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Large monochromatic components in colorings of complete 3-uniform hypergraphs

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ABSTRACT

Let $f(n, r)$ be the largest integer m with the following property: if the edges of the complete 3-uniform hypergraph K_n^3 are colored with r colors then there is a monochromatic component with at least m vertices. Here we show that $f(n, 5) \geq \frac{5n}{7}$ and $f(n, 6) \geq \frac{2n}{3}$. Both results are sharp under suitable divisibility conditions (namely if n is divisible by 7, or by 6 respectively).

1. Introduction

A first exercise in graph theory – in fact an old remark of Erdős and Rado – states that for any graph G, either G or its complement is connected. The following generalization (and the solution for $r = 3$) was suggested in [3]: suppose that the edges of K_n are colored with r colors in any fashion, what is the order of the largest monochromatic *connected* subgraph? The answer for general r, $\lceil \frac{n}{r-1} \rceil$, was given in [4] (it is sharp if $r - 1$ is a prime power and n is divisible by $(r - 1)^2$). This also follows from a result of Füredi [1] on fractional transversals of hypergraphs. The problem was generalized to hypergraphs in [2]. In the generalization, connectivity and components of hypergraphs are understood as follows. Let $\mathcal{H}$ be a hypergraph. We say that $\mathcal{H}$ is *connected* if the *shadow graph* of $\mathcal{H}$, with vertex set $V(\mathcal{H})$ and edge set $\{xy : xy \subset e \text{ for some } e \in E(\mathcal{H})\}$, is connected. A *component* of $\mathcal{H}$ is a maximal connected subhypergraph. The main result of [2] says that any r-coloring of the edges of the complete t-uniform hypergraph on n vertices contains a connected monochromatic subhypergraph on at least $\frac{n}{q}$ vertices, where q is the smallest integer satisfying $r \leq \sum_{i=0}^{t-1} q^i$. The result is best possible if q is a prime power and n is divisible by q^t. The case $t = 2$ (with $q = r - 1$) gives the graph case discussed above. This paper focuses on $t = 3$.

Let $f(n, r)$ be the largest integer m with the following property: if the edges of the complete 3-uniform hypergraph K_n^3 are colored with r colors then there is a monochromatic component with at least m vertices. Applying the result mentioned above for $t = 3$ we get that $f(n, r) = \frac{n}{q}$ if $r = q^2 + q + 1$ with a prime power q and n is divisible by q^3. The case $q = 2$ solves $r = 7$ and the cases $r \leq 4$ are also solved in [2] ($f(n, 3) = n$ and $f(n, 4) \geq \frac{3n}{4}$ with equality if n is divisible by 4). The cases $r = 5, 6$ are left open and the purpose of this note is to fill this gap. We apply the proof method of Füredi used first in [1] (see also in [2]) which connects $f(n, r)$ to fractional transversals of certain hypergraphs.

A hypergraph is *r-partite* if its vertices are partitioned into r classes and each edge intersects each class in exactly one vertex. A hypergraph is 3-*wise intersecting* if any three edges have nonempty intersection. A *fractional transversal* is a non-negative weighting of the vertices such that the sum of the weights over any edge is at least 1. The *value* of a fractional

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transversal is the sum of the weights over all vertices of the hypergraph. Finally, $\tau^*(\mathcal{H})$ is the minimum of the values over all fractional transversals of $\mathcal{H}$. We use the following lemma from [2].

Lemma 1. *Let $\tau^*(r)$ be defined as the maximum of $\tau^*(\mathcal{H})$ over all r-partite 3-wise intersecting hypergraphs $\mathcal{H}$. Then $f(n,r) \geq \frac{n}{\tau^*(r)}$.*

Theorem 1. *$f(n,5) \geq \frac{5n}{7}$ and this is sharp if n is divisible by 7.*

Proof. We start with a construction, showing that $f(n,5)$ is not larger than the claimed value if n is divisible by 7. Let $n = 7k$ and partition $[n] = \{1, \dots, n\}$ into seven k-element sets, X_i. We define five subsets $I_j \subset [7]$ as

$$I_1 = \{1,4,5,6,7\}, \qquad I_2 = \{2,4,5,6,7\}, \qquad I_3 = \{3,4,5,6,7\},$$
$$I_4 = \{1,2,3,6,7\}, \qquad I_5 = \{1,2,3,4,5\}.$$

Observe that every triple of [7] is covered by at least one I_j. Thus every triple $T \subset [n]$ is covered by at least one of the five sets $A_j = \{\cup_{i \in I_j} X_i\}$. Color T with color j where j is the smallest index such that $T \subset A_j$. Clearly each triple of $[n]$ is colored with one of five colors and there is no monochromatic component of size larger than $5k = \frac{5n}{7}$.

On the other hand, $f(n,5) \geq \frac{5n}{7}$ follows from Lemma 1 if we show that $\tau^*(\mathcal{H}) \leq \frac{7}{5}$ holds for every 5-partite 3-wise intersecting hypergraph $\mathcal{H}$. We shall define only the nonzero weights $w(x)$ for $x \in V(\mathcal{H})$. Let A_i denote the vertex classes of $\mathcal{H}$, vertices in A_i will be indexed with i. Note that if there are two edges $e, f \in E(\mathcal{H})$ with $|e \cap f| = 1$ then all edges of $\mathcal{H}$ intersect and $\tau^*(\mathcal{H}) = 1$ follows. Thus we may assume that any two edges of $\mathcal{H}$ intersect in at least two vertices.

Case (i): there exist $e, f \in E(\mathcal{H})$ with $|e \cap f| = 2$. Assume $e = \{x_1, x_2, y_3, y_4, y_5\}$, $f = \{x_1, x_2, z_3, z_4, z_5\}$. Set $Y = \{y_3, y_4, y_5\}$, $Z = \{z_3, z_4, z_5\}$. Using that $\mathcal{H}$ is 3-wise intersecting, it follows that the edge set of $\mathcal{H}$ can be partitioned into E_1, E_2, E_{12} where

$$E_{12} = \{h \in E(\mathcal{H}) : x_1, x_2 \in h\},$$
$$E_1 = \{h \in E(\mathcal{H}) : x_1 \in h, x_2 \notin h\}, \qquad E_2 = \{h \in E(\mathcal{H}) : x_2 \in h, x_1 \notin h\}.$$

We may assume that E_1, E_2 are both non-empty otherwise – as before – all edges of $\mathcal{H}$ intersect and $\tau^*(\mathcal{H}) = 1$.

Assume first that there is a pair of edges $e_1 \in E_1, e_2 \in E_2$ such that e_1, e_2 intersect on $A_3 \cup A_4 \cup A_5$ in a 3-element set $T = \{t_3, t_4, t_5\}$. Since e, e_1 and f, e_1 both intersect in at least two vertices, $T \cap Y$, $T \cap Z$ are nonempty sets, at least one of them, say $T \cap Z$ has exactly one element. We may suppose w.l.o.g. $t_3 = y_3$, $t_4 = z_4$.

If $t_5 \neq y_5$ ($t_5 \neq z_5$ also holds by assumption on $T \cap Z$) then the existence of the triple intersections

$$e \cap e_1 \cap b, \qquad e \cap e_2 \cap a, \qquad f \cap e_2 \cap a, \qquad f \cap e_1 \cap b$$

for $a \in E_1, b \in E_2$ imply that all edges of $E_1 \cup E_2$ contain both t_3 and t_4. If there exists an edge $e_{12} \in E_{12}$ such that neither t_3 nor t_4 is in e_{12} then the existence of the triple intersections $e_{12} \cap e_1 \cap b$, $e_{12} \cap e_2 \cap a$ for $a \in E_1, b \in E_2$ imply that all edges of $E_1 \cup E_2$ contain t_5 as well. Moreover, then all edges of E_{12} must also contain t_5. Now every edge in $E_1 \cup E_2$ intersects $\{x_1, x_2\}$ in one and intersects T in three elements; every edge of E_{12} intersects $\{x_1, x_2\}$ in two and T in at least one element. Thus the weight assignment $w(x_1) = w(x_2) = \frac{2}{5}$, $w(t_3) = w(t_4) = w(t_5) = \frac{1}{5}$ is a fractional transversal of $\mathcal{H}$ with value $\frac{7}{5}$. If every $e_{12} \in E_{12}$ intersects $\{t_3, t_4\}$ then every edge in $E_1 \cup E_2 \cup E_{12}$ intersects $S = \{x_1, x_2, t_3, t_4\}$ in at least three elements thus assigning $\frac{1}{3}$ to each element of S gives a fractional transversal of value $\frac{4}{3} < \frac{7}{5}$ finishing this part of the proof.

If $t_5 = y_5$ then, as in the argument above, the existence of the triple intersections $f \cap e_1 \cap b$, $f \cap e_2 \cap a$ for $a \in E_1, b \in E_2$ imply that all edges of $E_1 \cup E_2$ contain t_4. First suppose that there exist $b_1, b_2 \in E_2$ (not necessarily distinct) with $t_3 \notin b_1$ and $t_5 \notin b_2$. Then the triple intersections $e \cap a \cap b_1$ and $e \cap a \cap b_2$ show that for each $a \in E_1$ we have $t_3 \in a$ and $t_5 \in a$. Therefore we can conclude that all edges of E_1 or all edges of E_2- say all edges of E_1- contain both t_3 and t_5. Moreover, then the triple intersections $e \cap a \cap b$ show that each $b \in E_2$ contains either t_3 or t_5. Now if each $e_{12} \in E_{12}$ contains t_4 or both t_3, t_5 then we can assign $w(x_2) = w(t_4) = \frac{2}{5}$, $w(x_1) = w(t_3) = w(t_5) = \frac{1}{5}$ to get a fractional transversal of value $\frac{7}{5}$. Therefore we may assume (without loss of generality) that the set $E' = \{e_{12} \in E_{12} : \{t_4, t_5\} \cap e_{12} = \emptyset\}$ is nonempty. For any $e_{12} \in E'$, since $|e_{12} \cap e_1| \geq 2$ we know $t_3 \in e_{12}$. We know each $b \in E_2$ contains t_3 or t_5, and if $t_5 \in b$ then $|e_{12} \cap b| \geq 2$ implies $t_3 \in b$ also. Thus in this case t_3 is in every element of $E_1 \cup E_2 \cup E_{12}$. Now if $E'' = \{e_{12} \in E_{12} : \{t_3, t_4\} \cap e_{12} = \emptyset\} = \emptyset$ then the weight function $w(x_1) = w(x_2) = w(t_3) = w(t_4) = \frac{1}{3}$ is a fractional transversal of value $\frac{4}{3}$. If $E'' \neq \emptyset$ then as above t_5 is also in every element of $E_1 \cup E_2 \cup E_{12}$. Then $w(x_1) = w(x_2) = \frac{2}{5}$, $w(t_3) = w(t_4) = w(t_5) = \frac{1}{5}$ is a fractional transversal of value $\frac{7}{5}$.

Now we may assume that any pair of edges $e_1 \in E_1, e_2 \in E_2$ intersect on $A_3 \cup A_4 \cup A_5$ in a set of at most two elements. Fix $e_1 \in E_1, e_2 \in E_2$. In fact – since the triple intersections $e_1 \cap e_2 \cap e$, $e_1 \cap e_2 \cap f$ exist – e_1 and e_2 intersect on $A_3 \cup A_4 \cup A_5$ in a two-element set $T = \{t_3, t_4\}$, say $t_3 = y_3$, $t_4 = z_4$. Since e_1, e_2 do not intersect on A_5, w.l.o.g. $y_5 \notin e_1$, $z_5 \notin e_2$. The triple intersections $e_1 \cap e \cap b$, $e_2 \cap f \cap a$ imply $t_3 \in b$, $t_4 \in a$ for $a \in E_1, b \in E_2$. Since each intersection $a \cap b$ for $a \in E_1, b \in E_2$ has at least two elements, one of t_3, t_4, say t_3 is in all edges of $E_1 \cup E_2$. Moreover each $e_{12} \in E_{12}$ must intersect $\{t_3, t_4\}$ because of the triple intersection $e_{12} \cap e_1 \cap e_2$. If t_4 is also in all edges of $E_1 \cup E_2$ then $\{x_1, x_2, t_3, t_4\}$ intersects every edge of $E_1 \cup E_2 \cup E_{12}$ in at least three elements, implying a fractional transversal of value $\frac{4}{3}$. Otherwise $E' = \{b \in E_2 : t_4 \notin b\} \neq \emptyset$. In this case, since $|b \cap f| \geq 2$ we see that each $b \in E'$ contains z_5. Looking at $b \cap a$ for $a \in E_1, b \in E'$ tells us that $z_5 \in a$ as well. Finally $b \cap e_1 \cap e_{12}$

shows us that if $t_3 \notin e_{12}$ for some $e_{12} \in E_{12}$ then $z_5 \in e_{12}$ (and we know that $t_3 \notin e_{12}$ implies $t_4 \in e_{12}$). Summing up, we find that for each $a \in E_1, x_1, t_3, t_4, z_5 \in a$, and for each $b \in E_2, x_2, t_3 \in b$ and $(t_4 \cup z_5) \cap b$ is nonempty. For each $e_{12} \in E_{12}$, $x_1, x_2 \in e_{12}$ and either $t_3 \in e_{12}$ or $\{t_4, z_5\} \subset e_{12}$. Now the weighting $w(t_3) = w(x_2) = \frac{2}{5}$, $w(x_1) = w(t_4) = w(z_5) = \frac{1}{5}$ gives the required fractional transversal.

Case (ii): Any two distinct $e, f \in \mathcal{H}$ intersect in at least three vertices. Assume first that there is a pair $e, f \in \mathcal{H}$ intersecting in three elements, $e = \{x_1, x_2, x_3, x_4, x_5\}$, $f = \{x_1, x_2, x_3, y_4, y_5\}$. Observe then that every edge must intersect $\{x_1, x_2, x_3\}$ in at least two elements. Again, if the set of edges E_{ij} that intersect $\{x_1, x_2, x_3\}$ in $\{x_i, x_j\}$ is empty for some pair $i, j \in [3]$ then, for $k = [3] \setminus \{i, j\}$, all edges of $\mathcal{H}$ contain x_k and $\tau^*(\mathcal{H}) = 1$. Thus these sets E_{ij} are non-empty. Selecting $e_{12} \in E_{12}, e_{13} \in E_{13}, e_{23} \in E_{23}$, the assumptions on the intersection sizes imply that for each of the three pairs of indices $e_{ij} \cap (A_4 \cup A_5)$ is the same pair, say $\{x_4, y_5\}$. Any edge e_{123} that contains all of $\{x_1, x_2, x_3\}$ must also intersect $\{x_4, y_5\}$, otherwise $|e_{123} \cap e_{12}| \leq 2$. Now assigning $w(x_1) = w(x_2) = w(x_3) = \frac{1}{5}$, $w(x_4) = w(y_5) = \frac{2}{5}$ we have a fractional transversal of $\mathcal{H}$ with value $\frac{7}{5}$.

Finally, if each pair of edges of $\mathcal{H}$ intersect in at least four elements, we can assign weight $\frac{1}{4}$ to vertices of any fixed edge. This gives a fractional transversal of $\mathcal{H}$ with value $\frac{5}{4} < \frac{7}{5}$. □

Theorem 2. *$f(n, 6) \geq \frac{2n}{3}$ and this is sharp if n is divisible by 6.*

Proof. To show that $f(n, 6)$ is not larger than claimed value if n is divisible by 6, let $n = 6k$ and partition $[n]$ into six k-element sets, X_i. We define six subsets $I_j \subset [6]$ as

$$I_1 = \{3, 4, 5, 6\}, \quad I_2 = \{1, 4, 5, 6\}, \quad I_3 = \{2, 4, 5, 6\},$$
$$I_4 = \{1, 2, 3, 6\}, \quad I_5 = \{1, 2, 3, 4\}, \quad I_6 = \{1, 2, 3, 5\}.$$

Observe that every triple of [6] is covered by at least one I_j. Thus every triple $T \subset [n]$ is covered by at least one of the six sets $A_j = \{\cup_{i \in I_j} X_i\}$. Color T with color j where j is the smallest index such that $T \subset A_j$. Clearly each triple of $[n]$ is colored with one of six colors and there is no monochromatic component of size larger than $4k = \frac{2n}{3}$.

As in the proof of Theorem 1, $f(n, 6) \geq \frac{2n}{3}$ follows from Lemma 1 if we show that $\tau^*(\mathcal{H}) \leq \frac{3}{2}$ holds for every 6-partite 3-wise intersecting hypergraph $\mathcal{H}$. To see that, let A_i denote the vertex classes of $\mathcal{H}$. Note that if there are two edges $e, f \in E(\mathcal{H})$ with $|e \cap f| = 1$ then all edges of $\mathcal{H}$ intersect and $\tau^*(\mathcal{H}) = 1$ follows. Thus we may assume that any two edges of $\mathcal{H}$ intersect in at least two vertices. We basically follow the argument of the proof of Theorem 1.

Case (i): There exist $e, f \in E(\mathcal{H})$ with $|e \cap f| = 2$. Set $e \cap f = \{x_1, x_2\}$ and define

$$E_{12} = \{h \in E(\mathcal{H}) : x_1, x_2 \in h\},$$
$$E_1 = \{h \in E(\mathcal{H}) : x_1 \in h, x_2 \notin h\}, \qquad E_2 = \{h \in E(\mathcal{H}) : x_2 \in h, x_1 \notin h\}.$$

Then as before $\mathcal{H} = E_1 \cup E_2 \cup E_{12}$.

Let $E_1 = \{a_1, a_2, \ldots a_s\}$, $E_2 = \{b_1, b_2, \ldots, b_t\}$. We may assume that E_1, E_2 are both nonempty, otherwise – as before – all edges of $\mathcal{H}$ intersect and $\tau^*(\mathcal{H}) = 1$. Notice that $a_i \cap b_j \subset \cup_{k=3}^{6} A_k$ for any $a_i \in E_1, b_j \in E_2$.

If all edges of $E_1 \cup E_2$ have a common vertex v then assigning weight $\frac{1}{2}$ to the vertices in $\{x_1, x_2, v\}$ we have a fractional transversal of value $\frac{3}{2}$ and the proof is finished. Thus we may suppose that

$$\bigcap_{i \in [s]} a_i \cap \bigcap_{j \in [t]} b_j = \emptyset. \tag{1}$$

Lemma 2. *Suppose there exist distinct edges $a_1, a_2 \in E_1, b_1, b_2 \in E_2$ such that $a_1 \cap a_2 \cap b_1 \cap b_2 = \emptyset$. Then $\tau^*(\mathcal{H}) \leq \frac{3}{2}$.*

Proof. Observe that the four triple intersections among these edges are all disjoint (and nonempty). Let U denote the union over all four triple intersections, so $|U| \geq 4$. Note that if $x, x' \in U$ then one of (in fact, at least two of) a_1, a_2, b_1, b_2 contain both x and x'. Thus we cannot have distinct x, x' in the same partite class A_i. Therefore $U = \{x_3, x_4, x_5, x_6\}$ for some $x_i \in A_i$ for $i = 3, 4, 5, 6$, and we may assume without loss of generality that

$$x_3 \in (a_1 \cap b_1 \cap b_2) \setminus a_2, x_4 \in (a_2 \cap b_1 \cap b_2) \setminus a_1,$$

$$x_5 \in (a_1 \cap a_2 \cap b_1) \setminus b_2, x_6 \in (a_1 \cap a_2 \cap b_2) \setminus b_1. \tag{2}$$

We observe that – apart from the exceptional case when $a_i \cap U = \{x_3, x_4\}$ – each edge $a_i \in E_1$ intersects U in at least three vertices. Indeed, if $a_i \cap U \subseteq \{x_3, x_5\}$ then the triple intersection $a_i \cap a_2 \cap b_2$ is missing. If $a_i \cap U \subseteq \{x_4, x_6\}$ then $a_i \cap a_1 \cap b_1$ is missing. Similarly, $a_i \cap U \subseteq \{x_3, x_6\}, \{x_4, x_5\}, \{x_5, x_6\}$ in turn imply the missing intersections $a_i \cap a_2 \cap b_1$, $a_i \cap a_1 \cap b_2$, $a_i \cap b_1 \cap b_2$. (The argument in the exceptional case would require missing $a_i \cap a_1 \cap a_2$ but that intersection is present at x_1.)

Similarly, apart from the exceptional case when $b_j \cap U = \{x_5, x_6\}$, each edge of $b_j \in E_2$ intersects U in at least three vertices. Finally, observe that any $e_{12} \in E_{12}$ intersects U in at least two vertices. Indeed, $e_{12} \cap U \subset \{x_l\}$ for some $l \in \{3, 4, 5, 6\}$

would contradict the existence of the triple intersection $e_{12} \cap a_i \cap b_j$ where $i, j \in [2]$ such that one of a_i, b_j does not contain x_l. Consider $e_{12} \in E_{12}$ exceptional if $e_{12} \cap U = \{x_3, x_4\}$ or $e_{12} \cap U = \{x_5, x_6\}$.

Based on the above observations we can define the required fractional transversal as follows. If no edge in $E_1 \cup E_2$ is exceptional, $w(x_i) = \frac{1}{4}$ for $i = 1, 2, \dots 6$ is suitable. If there exists an exceptional edge in $E_1 \cup E_2$, say a_i, then no $b_j \in E_2$ can be exceptional (otherwise $a_i \cap b_j$ cannot exist) — in fact the following stronger statement is true for any b_j: if $\{x_5, x_6\} \subset b_j$ then $U \subset b_j$. Indeed, $U \cap b_j = \{x_4, x_5, x_6\}$ ($U \cap b_j = \{x_3, x_5, x_6\}$) contradicts the existence of $a_i \cap b_j \cap a_1$ ($a_i \cap b_j \cap a_2$). Moreover no $e_{12} \in E_{12}$ is exceptional with $e_{12} \cap U = \{x_5, x_6\}$ otherwise $e_{12} \cap a_i \cap b_1$ cannot exist. These properties ensure that $w(x_1) = w(x_3) = w(x_4) = \frac{1}{3}$, $w(x_2) = w(x_5) = w(x_6) = \frac{1}{6}$ is a suitable fractional transversal. $\square$

By Lemma 2, from now on we may suppose that

$$a_i \cap a_j \cap b_k \cap b_l \neq \emptyset$$

for every choice of the indices (if $i = j$ or $k = l$ the 3-wise intersecting property ensures it).

Because of (1) we can select a minimal nonintersecting subfamily of $E_1 \cup E_2$, that is $S \subseteq [s], T \subseteq [t]$ such that $\bigcap_{i \in S} a_i \cap \bigcap_{j \in T} b_j = \emptyset$ but for any proper subset $S_1 \cup T_1 \subset S \cup T$

$$\bigcap_{i \in S_1} a_i \cap \bigcap_{j \in T_1} b_j \neq \emptyset. \tag{3}$$

Since $A = \cap_{i \in [s]} a_i, B = \cap_{j \in [t]} b_j$ are both nonempty ($x_1 \in A, x_2 \in B$), it follows that S, T are nonempty. Moreover $|S \cup T| \geq 4$ because $\mathcal{H}$ is 3-wise intersecting. Set $u = |S \cup T|$. Then by choice of $S \cup T$, all $(u-1)$-wise intersections of elements of $S \cup T$ are disjoint and nonempty, so their union U has size at least u, and as in the proof of Lemma 2 no two vertices in U are in the same partite class A_i. Thus if $|S|, |T| \geq 2$ then $U \subset \cup_{k=3}^{6} A_k$, implying that $u = 4$. But then the assumptions of Lemma 2 hold, so the proof is done in this case.

Thus we may assume that one of S, T has one element only, say $T = \{1\}$. In this case $x_1 \in U$ and $x_2 \notin U$, so $U \subset \{x_1\} \cup \cup_{k=3}^{6} A_k$, implying that $u = 4$ or $u = 5$. In both cases, without loss of generality we may select three vertices $X = \{x_3, x_4, x_5\}$ from U with $x_i \in A_i$ for $i = 3, 4, 5$ as follows:

$$x_3 \in (a_1 \cap a_2 \cap b_1) \setminus a_3, x_4 \in (a_1 \cap a_3 \cap b_1) \setminus a_2, x_5 \in (a_2 \cap a_3 \cap b_1) \setminus a_1. \tag{4}$$

Lemma 3. *Suppose there exists $a_i \in E_1$ such that $|a_i \cap X| \leq 1$. Then $\tau^*(\mathcal{H}) \leq \frac{3}{2}$.*

Proof. Suppose without loss of generality that $a_i \cap \{x_4, x_5\} = \emptyset$. Then, for each $b_j \in E_2$, the (nonempty) quadruple intersection $a_3 \cap a_i \cap b_1 \cap b_j$ must be in A_6. This is possible only if all b_j-s intersect on A_6, say in a vertex $x_6 \in a_3 \cap a_i \cap B$. Because of (1) the set $K = \{k \in [s] | x_6 \notin a_k\}$ is nonempty. For every $k \in K, j \in [t]$ the quadruple intersection $a_k \cap a_i \cap b_1 \cap b_j$ contains x_3. This implies $x_3 \in B \cap (\cap_{k \in K} a_k)$. Reversing the argument, $L = \{l \in [s] | x_3 \notin a_l\}$ is nonempty implying that for every $l \in L, j \in [t]$ the quadruple intersection $a_l \cap a_i \cap b_1 \cap b_j$ contains x_6, implying $x_6 \in B \cap (\cap_{l \in L} a_l)$. Thus each edge in E_1 contains x_1 and at least one vertex of $\{x_3, x_6\}$. Every edge in E_2 contains *both* x_3, x_6 and every $e_{12} \in E_{12}$ contains x_1 and also at least one vertex of $\{x_3, x_6\}$ because the triple intersection $e_{12} \cap a_i \cap b_1$ is nonempty. Therefore $w(x_1) = w(x_3) = w(x_6) = \frac{1}{2}$ is a required fractional transversal. $\square$

By Lemma 3 we may suppose from now on that every edge $a_i \in E_1$ meets X in at least two elements.

Claim: Either $X \subset B$ or $B \cap A_6 \neq \emptyset$. Indeed, if an element of X, say $x_3 \notin b_i$ for some $i \in [t]$ then the quadruple intersection $a_1 \cap a_2 \cap b_i \cap b_m$ is in A_6 for all $m \in [t]$. This implies that $B \cap A_6 \neq \emptyset$. The argument works similarly if x_4 or x_5 plays the role of x_3 (considering $a_1 \cap a_3 \cap b_i \cap b_m$ or $a_2 \cap a_3 \cap b_i \cap b_m$), proving the claim.

We look at the two cases of the claim. If $X \subset B$ holds then $w(x_1) = \frac{1}{2}$, $w(x_2) = w(x_3) = w(x_4) = w(x_5) = \frac{1}{4}$ is a required fractional transversal. Indeed, each $a_i \in E_1$ contains x_1 and at least two elements of X, each $b_i \in E_2$ contains x_2 and all elements of X. Each $e_{12} \in E_{12}$ contains x_1, x_2 and at least one element of X otherwise – considering the triple intersections $e_{12} \cap a_i \cap b_j$ – all a_i, b_j should intersect in A_6, contradicting (1). Thus we may assume that $X \subset B$ does not hold.

Select $x_6 \in A_6 \cap B$. By definition of S, at least one a_j with $j \in S$ does not contain x_6, say $x_6 \notin a_3$. We show that $\{x_4, x_5\} \subset B$. Indeed, if $x_4 \notin b_j$ ($x_5 \notin b_j$)then the quadruple intersection $a_1 \cap a_3 \cap b_1 \cap b_j$ ($a_2 \cap a_3 \cap b_1 \cap b_j$) does not exist.

Therefore since $X \subset B$ does not hold, we know $x_3 \notin b_j$ for some $j \in [t]$. Define $K = \{k \in [s] | x_6 \notin a_k\}$ as before. We show that for each $k \in K$, $\{x_4, x_5\} \subset a_k$. Indeed, if $x_4 \notin a_k$ ($x_5 \notin a_k$) for some $k \in K$ then $a_1 \cap a_k \cap b_1 \cap b_j$ ($a_2 \cap a_k \cap b_1 \cap b_j$)does not exist.

Now we finish the proof by showing that $w(x_1) = \frac{1}{2}$, $w(x_2) = w(x_4) = w(x_5) = w(x_6) = \frac{1}{4}$ is a required fractional transversal. Notice that for every $a_i \in E_1$ either $x_6 \in a_i$ or $i \in K$ and $\{x_4, x_5\} \subset a_i$. This property and that every a_i contains at least one of x_4, x_5 ensures that the weight of a_i is at least one. The weighting is also good for every $b_j \in E_2$ since $\{x_2, x_4, x_5, x_6\} \subset B$. Finally, each $e_{12} \in E_{12}$ contains x_1, x_2 and at least one vertex of $\{x_4, x_5, x_6\}$ because $e_{12} \cap a_3 \cap b_1 \neq \emptyset$. Thus the weighting is a required fractional transversal. $\square$

Case (ii): $|e \cap f| \geq 3$ for each $e, f \in E(\mathcal{H})$. In this case let us first suppose that there exist e and f such that $e \cap f = M$ where $|M| = 3$. Then we define a fractional transversal by giving weight $\frac{1}{2}$ to each vertex in M. This is valid because every other edge g must intersect M in at least two vertices — otherwise either $|g \cap e| \leq 2$ or $|g \cap f| \leq 2$, contradicting the assumption for Case (ii). Thus we have a fractional transversal of value $\frac{3}{2}$. Thus we may suppose that every pair of edges intersects in at least four vertices. Let e and f be an arbitrary pair and let $M \subseteq e \cap f$ be a set of size four. Define a fractional transversal by weighting each vertex of M with $\frac{1}{3}$. Now every other edge g intersects M in at least three vertices — otherwise either $|g \cap e| \leq 3$ or $|g \cap f| \leq 3$, contradicting our assumption. Now we get a fractional transversal of value $\frac{4}{3} < \frac{3}{2}$. $\square$

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