central Sets Theorem [4, Proposition 8.21] dealt with finitely many sequences at a time. The versions in [12] dealt with countably many sequences at a time. The version in [1] dealt with all sequences in the semigroup S. The following theorem deals with the set of all sequences whose terms go to zero.

Theorem 4.6. *Let S be a dense subsemigroup of $((0,\infty),+)$. If there is a dense subsemigroup T of $(\mathbb{R},+)$ such that $S = T\cap(0,\infty)$, let $\mathcal{T}$ be the set of sequences $\langle y_n\rangle_{n=1}^\infty$ in $T\cup\{0\}$ such that $\lim_{n\to\infty} y_n = 0$. Otherwise let $\mathcal{T}$ be the set of sequences $\langle y_n\rangle_{n=1}^\infty$ in $S\cup\{0\}$ such that $\lim_{n\to\infty} y_n = 0$. Let C be a subset of S which is central near zero. Then there exist $\alpha : \mathcal{P}_f(\mathcal{T})\to S$ and $H : \mathcal{P}_f(\mathcal{T})\to\mathcal{P}_f(\mathbb{N})$ such that:*

(1) *for each $F\in\mathcal{P}_f(\mathcal{T})$, $\alpha(F)\in(0,\frac{1}{|F|})$;*
(2) *if $F, G\in\mathcal{P}_f(\mathcal{T})$ and $F\subsetneq G$, then $\max H(F) < \min H(G)$; and*
(3) *if $m\in\mathbb{N}$, $G_1, G_2,\ldots,G_m\in\mathcal{P}_f(\mathcal{T})$, $G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_m$, and for each $i\in\{1,2,\ldots,m\}$, $\langle y_{i,t}\rangle_{t=1}^\infty\in G_i$, then $\sum_{i=1}^m\left(\alpha(G_i)+\sum_{t\in H(G_i)} y_{i,t}\right)\in C$.*

Proof. Pick $p = p+p\in K\left(0^+(S)\right)$ such that $C\in p$. Let

$$C^\star = \{x\in C : -x+C\in p\}.$$

By [12, Lemma 4.14] $C^\star\in p$ and whenever $x\in C^\star$, $-x+C^\star\in p$. We define $\alpha(F)$ and $H(F)$ for $F\in\mathcal{P}_f(\mathcal{T})$ by induction on $|F|$ satisfying the following induction hypotheses:

(1) $\alpha(F) < \frac{1}{|F|}$.
(2) If $\emptyset\neq G\subsetneq F$, then $\max H(G) < \min H(F)$.
(3) If $m\in\mathbb{N}$, $\emptyset\neq G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_m = F$ and $\langle f_i\rangle_{i=1}^m\in\times_{i=1}^m G_i$, then $\sum_{i=1}^m\left(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\right)\in C^\star$.

Assume first that $F=\{f\}$. (It is more convenient here to write a sequence as a function.) Pick by Lemma 4.5, $a\in S\cap(0,1)$ and $L\in\mathcal{P}_f(\mathbb{N})$ such that $a+\sum_{t\in L} f(t)\in C^\star$. Let $\alpha(F)=a$ and $H(F)=L$. The hypotheses are satisfied, the second vacuously.

Now assume that $F\in\mathcal{P}_f(\mathcal{T})$, $|F|>1$, and $\alpha(G)$ and $H(G)$ have been defined for all nonempty $G\subsetneq F$. Let $K=\bigcup\{H(G) : \emptyset\neq G\subsetneq F\}$ and let $m=\max K$. Let $M=\left\{\sum_{i=1}^r\left(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\right) : \emptyset\neq G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_r\subsetneq F \text{ and } \langle f_i\rangle_{i=1}^r\in\times_{i=1}^r G_i\right\}$. Then M is a finite subset of $C^\star$. Let

$$B = C^\star\cap\bigcap_{x\in M}(-x+C^\star).$$

Then $B\in p$ so in particular B is piecewise syndetic near zero. Pick by Lemma 4.5, $a\in S\cap(0,\frac{1}{|F|})$ and $L\in\mathcal{P}_f(\mathbb{N})$ such that $\min L > m$ and for each $f\in F$, $a+\sum_{t\in L} f(t)\in B$. Let $\alpha(F)=a$ and let $H(F)=L$.

Hypotheses (1) and (2) are satisfied directly. To verify hypothesis (3), let $m\in\mathbb{N}$ and assume that $\emptyset\neq G_1\subsetneq G_2\subsetneq\cdots\subsetneq G_m = F$ and $\langle f_i\rangle_{i=1}^m\in\times_{i=1}^m G_i$. If $m=1$, then $f_m\in F$ and $\alpha(F)+\sum_{t\in H(F)} f_m(t)\in B\subseteq C^\star$. So assume that $m>1$ and let $x=\sum_{i=1}^{m-1}\left(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\right)$. Then $x\in M$ so

$$\alpha(F)+\sum_{t\in H(F)} f_m(t)\in B\subseteq(-x+C^\star)$$

and thus $\sum_{i=1}^m\left(\alpha(G_i)+\sum_{t\in H(G_i)} f_i(t)\right)\in C^\star$ as required. □

The following corollary resembles the original Central Sets Theorem.

Corollary 4.7. *Let S be a dense subsemigroup of $((0,\infty),+)$. If there is a dense subsemigroup T of $(\mathbb{R},+)$ such that $S=T\cap(0,\infty)$, let $\mathcal{T}$ be the set of sequences $\langle y_n\rangle_{n=1}^\infty$ in $T\cup\{0\}$ such that $\lim_{n\to\infty} y_n = 0$. Otherwise let $\mathcal{T}$ be the set of sequences $\langle y_n\rangle_{n=1}^\infty$ in $S\cup\{0\}$ such that $\lim_{n\to\infty} y_n=0$. Let C be a subset of S which is central near zero and let $F\in\mathcal{P}_f(\mathcal{T})$. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in S such that $\sum_{n=1}^\infty a_n$ converges and a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n\in\mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L\in\mathcal{P}_f(\mathbb{N})$ and each $f\in F$, $\sum_{n\in L}\left(a_n+\sum_{t\in H_n} f(t)\right)\in C$.*

Proof. Choose a sequence $\langle g_n\rangle_{n=1}^\infty$ of distinct members of $\mathcal{T}\setminus F$ and for each $n\in\mathbb{N}$, let $G_n = F\cup\{g_1,g_2,\ldots,g_n\}$. For $n\in\mathbb{N}$, let $a_n=\alpha(G_n)$ and let $H_n = H(G_n)$. By thinning the sequences, we may presume that $\sum_{n=1}^\infty a_n$ converges. □

We will show in Theorem 4.10 that for certain semigroups S, central sets characterize image partition regularity of finite matrices. We shall follow the custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case name of the matrix.

Definition 4.8. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is a *first entries matrix* if and only if no row of A is $\vec{0}$, and for each $j \in \{1, 2, \ldots, v\}$ there exists $c > 0$ such that for each $i \in \{1, 2, \ldots, u\}$, if the first nonzero entry of row i of A is in column j, then $a_{i,j} = c$. If c is the first nonzero entry of a row of A, then c is a *first entry* of A.

In the following lemma, note that we are demanding of T that it be a subgroup of $(\mathbb{R}, +)$, not just a dense subsemigroup.

Lemma 4.9. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix. Let S be a dense subsemigroup of $((0, \infty), +)$. If there is a subgroup T of $(\mathbb{R}, +)$ such that $S = T \cap (0, \infty)$, assume that the entries of A come from $\mathbb{Z}$. Otherwise, assume that the entries of A come from ω. Let $C \subseteq S$ be central near zero. Assume that for each first entry c of A, $C \cap cS$ is central near zero. Then there exist for each $j \in \{1, 2, \ldots, v\}$ a sequence $\langle x_{j,t}\rangle_{t=1}^{\infty}$ in S such that $\sum_{t=1}^{\infty} x_{j,t}$ converges and for each $F \in \mathcal{P}_f(\mathbb{N})$, $A\vec{x}_F \in C^u$ where*

$$\vec{x}_F = \begin{pmatrix} \sum_{t\in F} x_{1,t} \\ \sum_{t\in F} x_{2,t} \\ \vdots \\ \sum_{t\in F} x_{v,t} \end{pmatrix}.$$

Proof. We proceed by induction on v. Assume first that $v = 1$. We may presume that A has no repeated rows, so there is some $c \in \mathbb{N}$ such that $A = (c)$. Pick a sequence $\langle w_n\rangle_{n=1}^{\infty}$ in S such that $\sum_{n=1}^{\infty} w_n$ converges. Pick by Corollary 4.7 sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S such that $\sum_{n=1}^{\infty} a_n$ converges and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$, $\sum_{n\in L}\left(a_n + \sum_{t\in H_n} w_t\right) \in C \cap cS$. For $n \in \mathbb{N}$, let $y_n = a_n + \sum_{t\in H_n} w_t$. Let $x_{1,n} = y_n/c$ for each $n \in \mathbb{N}$.

Now let $v \in \mathbb{N}$ and assume the result holds for v. Let A be a $u \times (v + 1)$ matrix with entries from $\mathbb{Z}$ or ω as appropriate. We may assume that we have $c \in \mathbb{N}$ and $k \in \{1, 2, \ldots, u - 1\}$ such that $a_{i,1} = 0$ if $i \in \{1, 2, \ldots, k\}$ and $a_{i,1} = c$ if $i \in \{k + 1, k + 2, \ldots, u\}$.

Let B be the $k \times v$ matrix defined by $b_{i,j} = a_{i,j+1}$ for $i \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, v\}$. Pick a sequence $\langle x_{j,t}\rangle_{t=1}^{\infty}$ for each $j \in \{1, 2, \ldots, v\}$ as guaranteed by the induction hypothesis for the matrix B and C. For $i \in \{k + 1, k + 2, \ldots, u\}$ and $t \in \mathbb{N}$, let $y_{i,t} = \sum_{j=2}^{v+1} a_{i,j}x_{j-1,t}$. If there is a subgroup T of $(\mathbb{R}, +)$ such that $S = T \cap (0, \infty)$, then each $y_{i,t} \in T$ and otherwise (since the entries of A are in ω) each $y_{i,t} \in S \cup \{0\}$. In any event for each $i \in \{k + 1, k + 2, \ldots, u\}$, $\sum_{t=1}^{\infty} y_{i,t}$ converges. For each $t \in \mathbb{N}$, let $y_{u+1,t} = 0$.

Pick by Corollary 4.7 a sequence $\langle d_n\rangle_{n=1}^{\infty}$ in S such that $\sum_{n=1}^{\infty} d_n$ converges and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $i \in \{k + 1, k + 2, \ldots, u + 1\}$,

$$\sum_{n\in L}\left(d_n + \sum_{t\in H_n} y_{i,t}\right) \in C \cap cS.$$

For each $n \in \mathbb{N}$, let $z_{1,n} = d_n/c$ and for $j \in \{2, 3, \ldots, v + 1\}$ let $z_{j,n} = \sum_{t\in H_n} x_{j-1,t}$. Since $d_n = d_n + \sum_{t\in H_n} y_{u+1,t} \in cS$, we have that $d_n \in S$.

One has immediately that for each $j \in \{1, 2, \ldots, v + 1\}$, $\sum_{n=1}^{\infty} z_{j,n}$ converges. Now let $L \in \mathcal{P}_f(\mathbb{N})$ and $i \in \{1, 2, \ldots, u\}$ be given. If $i \le k$, let $K = \bigcup_{n\in L} H_n$. Then $\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in L} z_{j,n} = \sum_{j=1}^{v} b_{i,j} \sum_{t\in K} x_{j,t} \in C$. So assume that $i > k$. Then $\sum_{j=1}^{v+1} a_{i,j} \sum_{n\in L} z_{j,n} = \sum_{n\in L}(d_n + \sum_{t\in H_n} y_{i,t}) \in C$. □

We now see that for certain semigroups, sets central near zero contain solutions to all image partition regular matrices. A subset D of S is *central* near zero* if and only if for every subset C of S which is central near zero, $C \cap D$ is central near zero. (Equivalently, D is a member of every idempotent in $K\left(0^+(S)\right)$.)

Theorem 4.10. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix. Let S be a dense subsemigroup of $((0, \infty), +)$. If there is a subgroup T of $(\mathbb{R}, +)$ such that $S = T \cap (0, \infty)$, assume that the entries of A come from $\mathbb{Z}$. Otherwise, assume that the entries of A come from ω. Assume that for every first entry c of A, cS is central* near zero. Then A is IPR/S_0 if and only if for every set C which is central near zero there exists $\vec{x} \in S^v$ such that $A\vec{x} \in C^u$.*

Proof. Sufficiency. Let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{r} C_i$. Pick an idempotent $p \in K\left(0^+(S)\right)$ and pick $i \in \{1, 2, \ldots, r\}$ such that $C_i \in p$. Then for each $\delta > 0$, $C_i \cap (0, \delta)$ is central near zero.
Necessity. We have by Theorem 2.3 that A is IPR/$\mathbb{R}_+$ so by Theorem 2.6, A is IPR/$\mathbb{N}$. By [10, Theorem 2.10], choose some $m \in \mathbb{N}$ and a $u \times m$ first entries matrix B such that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$. Let C be a subset of S which is central near zero. Pick by Lemma 4.9 some $\vec{y} \in \mathbb{N}^m$ such that $B\vec{y} \in C^u$. Pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$. □

Notice that if for some $c \in \mathbb{N}$, cS is not central* near zero, then $C = S \setminus cS$ is central near zero and (c) is a first entries matrix all of whose images miss C so that requirement is needed in Theorem 4.10. We do not have an example of a dense subgroup S of $(\mathbb{R}, +)$ for which some cS is not central* near zero. But we do have the following.

Theorem 4.11. *Let κ be an infinite cardinal with $\kappa \leq \mathfrak{c}$. There is a dense subsemigroup S of $((0,\infty),+)$ such that $|S| = \kappa$ and for every $c \in \mathbb{N} \setminus \{1\}$, cS is not central near zero.*

Proof. Choose a subset I of $(0,\infty)$ such that $|I| = \kappa$, I is linearly independent over $\mathbb{Q}$, and $0 \in c\ell I$. Let

$$S = \left\{ \sum_{t\in F} m_t \cdot t : F \in \mathcal{P}_f(I) \text{ and for each } t \in F,\ m_t \in \mathbb{N} \right\}.$$

Let $c \in \mathbb{N} \setminus \{1\}$ and let $B = \{\sum_{t\in F} m_t \cdot t : F \in \mathcal{P}_f(I)$ and for each $t \in F, m_t \in \mathbb{N}$ and some $m_t \equiv 1 \pmod{c}\}$. Then $B \cap cS = \emptyset$. We show that B is central* near zero (and thus cS is not central near zero) by showing that $\overline{B} \cap 0^+(S)$ is an ideal of $0^+(S)$ and so $K\left(0^+(S)\right) \subseteq \overline{B}$. To this end, let $p \in 0^+(S) \cap \overline{B}$ and let $q \in 0^+(S)$. We show that $B \in p+q$ and $B \in q+p$. To see that $B \in p+q$, we show that $B \subseteq \{y \in S : -y+B \in q\}$. So let $y \in B$ and pick $F \in \mathcal{P}_f(I)$ and $\langle m_x\rangle_{x\in F}$ in $\mathbb{N}$ such that $y = \sum_{x\in F} m_x \cdot x$ and some $m_x \equiv 1 \pmod{c}$. Let $\delta = \min F$. Then $(0,\delta) \cap S \in q$ and $(0,\delta) \cap S \subseteq -y+B$.

To see that $B \in q+p$ we show that $S \subseteq \{y \in S : -y+B \in p\}$. So let $y \in S$ and pick $F \in \mathcal{P}_f(I)$ and $\langle m_x\rangle_{x\in F}$ in $\mathbb{N}$ such that $y = \sum_{x\in F} m_x \cdot x$. Let $\delta = \min F$. Then $(0,\delta) \cap B \in p$ and $(0,\delta) \cap B \subseteq -y+B$. □

5. Milliken–Taylor matrices

Milliken [14, Theorem 2.2] and Taylor [18, Lemma 2.2] independently proved a theorem which implies that certain matrices, which we now introduce, are image partition regular over $\mathbb{N}$.

Definition 5.1. Let $m \in \omega$, let $\vec{a} = \langle a_i\rangle_{i=0}^m$ be a sequence in $\mathbb{Z}\setminus\{0\}$, and let $\vec{x} = \langle x_n\rangle_{n=0}^\infty$ be a sequence in $\mathbb{R}$. The *Milliken–Taylor system* determined by $\vec{a}$ and $\vec{x}$ is defined by $MT(\vec{a},\vec{x}) = \{\sum_{i=0}^m a_i \cdot \sum_{t\in F_i} x_t :$ each $F_i \in \mathcal{P}_f(\omega)$ and if $i < m$, then $\max F_i < \min F_{i+1}\}$

Notice that if $\vec{a}$ has adjacent repeated entries and $\vec{c}$ is obtained from $\vec{a}$ by deleting such repetitions, then for any infinite sequence $\vec{x}$, one has $MT(\vec{a},\vec{x}) \subseteq MT(\vec{c},\vec{x})$, so it suffices to consider sequences $\vec{c}$ without adjacent repeated entries.

Definition 5.2. Let $\vec{a}$ be a finite or infinite sequence in $\mathbb{Z}$ with only finitely many nonzero entries. Then $c(\vec{a})$ is the sequence obtained from $\vec{a}$ by deleting all zeroes and then deleting all adjacent repeated entries. The sequence $c(\vec{a})$ is the *compressed form* of $\vec{a}$. If $\vec{a} = c(\vec{a})$, then $\vec{a}$ is a *compressed sequence*.

For example, if $\vec{a} = \langle 0,1,0,0,1,2,0,0,2,2,0,0,\ldots\rangle$, then $c(\vec{a}) = \langle 1,2\rangle$.

Definition 5.3. Let $\vec{a}$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$. A *Milliken–Taylor matrix determined by* $\vec{a}$ is an $\omega \times \omega$ matrix A such that the rows of A are all possible rows with finitely many nonzero entries and compressed form equal to $\vec{a}$.

Notice that if A is a Milliken–Taylor matrix whose rows all have compressed form $\vec{a}$ and $\vec{x}$ is an infinite sequence in $\mathbb{R}$, then the set of entries of $A\vec{x}$ is precisely $MT(\vec{a},\vec{x})$.

When the partition regularity of Milliken–Taylor systems was first considered in [2] the sequence $\vec{a}$ was required to have entries from $\mathbb{N}$. Later it was shown that as long as the last entry was positive, the sequence could have negative entries as well.

Theorem 5.4. *Let $m \in \omega$, let $\vec{a} = \langle a_i\rangle_{i=0}^m$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$, and let A be a Milliken–Taylor matrix determined by $\vec{a}$. If $a_m > 0$, then A is* IPR/$\mathbb{N}$.

Proof. [11, Corollary 3.6]. □

We show in this section that if T is any dense subgroup of $(\mathbb{R},+)$, $\vec{a} = \langle a_i\rangle_{i=0}^m$ is a compressed sequence in $\mathbb{Z}\setminus\{0\}$ with $a_0 > 0$, and A is a Milliken–Taylor matrix determined by $\vec{a}$, then A is IPR/T^+_{0s}, where $T^+ = T \cap (0,\infty)$. Notice that, unlike the result in Theorem 5.4, it is the first rather than the last entry of $\vec{a}$ which is required to be positive. The reason for the difference is that $\beta\mathbb{N}\setminus\mathbb{N}$ is a left ideal of $\beta\mathbb{Z}$ while $0^+(T)$ is a right ideal of $0^+(T) \cup 0^-(T)$ as is $0^-(T)$ [8, Lemma 2.5].

Given $c \in \mathbb{R}\setminus\{0\}$ and $p \in \beta\mathbb{R}_d \setminus \{0\}$, the product $c \cdot p$ is defined in $(\beta\mathbb{R}_d,\cdot)$. One has that $A \subseteq \mathbb{R}$ is a member of $c\cdot p$ if and only if $c^{-1}A = \{x \in \mathbb{R} : c \cdot x \in A\}$ is a member of p.

Lemma 5.5. *Let T be a dense subgroup of $(\mathbb{R},+)$, let $p \in 0^+(T)$, and let $c \in \mathbb{N}$. Then $c \cdot p \in 0^+(T)$ and $(-c)\cdot p \in 0^-(T)$.*

Proof. The two proofs are similar. We do the second, which is the one that uses the fact that T is a subgroup rather than just a subsemigroup. Let $\epsilon > 0$. We need to show that $(-\epsilon,0) \cap T \in (-c)\cdot p$. Now $(0,\epsilon/c) \cap T \in p$, so it suffices to show that $(0,\epsilon/c) \cap T \subseteq (-c)^{-1}((-\epsilon,0)\cap T)$. So let $x \in (0,\epsilon/c) \cap T$. Then $(-c)\cdot x \in (-\epsilon,0)$ and, since $(T,+)$ is a group, $(-c)\cdot x \in T$ □

Definition 5.6. Let $\langle w_n\rangle_{n=0}^\infty$ be a sequence in $\mathbb{R}$. A *sum subsystem* of $\langle w_n\rangle_{n=0}^\infty$ is a sequence $\langle x_n\rangle_{n=0}^\infty$ such that there exists a sequence $\langle H_n\rangle_{n=0}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \omega$, $\max H_n < \min H_{n+1}$ and $x_n = \sum_{t\in H_n} w_t$.

Notice that if $\langle w_n\rangle_{n=0}^{\infty}$ is a sequence in $\mathbb{R}^+$ such that $\sum_{n=0}^{\infty} w_n$ converges and $\langle x_n\rangle_{n=0}^{\infty}$ is a sum subsystem of $\langle w_n\rangle_{n=0}^{\infty}$, then $\sum_{n=0}^{\infty} x_n$ also converges.

The proof of the following theorem is similar to that of [11, Theorem 3.3]. Given a sequence $\langle x_n\rangle_{n=0}^{\infty}$ and $k \in \omega$ we let $FS(\langle x_n\rangle_{n=k}^{\infty}) = \{\sum_{n\in F} x_n : F \in \mathcal{P}_f(\omega) \text{ and } \min F \geq k\}$.

Theorem 5.7. *Let T be a dense subgroup of $(\mathbb{R}, +)$, let $\vec{a} = \langle a_i\rangle_{i=0}^{m}$ be a compressed sequence in $\mathbb{Z}\setminus\{0\}$ with $a_0 > 0$, and let A be a Milliken–Taylor matrix determined by $\vec{a}$. Then A is IPR/T_{0s}^{+}. In fact, given any sequence $\langle w_n\rangle_{n=0}^{\infty}$ in T^+ such that $\lim_{n\to\infty} w_n = 0$, whenever $r \in \mathbb{N}$, $T^+ = \bigcup_{i=1}^{r} C_i$, and $\delta > 0$, there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_n\rangle_{n=0}^{\infty}$ of $\langle w_n\rangle_{n=0}^{\infty}$ such that $MT(\vec{a}, \vec{x}) \subseteq C_i \cap (0, \delta)$.*

Proof. By passing to a subsequence, we may presume that $\sum_{n=0}^{\infty} w_n$ converges. Pick by [12, Lemma 5.11] an idempotent $p \in \bigcap_{k=0}^{\infty} c\ell_{\beta T_d} FS(\langle w_n\rangle_{n=k}^{\infty})$. Note that since $\sum_{n=0}^{\infty} w_n$ converges, $p \in 0^+(T)$. Let $q = a_0 \cdot p + a_1 \cdot p + \cdots + a_m \cdot p$. Then by Lemma 5.5 and the previously mentioned fact that $0^+(T)$ and $0^-(T)$ are both right ideals of $0^+(T) \cup 0^-(T)$, we have that $q \in 0^+(T)$. So it suffices to show that whenever $Q \in q$, there is a sum subsystem $\langle x_n\rangle_{n=0}^{\infty}$ of $\langle w_n\rangle_{n=0}^{\infty}$ such that $MT(\vec{a}, \vec{x}) \subseteq Q$.

Let $Q \in q$ be given. Assume first that $m = 0$. Then $(a_0)^{-1}Q \in p$, so by [12, Theorem 5.14] there is a sum subsystem $\langle x_n\rangle_{n=0}^{\infty}$ of $\langle w_n\rangle_{n=0}^{\infty}$ such that $FS(\langle x_n\rangle_{n=0}^{\infty}) \subseteq (a_0)^{-1}Q$. Then $MT(\vec{a}, \vec{x}) \subseteq Q$.

Now assume that $m > 0$. Define

$$P(\emptyset) = \{x \in T : -(a_0 \cdot x) + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p\}.$$

We claim that $P(\emptyset) \in p$. To see this let

$$D = \{y \in T : -y + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p\}.$$

Then $D \in a_0 \cdot p$, so $(a_0)^{-1}D \in p$ and $(a_0)^{-1}D \subseteq P(\emptyset)$. Given x_0 define $P(x_0) = \{y \in T : -(a_0 \cdot x_0 + a_1 \cdot y) + Q \in a_2 \cdot p + a_3 \cdot p + \ldots + a_m \cdot p\}$. If $x_0 \in P(\emptyset)$, then $-(a_0 \cdot x_0) + Q \in a_1 \cdot p + a_2 \cdot p + \ldots + a_m \cdot p$ and so

$$\{y \in T : -(a_1 \cdot y) + (-(a_0 \cdot x_0) + Q) \in a_2 \cdot p + a_3 \cdot p + \ldots + a_m \cdot p\} \in p$$

and thus $P(x_0) \in p$.

Given $n \in \{1, 2, \ldots, m-1\}$ and $x_0, x_1, \ldots, x_{n-1}$, let $P(x_0, x_1, \ldots, x_{n-1}) = \{y \in T : -(a_0 \cdot x_0 + \ldots + a_{n-1} \cdot x_{n-1} + a_n \cdot y) + Q \in a_{n+1} \cdot p + \ldots + a_m \cdot p\}$. If $x_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, n-1\}$, $x_i \in P(x_0, x_1, \ldots, x_{i-1})$, then $P(x_0, x_1, \ldots, x_{n-1}) \in p$. Given $x_0, x_1, \ldots, x_{m-1}$, let $P(x_0, x_1, \ldots, x_{m-1}) = \{y \in T : a_0 \cdot x_0 + a_1 \cdot x_1 + \ldots + a_{m-1} \cdot x_{m-1} + a_m \cdot y \in Q\}$. If $x_0 \in P(\emptyset)$ and for each $i \in \{1, 2, \ldots, m-1\}$, $x_i \in P(x_0, x_1, \ldots, x_{i-1})$, then $P(x_0, x_1, \ldots, x_{m-1}) \in p$.

Given any $B \in p$, let $B^\star = \{x \in B : -x + B \in p\}$. Then $B^\star \in p$ and by [12, Lemma 4.14], for each $x \in B^\star$, $-x + B^\star \in p$.

Choose $x_0 \in P(\emptyset)^\star \cap FS(\langle w_n\rangle_{n=0}^{\infty})$ and choose $H_0 \in \mathcal{P}_f(\mathbb{N})$ such that $x_0 = \sum_{t\in H_0} w_t$. Let $n \in \omega$ and assume that we have chosen $x_0, x_1, \ldots, x_n$ and $H_0, H_1, \ldots, H_n$ such that:

(1) if $k \in \{0, 1, \ldots, n\}$, then $H_k \in \mathcal{P}_f(\omega)$ and $x_k = \sum_{t\in H_k} w_t$,
(2) if $k \in \{0, 1, \ldots, n-1\}$, then $\max H_k < \min H_{k+1}$,
(3) if $\emptyset \neq F \subseteq \{0, 1, \ldots, n\}$, then $\sum_{t\in F} x_t \in P(\emptyset)^\star$, and
(4) if $k \in \{1, 2, \ldots, \min\{m, n\}\}$, $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, n\})$, and for each $j \in \{0, 1, \ldots, k-1\}$, $\max F_j < \min F_{j+1}$, then $\sum_{t\in F_k} x_t \in P(\sum_{t\in F_0} x_t, \sum_{t\in F_1} x_t \ldots, \sum_{t\in F_{k-1}} x_t)^\star$.

All hypotheses hold at $n = 0$, (2) and (4) vacuously.

Let $v = \max H_n$. For $r \in \{0, 1, \ldots, n\}$, let

$$E_r = \left\{\sum_{t\in F} x_t : \emptyset \neq F \subseteq \{r, r+1, \ldots, n\}\right\}.$$

For $k \in \{0, 1, \ldots, m-1\}$ and $r \in \{0, 1, \ldots, n\}$, let

$$W_{k,r} = \left\{\left(\sum_{t\in F_0} x_t, \ldots, \sum_{t\in F_k} x_t\right) : F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, r\}) \text{ and for each } i \in \{0, 1, \ldots, k-1\}, \max F_i < \min F_{i+1}\right\}.$$

Note that $W_{k,r} \neq \emptyset$ if and only if $k \leq r$.

If $y \in E_0$, then $y \in P(\emptyset)^\star$, so $-y + P(\emptyset)^\star \in p$ and $P(y) \in p$. If $k \in \{1, 2, \ldots, m-1\}$ and $(y_0, y_1, \ldots, y_k) \in W_{k,m}$, then $y_k \in P(y_0, y_1, \ldots, y_{k-1})$, so $P(y_0, y_1, \ldots, y_k) \in p$ and thus $P(y_0, y_1, \ldots, y_k)^\star \in p$. If $r \in \{0, 1, \ldots, n-1\}$, $k \in \{0, 1, \ldots, \min\{m-1, r\}\}$, $(y_0, y_1, \ldots, y_k) \in W_{k,r}$, and $z \in E_{r+1}$, then $z \in P(y_0, y_1, \ldots, y_k)^\star$, so $-z + P(y_0, y_1, \ldots, y_k)^\star \in p$.

If $n = 0$, let $x_1 \in FS(\langle w_t\rangle_{t=v+1}^{\infty}) \cap P(\emptyset)^\star \cap (-x_0 + P(\emptyset)^\star) \cap P(x_0)^\star$ and pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_1 > v$ and $x_1 = \sum_{t\in H_1} w_t$. The hypotheses are satisfied.

Now assume that $n \geq 1$ and pick

$$x_{n+1} \in FS(\langle w_t \rangle_{t=v+1}^{\infty}) \cap P(\emptyset)^{\star} \cap \bigcap_{y \in E_0} \left(-y + P(\emptyset)^{\star}\right) \cap \bigcap_{k=0}^{\min\{m-1,n\}} \bigcap_{(y_0,y_1,\ldots,y_k) \in W_{k,m}} P(y_0, y_1, \ldots, y_k)^{\star}$$
$$\cap \bigcap_{r=0}^{n-1} \bigcap_{k=0}^{\min\{m-1,r\}} \bigcap_{(y_0,y_1,\ldots,y_k) \in W_{k,r}} \bigcap_{z \in E_{r+1}} (-z + P(y_0, y_1, \ldots, y_k)^{\star}).$$

Pick $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{n+1} > v$ and $x_{n+1} = \sum_{t \in H_{n+1}} w_t$.

Hypotheses (1) and (2) hold directly. For hypothesis (3) assume that $\emptyset \neq F \subseteq \{0, 1, \ldots, n+1\}$ and $n+1 \in F$. If $F = \{n+1\}$ we have directly that $x_{n+1} \in P(\emptyset)^{\star}$, so assume that $\{n+1\} \subsetneq F$ and let $G = F \setminus \{n+1\}$. Let $y = \sum_{t \in G} x_t$. Then $y \in E_0$, so $x_{n+1} \in -y + P(\emptyset)^{\star}$ and so $\sum_{t \in F} x_t \in P(\emptyset)^{\star}$.

To verify hypothesis (4), let $k \in \{1, 2, \ldots, \min\{m, n+1\}\}$ and assume that $F_0, F_1, \ldots, F_k \in \mathcal{P}_f(\{0, 1, \ldots, n+1\})$ and for each $j \in \{0, 1, \ldots, k-1\}$, $\max F_j < \min F_{j+1}$. We can assume that $n+1 \in F_k$. For $l \in \{0, 1, \ldots, k-1\}$ let $y_l = \sum_{t \in F_l} x_t$. Then $k-1 \leq \min\{m-1, n\}$ and $(y_0, y_1, \ldots, y_{k-1}) \in W_{k-1,m}$. If $F_k = \{n+1\}$, then $\sum_{t \in F_k} x_t = x_{n+1} \in P(y_0, y_1, \ldots, y_{k-1})^{\star}$. So assume that $\{n+1\} \subsetneq F_k$ and let $F_k' = F_k \setminus \{n+1\}$. Let $r = \max F_{k-1}$. Then $r < \min F_k'$, so $r \leq n-1$, $k-1 \leq \min\{m-1, r\}$, and $(y_0, y_1, \ldots, y_{k-1}) \in W_{k-1,r}$. Let $z = \sum_{t \in F_k'} x_t$. Then $z \in E_{r+1}$, so $x_{n+1} \in -z + P(y_0, y_1, \ldots, y_{k-1})^{\star}$ and hence $\sum_{t \in F_k} x_t \in P(\sum_{t \in F_0} x_t, \sum_{t \in F_1} x_t \ldots, \sum_{t \in F_{k-1}} x_t)^{\star}$. □

Acknowledgements

The second author acknowledges support received from the National Science Foundation (USA) via Grant DMS-0554803. The authors would like to thank the referee for a constructive and helpful report.

References

[1] D. De, N. Hindman, D. Strauss, A new and stronger central sets theorem, Fund. Math. 199 (2008) 155–175.
[2] W. Deuber, N. Hindman, I. Leader, H. Lefmann, Infinite partition regular matrices, Combinatorica 15 (1995) 333–355.
[3] P. Erdős, R. Rado, A partition calculus in set theory, Bull. Amer. Math. Soc. 62 (1956) 427–489.
[4] H. Furstenberg, Recurrence in Ergodic Theory and Combinatorical Number Theory, Princeton University Press, Princeton, 1981.
[5] F. Galvin, On a partition theorem of Baumgartner and Hajnal, in: Colloq. Math. Soc. János Bolyai, vol. 10, North Holland, Amsterdam, 1975, pp. 711–729.
[6] R. Graham, B. Rothschild, J. Spencer, Ramsey Theory, Wiley, New York, 1990.
[7] N. Hindman, Partition regularity of matrices, in: B. Landman, M. Nathanson, J. Nesetril, R. Nowakowski, C. Pomerance (Eds.), Combinatorial Number Theory, deGruyter, Berlin, 2007, pp. 265–298.
[8] N. Hindman, I. Leader, Image partition regularity of matrices, Combin. Probab. Comput. 2 (1993) 437–463.
[9] N. Hindman, I. Leader, The semigroup of ultrafilters near 0, Semigroup Forum 59 (1999) 33–55.
[10] N. Hindman, I. Leader, D. Strauss, Image partition regular matrices — bounded solutions and preservation of largeness, Discrete Math. 242 (2002) 115–144.
[11] N. Hindman, I. Leader, D. Strauss, Infinite partition regular matrices — solutions in central sets, Trans. Amer. Math. Soc. 355 (2003) 1213–1235.
[12] N. Hindman, D. Strauss, Algebra in the Stone-Čech Compactification: Theory and Applications, de Gruyter, Berlin, 1998.
[13] N. Hindman, D. Strauss, Image partition regularity over the integers, rationals, and reals, New York J. Math. 11 (2005) 519–538.
[14] K. Milliken, Ramsey's Theorem with sums or unions, J. Combin. Theory Ser. A 18 (1975) 276–290.
[15] R. Rado, Studien zur Kombinatorik, Math. Zeit. 36 (1933) 242–280.
[16] R. Rado, Note on combinatorial analysis, Proc. London Math. Soc. 48 (1943) 122–160.
[17] I. Schur, Über die Kongruenz $x^m + y^m = z^m \pmod p$, Jahresbericht der Deutschen Math.-Verein. 25 (1916) 114–117.
[18] A. Taylor, A canonical partition relation for finite subsets of ω, J. Combin. Theory Ser. A 21 (1976) 137–146.
[19] B. van der Waerden, Beweis einer Baudetschen Vermutung, Nieuw Arch. Wiskunde 19 (1927) 212–216.