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M-alternating Hamilton paths and M-alternating Hamilton cycles[☆]

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ABSTRACT

We study M-alternating Hamilton paths, and M-alternating Hamilton cycles in a simple connected graph G on ν vertices with a perfect matching M. Let G be a bipartite graph, we prove that if for any two vertices x and y in different parts of G, $d(x)+d(y) \geq \nu/2+2$, then G has an M-alternating Hamilton cycle. For general graphs, a condition for the existence of an M-alternating Hamilton path starting and ending with edges in M is put forward. Then we prove that if $\kappa(G) \geq \nu/2$, where $\kappa(G)$ denotes the connectivity of G, then G has an M-alternating Hamilton cycle or belongs to one class of exceptional graphs. Lou and Yu [D. Lou, Q. Yu, Connectivity of k-extendable graphs with large k, Discrete Appl. Math. 136 (2004) 55–61] have proved that every k-extendable graph H with $k \geq \nu/4$ is bipartite or satisfies $\kappa(H) \geq 2k$. Combining our result with theirs we obtain we prove the existence of M-alternating Hamilton cycles in H.

1. Introduction, terminologies and preliminary results

All graphs considered in this paper are finite, undirected, connected and simple. For the terminologies and notations not defined in this paper, the reader is referred to [4].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by ν or $|G|$ the order of $V(G)$, κ the connectivity of G, and δ the minimum degree of G. For $u \in V(G)$, we denote by $d(u)$ the degree of u and $N(u)$ the set of neighbors of u in G. For a subgraph H of G and a vertex set $U \subseteq V(G-H)$, we denote by $N_H(U)$, or $N_H(u)$ if U contains only one vertex u, the set of neighbors of U in H. For any two disjoint vertex sets X, Y in G we denote by $e(X, Y)$ the number of edges of G from X to Y.

Let $C = u_0u_1 \ldots u_{m-1}u_0$ be a cycle in G. Throughout this paper, the subscripts of u_i will be reduced modulo m. We always orient C such that u_{i+1} is the successor of u_i. Let $U \subseteq V(C)$, the set of predecessors and successors of U on C is denoted by U^- and U^+ respectively, or u^- and u^+ when U contains only one vertex u. For $0 \leq i, j \leq m-1$, the path $u_iu_{i+1} \ldots u_j$ is denoted by $u_iC^+u_j$, while the path $u_iu_{i-1} \ldots u_j$ is denoted by $u_iC^-u_j$. For a path $P = v_0v_1 \ldots v_{q-1}$ and $0 \leq i, j \leq q-1$, the segment of P from v_i to v_j is denoted by v_iPv_j.

A *matching* M of G is a subset of $E(G)$ in which no two elements are adjacent. If every $v \in V(G)$ is covered by an edge in M then M is said to be a *perfect matching* of G. An *M-alternating path* P is a path of which the edges appear alternately in M and $E(G) \setminus M$. An *M-alternating cycle* C is a cycle of which the edges appear alternately in M and $E(G) \setminus M$. We call an edge in a matching M or an M-alternating path starting and ending with edges in M a *closed M-alternating path*, while an edge in $E(G) \setminus M$ or an M-alternating path starting and ending with edges in $E(G) \setminus M$ an *open M-alternating path*. An M-alternating path whose starting and ending vertices are not covered by M are called an *M-augmenting path*.

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A graph G is said to be *k-extendable* for $0 \leq k \leq (\nu-2)/2$ if there exists a matching of size k in G, and any such matching is contained in a perfect matching of G. The concept of k-extendable was introduced by Plummer in [7]. In the same paper a relationship between extendability and connectivity is showed.

Theorem 1.1. *If G is a k-extendable graph, then $\kappa \geq k+1$.*

When k is large and G is not bipartite, the lower bound of connectivity can be raised.

Theorem 1.2 (*Lou and Yu [6]*). *If G is a k-extendable graph with $k \geq \nu/4$, then either G is bipartite or $\kappa \geq 2k$.*

M-alternating paths and M-alternating cycles play important roles in matching theory. Berge's well-known theory [3] on maximum matchings and M-augmenting paths is a good demonstration. In [1,2], M-alternating paths are used to characterize k-extendable and n-factor-critical graphs. In this paper, we study the existence of M-alternating Hamilton paths and M-alternating Hamilton cycles in graphs with a perfect matching. The following two lemmas will be useful to obtain our main results.

Lemma 1.3. *Let G be a graph with a perfect matching M. Let $C = u_0u_1\ldots u_{2m-1}u_0$ be a longest M-alternating cycle in G, where $u_{2i-1}u_{2i} \in M$, $0 \leq i \leq m-1$. Let v, w be the endvertices of a closed M-alternating path in $G - C$. For any vertex set $\{u_{2i}, u_{2i+1}\}$, $0 \leq i \leq m-1$, if G is bipartite then $e(\{u_{2i}, u_{2i+1}\}, \{v, w\}) \leq 1$, otherwise $e(\{u_{2i}, u_{2i+1}\}, \{v, w\}) \leq 2$.*

Proof. Let P be a closed M-alternating path connecting v and w in $G - C$. If $u_{2i}v, u_{2i+1}w \in E(G)$, then $u_{2i}vPwu_{2i+1}C^+u_{2i}$ is an M-alternating cycle longer than C, contradicting the maximality of C. Thus $|\{u_{2i}v, u_{2i+1}w\} \cap E(G)| \leq 1$. Similarly $|\{u_{2i}w, u_{2i+1}v\} \cap E(G)| \leq 1$. So $e(\{u_{2i}, u_{2i+1}\}, \{v, w\}) \leq 2$. If G is bipartite, then $|\{u_{2i}v, u_{2i+1}w\} \cap E(G)| = 0$ or $|\{u_{2i}w, u_{2i+1}v\} \cap E(G)| = 0$, so $e(\{u_{2i}, u_{2i+1}\}, \{v, w\}) \leq 1$. □

Lemma 1.4. *Let G be a graph with a perfect matching M. Let $P = u_0u_1\ldots u_{2p-1}$ be a longest closed M-alternating path in G. Let v, w be the endvertices of a closed M-alternating path in $G - P$. For any vertex set $\{u_{2i-1}, u_{2i}\}$, $1 \leq i \leq p-1$, if G is bipartite then $e(\{u_{2i-1}, u_{2i}\}, \{v, w\}) \leq 1$, otherwise $e(\{u_{2i-1}, u_{2i}\}, \{v, w\}) \leq 2$.*

Proof. The proof is similar to that of Lemma 1.3. □

2. *M*-alternating cycles in bipartite graphs

Theorem 2.1. *Let G be a bipartite graph and M a perfect matching of G. For any two vertices x and y in different parts of G, $d(x) + d(y) \geq \nu/2 + 2$. Then G has an M-alternating Hamilton cycle.*

Proof. Let G' be a graph, with a perfect matching M, which satisfies the conditions of the theorem but does not have an M-alternating Hamilton cycle. We add edges to G' until the addition of any more edge results in an M-alternating Hamilton cycle. Let the graph obtained finally be G.

Let the bipartition of G be (A, B). G cannot be complete bipartite, or an M-alternating Hamilton cycle exists. So there are two nonadjacent vertices $w_0 \in A$ and $w_{\nu-1} \in B$. By our assumption on G, $G + w_0w_{\nu-1}$ has an M-alternating Hamilton cycle. Hence, there is a closed M-alternating Hamilton path in G connecting w_0 and $w_{\nu-1}$. Let the path be $P' = w_0w_1\ldots w_{\nu-1}$, where $w_{2i} \in A$ and $w_{2i-1} \in B$, $0 \leq i \leq \nu/2$. Since $d(w_0)+d(w_{\nu-1}) \geq \nu/2+2$, without loss of generality, let $d(w_0) \geq d(w_{\nu-1})$, we have $d(u_0) \geq \nu/4 + 1$. Hence the neighbor w_i of w_0 with the maximum subscript i satisfies $i \geq 2(\nu/4 + 1) = \nu/2 + 2$. Then $w_0P'w_iw_0$ is an M-alternating cycle with length at least $\nu/2 + 2$.

Let $C = u_0u_1\ldots u_{2m-1}u_0$ be one longest M-alternating cycle in G, where $u_{2i} \in A$, $u_{2i+1} \in B$ and $u_{2i-1}u_{2i} \in M$, $0 \leq i \leq m-1$. Then $2m < \nu$. By above discussion, $2m \geq \nu/2 + 2$. Let $G_1 = G - C$, we have $|G_1| \leq \nu/2 - 2$. Denote the degree of a vertex x in G_1 by $d_1(x)$.

Let v_0 be a vertex in G_1 who sends some edges to C. Without loss of generality let $v_0 \in A$. Let $P = v_0v_1\ldots v_{2p-1}$ be a maximal closed M-alternating path in G_1 starting with v_0. Then v_{2p-1} cannot be adjacent to any vertex in $G_1 - P$. So $d_1(v_{2p-1}) \leq p$.

Assume that v_{2p-1} also sends some edges to C. Since G is bipartite, v_0 and v_{2p-1} can only be adjacent to u_{2i+1} and u_{2j}, $0 \leq i, j \leq m-1$, respectively. Let u_{2r+1} and u_{2s} be the neighbors of v_0 and v_{2p-1} on C such that the path $P_1 = u_{2s}C^+u_{2r+1}$ is the shortest. Then any internal vertex of P_1 cannot be adjacent to v_0 or v_{2p-1}. Consider the M-alternating cycle $C_1 = u_{2r+1}C^+u_{2s}v_{2p-1}Pv_0u_{2r+1}$. Since C is the longest M-alternating cycle in G, $|C_1| \leq |C|$, so $|P| \leq |P_1| - 2$.

By Lemma 1.3, for any vertex set $\{u_{2i}, u_{2i+1}\}$ on P_2, $e(\{u_{2i}, u_{2i+1}\}, \{v_0, v_{2p-1}\}) \leq 1$. The number of such sets is

$$(|P_2| - 2)/2 = (|C| - |P_1| + 2 - 2)/2 \leq (|C| - (|P| + 2))/2 = (|C| - |P|)/2 - 1.$$

So

$$\begin{aligned} d(v_0) + d(v_{2p-1}) &= |N_C(v_0)| + |N_C(v_{2p-1})| + d_1(v_0) + d_1(v_{2p-1}) \\ &\leq ((|C| - |P|)/2 - 1 + 2) + |G_1|/2 + p \\ &= (2m - 2p)/2 + 1 + (\nu - 2m)/2 + p \\ &= \nu/2 + 1, \end{aligned}$$

contradicting $d(v_0)+d(v_{2p-1})\geq \nu/2+2$. Therefore, v_{2p-1} sends no edges to C. Similarly, for any vertex $x\in G_1$ who sends some edges to C, and any maximal close M-alternating path P_0 in G_1 starting with x, the other endvertex y of P_0 sends on edge to C.

We also have $d(v_{2p-1})\leq p\leq |G_1|/2\leq \nu/4-1$. For any vertex $x\in A\cap V(G_1)$, $d(x)\geq \nu/2+2-d(v_{2p-1})\geq \nu/2+2-(\nu/4-1)=\nu/4+3$. Since $d_1(x)\leq |G_1|/2\leq \nu/4-1$, x must send some edges to C.

Suppose that $y\in B\cap V(G_1)$ sends some edges to C. Let $P(y)$ be a maximal closed M-alternating path in G_1 starting with y. Then, the other endvertex x of $P(y)$ sends on edge to C. However $x\in A\cap V(G_1)$, a contradiction. So for any $y\in B\cap V(G_1)$, y sends no edge to C. Hence $d(y)\leq |G_1|/2$. Correspondingly, for any u_{2i}, $0\leq i\leq m-1$, u_{2i} sends no edge to G_1, so $d(u_{2i})\leq |C|/2$. But then $d(u_{2i})+d(y)\leq |C|/2+|G_1|/2=\nu/2$, contradicting the conditions of our theorem. So G, and therefore G', must have an M-alternating Hamilton cycle. □

Remark 2.2. The lower bound of degree sum in Theorem 2.1 is best possible. Let H_0 and H_1 be two disjoint complete bipartite with bipartition $(U_0,\ V_0)$ and $(U_1,\ V_1)$ respectively, where $|U_0|=|U_1|=|V_0|=|V_1|$. Let $u,\ v\notin V(H_0)\cup V(H_1)$ be two different vertices. We construct graph G by joining u to every vertex in V_i, v to every vertex in U_i, $i=0,\ 1$, and u to v. For any x and y in different parts of G, we have $d(x)+d(y)\geq \nu/2+1$. Let M be a perfect matching containing the edge uv, G does not have an M-alternating Hamilton cycle.

3. *M*-alternating paths in general graphs

In this section we bring forward a result on the relationship between degree sums and M-alternating Hamilton paths, which will be used in the next section as well.

Theorem 3.1. *Let G be a graph with a perfect matching M. For any $x, y\in V(G)$ connected by a closed M-alternating path, $d(x)+d(y)\geq \nu-1$. Then G has a closed M-alternating Hamilton path.*

Proof. Suppose that G does not have a closed M-alternating Hamilton path. Let $P=u_0u_1\dots u_{2m-1}$ be a longest closed M-alternating path in G. Then $|P|\leq \nu-2$.

By the choice of P, $N(u_0), N(u_{2m-1})\subseteq V(P)$. So

$$|P|\geq \max(d(u_0), d(u_{2m-1}))+1\geq (d(u_0)+d(u_{2m-1}))/2+1\geq (\nu-1)/2+1=(\nu+1)/2.$$

Let $N_0(u_0)$ and $N_1(u_0)$ be the set of the neighbors of u_0 whose indices are even and odd, $N_0(u_{2m-1})$ and $N_1(u_{2m-1})$ be the set of the neighbors of u_{2m-1} whose indices are even and odd, respectively. Let $S=M\setminus E(P)$. Denoted by $V(S)$ the set of vertices associated with the edges in S. Then

$$|N_0(u_0)|+|N_1(u_0)|+|N_0(u_{2m-1})|+|N_1(u_{2m-1})|=d(u_0)+d(u_{2m-1})\geq \nu-1. \tag{1}$$

Claim 1. There does not exist an M-alternating cycle C in G such that $V(P)\subseteq V(C)$.

Suppose that such a cycle C exists. Then for an edge $xy\in M\setminus E(C)$, each of x and y cannot be adjacent to any vertex on C, or we can obtain a closed M-alternating path longer than P, by going through xy, then all vertices on C. So

$$d(x)+d(y)\leq 2(\nu-1)-2|C|\leq 2(\nu-1)-2|P|\leq 2(\nu-1)-(\nu+1)=\nu-3,$$

contradicting the condition of the theorem. Thus Claim 1 holds. □

For any edge $u_{2i-1}u_{2i}$, $1\leq i\leq m-1$, if $u_0u_{2i}, u_{2i-1}u_{2m-1}\in E(G)$, then we obtain an M-alternating cycle $u_0u_{2i}Pu_{2m-1}u_{2i-1}Pu_0$ containing all vertices on P, contradicting Claim 1. So

$$|N_0(u_0)|+|N_1(u_{2m-1})|\leq m-1. \tag{2}$$

By Claim 1, u_0 and u_{2m-1} cannot be adjacent to each other, so $|N_1(u_0)|\leq m-1$ and $|N_0(u_{2m-1})|\leq m-1$. Together with (1), we have

$$|N_0(u_0)|+|N_1(u_{2m-1})|\geq (\nu-1)-(|N_1(u_0)|+|N_0(u_{2m-1})|)\geq \nu-2m+1. \tag{3}$$

By (2) and (3), $m-1\geq \nu-2m+1$, that is,

$$m\geq (\nu+2)/3. \tag{4}$$

By (1) and (2),

$$|N_1(u_0)|+|N_0(u_{2m-1})|\geq \nu-m. \tag{5}$$

We classify all sets $\{u_{2i-1}, u_{2i}\}$, $1\leq i\leq m-1$ as following. If $|\{u_0u_{2i-1}, u_{2m-1}u_{2i}\}\cap E(G)|=0$, 1 or 2, then let $\{u_{2i-1}, u_{2i}\}\in \mathscr{C}_0, \mathscr{C}_1$ or $\mathscr{C}_2$. Let $|\mathscr{C}_1|=r_1$ and $|\mathscr{C}_2|=r_2$. Then

$$r_1+r_2\leq m-1, \tag{6}$$

and

$$r_1 + 2r_2 = |N_1(u_0)| + |N_0(u_{2m-1})| \geq \nu - m. \quad (7)$$

By (6) and (7), we have $r_2 \geq \nu - 2m + 1$.

Claim 2. For any $xy \in S$, $N_P(x) \neq \phi$ and $N_P(y) \neq \phi$.

Suppose that the claim is not true and without loss of generality let $N_P(y) = \phi$. For any edge $u_{2i-1}u_{2i}$, $1 \leq i \leq m-1$, if $u_0u_{2i} \in E(G)$, then x cannot be adjacent to u_{2i-1}, or $yxu_{2i-1}Pu_0u_{2i}Pu_{2m-1}$ is a closed M-alternating path longer than P, contradicting the maximality of P. Similarly, if $u_{2m-1}u_{2i-1} \in E(G)$, then x cannot be adjacent to u_{2i}. Furthermore x cannot be adjacent to u_0 and u_{2m-1}. Thus $|N_P(x)| \leq 2m - (|N_0(u_0)| + |N_1(u_{2m-1})|) - 2 \leq 2m - (\nu - 2m + 1) - 2 = 4m - \nu - 3$.

Since $|N(x) \cap V(S)| \leq |V(S)| - 1 = \nu - 2m - 1$ and similarly $|N(y) \cap V(S)| \leq \nu - 2m - 1$. We have $d(x) + d(y) \leq 4m - \nu - 3 + 2(\nu - 2m - 1) \leq \nu - 5$, contradicting the condition of the theorem. So Claim 2 must hold. □

We call an edge $u_{2i-1}u_{2i}$, $1 \leq i \leq m-1$, removable if $\{u_{2i-1}, u_{2i}\} \in \mathscr{C}_2$. For every removable edge $u_{2i-1}u_{2i}$ we get two M-alternating cycles containing all vertices of P, that is, $C_0 = u_0Pu_{2i-1}u_0$ and $C_1 = u_{2i}Pu_{2m-1}u_{2i}$. For any edge $xy \in S$, if $N_{C_0}(x) \neq \phi \neq N_{C_1}(y)$, or $N_{C_1}(x) \neq \phi \neq N_{C_0}(y)$, then we obtain a closed M-alternating path longer than P, by traversing all vertices on C_0, followed by x and y and those on C_1, contradicting the maximality of P. But by Claim 2, $N_P(x) \neq \phi \neq N_P(y)$. So either $N_P(x), N_P(y) \subseteq V(C_0)$ or $N_P(x), N_P(y) \subseteq V(C_1)$.

Let $r = r_2$, $\{e_1, e_2, \ldots, e_r\}$ the set of removable edges, $P_0, P_1, \ldots, P_r$ the $r+1$ segments of P obtained by removing all removable edges. Then $P = P_0e_1P_1e_2 \ldots e_rP_r$ and $V(P) = \cup_{i=0}^{r} V(P_i)$. Note here that the length of P_i $(0 \leq i \leq r)$ is at least 1.

For any edge $xy \in S$, suppose that there exist integers s, t, $0 \leq s \neq t \leq r$, such that $N_{P_s}(x) \neq \phi \neq N_{P_t}(y)$. Without loss of generality, suppose that $s < t$. Let $e_t = u_{2h-1}u_{2h}$. Then x and y are adjacent to vertices on two M-alternating cycles $u_0Pu_{2h-1}u_0$ and $u_{2h}Pu_{2m-1}u_{2h}$ respectively, contradicting our conclusion above. So there must exist an integer l, $1 \leq l \leq r$, such that all neighbors of x, y on P be on P_l.

Let $P_l = u_{2g}u_{2g+1} \ldots u_{2g+2p-1}$. Counting the vertices on P_l, we have

$$2p = |P_l| = |E(P_l)| + 1 \leq (|E(P)| - 2r) + 1 = 2m - 2r \leq 2m - 2(\nu - 2m + 1) = 6m - 2\nu - 2.$$

Note that by (4) the last value is positive. By Lemma 1.4, $e(\{x, y\}, \{u_{2g+2j-1}, u_{2g+2j}\}) \leq 2$ for $1 \leq j \leq p-1$. So $e(\{x, y\}, \{u_{2g+1}, u_{2g+2}, \ldots, v_{2g+2p-2}\}) \leq 2(p-1)$. Then

$$|N_P(x)| + |N_P(y)| \leq 2(p-1) + 4 = 2p + 2 \leq 6m - 2\nu - 2 + 2 = 6m - 2\nu.$$

Since $|N(x) \cap V(S)|, |N(y) \cap V(S)| \leq \nu - 2m - 1$, we have

$$\begin{aligned} d(x) + d(y) &= |N_P(x)| + |N_P(y)| + |N(x) \cap V(S)| + |N(y) \cap V(S)| \\ &\leq 6m - 2\nu + 2(\nu - 2m - 1) \\ &= 2m - 2 \\ &< \nu - 2, \end{aligned}$$

again contradicting the condition of our theorem. □

4. *M*-alternating cycles in general graphs

In this section, we prove that except for one class of graphs, every graph G with $\kappa \geq \nu/2$ and a perfect matching M has an M-alternating Hamilton cycle. Firstly we construct the exceptional graphs.

We define $\mathcal{G}_1$ as the class of graphs constructed by taking two copies of the complete graph K_{2n+1}, $n \geq 1$, with vertex sets $\{x_1, x_2, \ldots, x_{2n+1}\}$ and $\{y_1, y_2, \ldots, y_{2n+1}\}$, and joining every x_i to y_i, $1 \leq i \leq 2n+1$. It is easy to check that any graph $G \in \mathcal{G}_1$ with size $4n+2$ $(n \geq 1)$ is $(2n+1)$-connected, but if we take the perfect matching $M = \{x_iy_i : 1 \leq i \leq 2n+1\}$, then there is no M-alternating Hamilton cycle in G. We call M the jointing matching of G. Note that the jointing matching of G is unique.

Lemma 4.1. *Let G be a graph with $\kappa \geq \nu/2$ and M a perfect matching of G. Then G has an M-alternating cycle C such that $|C| \geq \nu/2 + 1$.*

Proof. Suppose that there is no M-alternating cycle C with $|C| \geq \nu/2 + 1$ in G. By $\kappa \geq \nu/2$ we have $\delta \geq \nu/2$, so $d(x) + d(y) \geq \nu$ for any $x, y \in V(G)$. By Theorem 3.1, there is an M-alternating Hamilton path in G. Let the path be $P = u_0u_1 \ldots u_{2m-1}$, where $2m = \nu$. We follow the notations $N_0(u_0), N_1(u_0), N_0(u_{2m-1}), N_1(u_{2m-1})$ in Theorem 3.1.

Obviously $u_0u_{2m-1} \notin E(G)$, or we have an M-alternating Hamilton cycle, contradicting our assumption. For any $1 \leq i \leq m-1$, if $u_0u_{2i}, u_{2m-1}u_{2i-1} \in E(G)$, then $u_0Pu_{2i-1}u_{2m-1}Pu_{2i}u_0$ is an M-alternating Hamilton cycle, again contradicting our assumption. So $u_0u_{2i} \notin E(G)$ or $u_{2m-1}u_{2i-1} \notin E(G)$. Hence $|N_0(u_0)| + |N_1(u_{2m-1})| \leq \nu/2 - 1$. Therefore,

$$|N_1(u_0)| + |N_0(u_{2m-1})| = d(u_0) + d(u_{2m-1}) - (|N_0(u_0)| + |N_1(u_{2m-1})|) \geq \nu/2 + 1.$$

Without loss of generality suppose that $|N_1(u_0)| \geq |N_0(u_{2m-1})|$. Then $|N_1(u_0)| \geq \nu/4 + 1/2$. Thus there exists an integer l, $1 \leq l \leq m$, such that $2l - 1 \geq 2(\nu/4 + 1/2) - 1 = \nu/2$ and $u_0u_{2l-1} \in E(G)$. Then $u_0Pu_{2l-1}u_0$ is an M-alternating cycle with length at least $\nu/2 + 1$, again contradicting our assumption. □

Theorem 4.2. *Let G be a graph with $\kappa \geq \nu/2$ and M a perfect matching of G. Then either G has an M-alternating Hamilton cycle or $G \in \mathcal{G}_1$ and M is the jointing matching of G.*

Proof. Suppose that G does not have an M-alternating Hamilton cycle. Let $C = u_0u_1 \ldots u_{2m-1}u_0$ be the longest M-alternating cycle in G, where $u_{2i-1}u_{2i} \in M$ and $m < \nu/2$. By $\kappa \geq \nu/2$ we have $\delta \geq \nu/2$.

Let $w \in V(G - C)$, we let $N_0(w) = \{u_{2i} : u_{2i} \in N_C(w), 0 \leq i \leq m-1\}$ and $N_1(w) = \{u_{2i+1} : u_{2i+1} \in N_C(w), 0 \leq i \leq m-1\}$. Let $W \subseteq V(G - C)$, we let $N_0(W) = \{u_{2i} : u_{2i} \in N_C(W), 0 \leq i \leq m-1\}$ and $N_1(W) = \{u_{2i+1} : u_{2i+1} \in N_C(W), 0 \leq i \leq m-1\}$.

Firstly we prove that $G - C$ is connected. Suppose to the contrary that there are at least two components in $G - C$, say G_1 and G_2 with $|G_1| \leq |G_2|$. There is at least one edge $v_0v_1 \in M \cap E(G_1)$. By Lemma 1.3 $e(\{u_{2i}, u_{2i+1}\}, \{v_0, v_1\}) \leq 2$ for every $0 \leq i \leq m-1$. So $|N_C(v_0)| + |N_C(v_1)| \leq 2m$. Let $d_1(v)$ denote the degree of $v \in V(G_1)$ in G_1. Then

$$d(v_0) + d(v_1) = d_1(v_0) + d_1(v_1) + |N_C(v_0)| + |N_C(v_1)| \leq 2(|G_1| - 1) + 2m \leq |G_1| + |G_2| - 2 + 2m \leq \nu - 2,$$

contradicting $d(v_0) + d(v_1) \geq \nu$. Hence $G - C$ is connected. Let $G_1 = G - C$.

Consider any closed M-alternating paths in G_1 with endvertices w and z. By Lemma 1.3, $e(\{u_{2i}, u_{2i+1}\}, \{w, z\}) \leq 2$ for every $0 \leq i \leq m-1$. Thus

$$|N_C(w)| + |N_C(z)| \leq 2m.$$

Since $|N_C(w)| + |N_C(z)| + d_1(w) + d_1(z) = d(w) + d(z) \geq \nu$, we have

$$d_1(w) + d_1(z) \geq \nu - (|N_C(w)| + |N_C(z)|) \geq \nu - 2m = |G_1|.$$

Let $M_1 = M - E(C)$, then M_1 is a perfect matching of G_1 and any closed M-alternating path in G_1 is a closed M_1-alternating path. G_1 with M_1 satisfies the condition of Theorem 3.1, so there is a closed M_1-alternating Hamilton path in G_1, or equally, a closed M-alternating path in G containing all vertices in G_1. Let such a path be $P = v_0v_1 \ldots v_{2q-1}$, where $2q = \nu - 2m$. We have the following cases to discuss.

Case 1. There exist r, s, $0 \leq r, s \leq q-1$, such that there are no closed M-alternating path in G_1 connecting v_{2r} and v_{2s+1}.

Obviously $2s + 1 < 2r$, or $v_{2r}Pv_{2s+1}$ is a closed M-alternating path in G_1 connecting v_{2r} and v_{2s+1}. Thus we have $s < r$ and $|G_1| \geq 4$. Consider v_{2s} and v_{2r+1}. They are the endvertices of a closed M-alternating path in G_1. By the discussion above,

$$d_1(v_{2s}) + d_1(v_{2r+1}) \geq |G_1| = 2q. \tag{8}$$

For any vertex set $\{v_{2i}, v_{2i+1}\}$, $0 \leq i \leq q-1$, $i \neq r, s$, if $v_{2s}v_{2i+1}, v_{2i}v_{2r+1} \in E(G)$, then

$$v_{2s+1}v_{2s}v_{2i+1}v_{2i}v_{2r+1}v_{2r}$$

is a closed M-alternating path in G_1 connecting v_{2r} and v_{2s+1}, contradicting the assumption of Case 1. So $|\{v_{2s}v_{2i+1}, v_{2i}v_{2r+1}\} \cap E(G)| \leq 1$. Similarly $|\{v_{2s}v_{2i}, v_{2i+1}v_{2r+1}\} \cap E(G)| \leq 1$. So

$$e(\{v_{2s}, v_{2r+1}\}, \{v_{2i}, v_{2i+1}\}) \leq 2.$$

Furthermore, v_{2s} and v_{2r+1} cannot be adjacent or $v_{2s+1}v_{2s}v_{2r+1}v_{2r}$ is a closed M-alternating path in G_1 connecting v_{2r} and v_{2s+1}. So

$$d_1(v_{2s}) + d_1(v_{2r+1}) \leq 2(q-2) + 4 = 2q. \tag{9}$$

Thus equalities in (8) and (9) must hold. Furthermore $|N_C(v_{2s})| + |N_C(v_{2r+1})| = 2m$ and

$$e(\{u_{2j}, u_{2j+1}\}, \{v_{2s}, v_{2r+1}\}) = 2$$

for every $0 \leq j \leq m-1$.

We classify the sets $\{u_{2j}, u_{2j+1}\}$, $0 \leq j \leq m-1$ into four classes, by the distribution of the 2 edges between $\{u_{2j}, u_{2j+1}\}$ and $\{v_{2s}, v_{2r+1}\}$. That is,

$$\{u_{2j}, u_{2j+1}\} \in \begin{cases} \mathcal{C}_1, & \text{if } u_{2j}v_{2s}, u_{2j+1}v_{2s} \in E(G), \\ \mathcal{C}_2, & \text{if } u_{2j}v_{2s}, u_{2j}v_{2r+1} \in E(G), \\ \mathcal{C}_3, & \text{if } u_{2j+1}v_{2s}, u_{2j+1}v_{2r+1} \in E(G), \\ \mathcal{C}_4, & \text{if } u_{2j}v_{2r+1}, u_{2j+1}v_{2r+1} \in E(G). \end{cases}$$

Let $|\mathcal{C}_i| = t_i$, $1 \leq i \leq 4$. We have $t_1 + t_2 + t_3 + t_4 = m$, $|N_C(v_{2s})| = t_2 + t_3 + 2t_1$, $|N_C(v_{2r+1})| = t_2 + t_3 + 2t_4$, $|N_0(v_{2s})| = t_1 + t_2$, $|N_1(v_{2s})| = t_1 + t_3$, $|N_0(v_{2r+1})| = t_2 + t_4$ and $|N_1(v_{2r+1})| = t_3 + t_4$.

Case 1.1. t_2 or $t_3 \neq 0$. Without loss of generality let $t_2 > 0$.

Case 1.1.1. $t_2 = m$.

For any $0 \leq i < j \leq m-1$, $u_{2i+1}u_{2j+1} \notin E(G)$, or $u_{2j}v_{2s}Pv_{2r+1}u_{2i}C^{-}u_{2j+1}u_{2i+1}C^{+}u_{2j}$ is an M-alternating cycle longer than C, a contradiction. Therefore any u_{2l+1}, $0 \leq l \leq m-1$, has at most $|C|/2 = m$ neighbors on C. Thus $|N_{G_1}(u_{2l+1})| \geq d(u_{2l+1}) - m \geq \nu/2 - m = q$.

Since u_{2l} is adjacent to v_{2r+1}, u_{2l+1} cannot be adjacent to any vertex v_{2i}, $0 \le 2i \le 2r$, or $u_{2l+1}v_{2i}Pv_{2r+1}u_{2l}C^-u_{2l+1}$ is an M-alternating cycle longer than C, a contradiction. Similarly, u_{2l+1} cannot be adjacent to any vertex v_{2j+1}, $2s+1 \le 2j+1 \le 2q-1$. So $N_{G_1}(u_{2l+1}) \le 2q-(r+1)-(q-s) = q-(r+1-s) \le q-2$, contradicting $N_{G_1}(u_{2l+1}) \ge q$.

Case 1.1.2. $0 < t_2 < m$.

There exists an integer h, $0 \le h \le m-1$, such that $\{u_{2h}, u_{2h+1}\} \in \mathscr{C}_2$, while $\{u_{2h+2}, u_{2h+3}\} \in \mathscr{C}_i$, $i = 1$, 3 or 4. Then u_{2h+3} is adjacent to v_{2s} or v_{2r+1}. Without loss of generality assume that $u_{2h+3}v_{2s} \in E(G)$. Since $v_{2s}Pv_{2r+1}$ has length greater or equal to 3. The M-alternating cycle $u_{2h}v_{2r+1}Pv_{2s}u_{2h+3}C^+u_{2h}$ is longer than C, contradicting the maximality of C.

Case 1.2. $t_2 = t_3 = 0$.

If $t_1 \ne 0 \ne t_4$, then there exists an integer h, $0 \le h \le m-1$, such that $\{u_{2h}, u_{2h+1}\} \in \mathscr{C}_1$ and $\{u_{2h+2}, u_{2h+3}\} \in \mathscr{C}_4$. Similar to Case 1.1.2 we get an M-alternating cycle $u_{2h}v_{2s}Pv_{2r+1}u_{2h+3}C^+u_{2h}$ which is longer than C, a contradiction.

If t_1 or $t_4 = 0$, say $t_1 = 0$, then $t_4 = m$ and $N_C(v_{2s}) = 0$. By Lemma 4.1, $|C| \ge \nu/2+1$, hence $d(v_{2s}) \le \nu - 1 - |V(C)| \le \nu/2 - 2$, contradicting $d(v_{2s}) \ge \nu/2$.

Case 2. For any vertex set $\{v_{2i}, v_{2j+1}\}$, $0 \le i, j \le q-1$, there is a closed M-alternating path in G_1 connecting them.

Let $V_0 = \{v_{2i} : v_{2i} \in V(P)\}$ and $V_1 = \{v_{2i+1} : v_{2i+1} \in V(P)\}$. For any vertex set $\{u_{2l}, u_{2l+1}\}$, $0 \le l \le m-1$, suppose that there exist two integers $0 \le i, j \le q-1$, $u_{2l}v_{2i}, u_{2l+1}v_{2j+1} \in E(G)$. By the condition of Case 2 there is a closed M-alternating path P_1 in G_1 connecting v_{2i} and v_{2j+1}, thus we obtain an M-alternating cycle $u_{2l}v_{2i}P_1v_{2j+1}u_{2l+1}C^+u_{2l}$ which is longer than C, a contradiction. Therefore $u_{2l} \notin N_C(V_0)$ or $u_{2l+1} \notin N_C(V_1)$. Similarly $u_{2l} \notin N_C(V_1)$ or $u_{2l+1} \notin N_C(V_0)$. Hence

$$|N_C(V_0) \cap \{u_{2l}, u_{2l+1}\}| + |N_C(V_1) \cap \{u_{2l}, u_{2l+1}\}| \le 2 \tag{10}$$

and

$$|N_C(V_0)| + |N_C(V_1)| \le 2m. \tag{11}$$

We classify all sets $\{u_{2l}, u_{2l+1}\}$ for which the equality in (10) holds into four classes. Let

$$\{u_{2l}, u_{2l+1}\} \in \begin{cases} \mathscr{C}_1, & \text{if } u_{2l}, u_{2l+1} \text{ send edges to } V_0, \\ \mathscr{C}_2, & \text{if } u_{2l} \text{ sends edges to } V_0 \text{ and } V_1, \\ \mathscr{C}_3, & \text{if } u_{2l+1} \text{ sends edges to } V_0 \text{ and } V_1, \\ \mathscr{C}_4, & \text{if } u_{2l}, u_{2l+1} \text{ send edges to } V_1. \end{cases}$$

If $|N_C(V_0)| < m$, then $N_C(V_0) \cup V_1$ is a cut set of G with size less than $q+m = \nu/2$, contradicting $\kappa(G) \ge \nu/2$. So $|N_C(V_0)| \ge m$. Similarly $|N_C(V_1)| \ge m$. We then have $|N_C(V_0)| + |N_C(V_1)| \ge 2m$. By (11) the equality must hold and $|N_C(V_0)| = |N_C(V_1)| = m$. Meanwhile, for every vertex set $\{u_{2l}, u_{2l+1}\}$, $0 \le l \le m-1$, equality in (10) must hold, so $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_i$, $i = 1, 2, 3$ or 4.

Let $|\mathscr{C}_i| = t_i$, $1 \le i \le 4$. Then $t_1+t_2+t_3+t_4 = m$, $|N_0(V_0)| = t_1+t_2$, $|N_1(V_0)| = t_1+t_3$, $|N_0(V_1)| = t_2+t_4$, $|N_1(V_1)| = t_3+t_4$, $2t_1+t_2+t_3 = |N_C(V_0)| = m = |N_C(V_1)| = 2t_4+t_2+t_3$ and $t_1 = t_4$.

Claim 1. $e(N_0(V_0)^+, N_0(V_1)^+) = 0$ and $e(N_1(V_0)^-, N_1(V_1)^-) = 0$.

Suppose the claim does not hold and there exist integers r, s, g, h, $0 \le r, s \le m-1$, $0 \le g, h \le q-1$, such that $u_{2r}v_{2g} \in E(G)$, $u_{2s}v_{2h+1} \in E(G)$ and $u_{2r+1}u_{2s+1} \in E(G)$. By the condition of Case 2 there is a closed M-alternating path P_2 in G_1 connecting v_{2g} and v_{2h+1}. Then $u_{2r}v_{2g}P_2v_{2h+1}u_{2s}C^-u_{2r+1}u_{2s+1}C^+u_{2r}$ is an M-alternating cycle longer than C, contradicting the maximality of C. Thus $e(N_0(V_0)^+, N_0(V_1)^+) = 0$. Similarly $e(N_1(V_0)^-, N_1(V_1)^-) = 0$ and Claim 1 holds. □

Case 2.1. t_2 or $t_3 > 0$. Without loss of generality suppose $t_2 > 0$.

Case 2.1.1. $t_2 = m$.

The vertex set $\{u_{2i}, 0 \le i \le m-1\}$ is a cut set of G with size $m < \nu/2$, contradicting $\kappa(G) \ge \nu/2$.

Case 2.1.2. $0 < t_2 < m$.

There must exist an r, such that $\{u_{2r}, u_{2r+1}\} \in \mathscr{C}_2$, $\{u_{2r+2}, u_{2r+3}\} \in \mathscr{C}_i$, $i = 1$, 3 or 4. Hence u_{2r+3} sends some edges to V_0 or V_1. Without loss of generality, suppose u_{2r+3} sends some edges to V_1, say $u_{2r+3}v_{2g+1} \in E(G)$, $0 \le g \le q-1$. Let $0 \le h \le q-1$ be such that $u_{2r}v_{2h} \in E(G)$. By the condition of Case 2, there is a closed M-alternating path P_3 in G_1 connecting v_{2h} and v_{2g+1}.

Now let's estimate the sum of the degrees of u_{2r+1} and u_{2r+2}.

Since $\{u_{2r}, u_{2r+1}\} \in \mathscr{C}_2$, u_{2r+1} sends no edge to G_1, the number of vertices in which is $2q$. Since u_{2r+3} sends edges to V_1, $\{u_{2r+2}, u_{2r+3}\} \in \mathscr{C}_3$ or $\mathscr{C}_4$, so u_{2r+2} sends no edge to V_0, the number of vertices in which is q.

Note that $u_{2r+1} \in N_0(V_0)^+ \cap N_0(V_1)^+$, by Claim 1, u_{2r+1} cannot be adjacent to any other vertex in $N_0(V_0)^+ \cup N_0(V_1)^+$, the number of which is equal to $|N_0(V_0) \cup N_0(V_1)| - 1$, that is, $t_1+t_2+t_4-1$.

If u_{2r+3} sends no edge to V_0, then $u_{2r+2} \in N_1(V_1)^-$ and $u_{2r+2} \notin N_1(V_0)^-$. By Claim 1, u_{2r+2} cannot be adjacent to any vertex in $N_1(V_0)^-$, the number of which is t_1+t_3. If u_{2r+3} sends some edges to V_0, then $u_{2r+2} \in N_1(V_0)^- \cap N_1(V_1)^-$. Again by Claim 1, u_{2r+2} cannot be adjacent to any other vertices in $N_1(V_0)^- \cup N_1(V_1)^-$, the number of which is equal to $t_1+t_3+t_4-1$.

Suppose there exists an integer l, $0 \le l \le m-1$, $l \ne r, r+1$, such that $u_{2l}u_{2r+1}, u_{2l+1}u_{2r+2} \in E(G)$. Then

$$u_{2r}v_{2h}P_3v_{2g+1}u_{2r+3}C^+u_{2l}u_{2r+1}u_{2r+2}u_{2l+1}C^+u_{2r}$$

is an M-alternating cycle longer than C, a contradiction. Thus for any $0 \le i \le m-1$, $i \ne r, r+1$, $u_{2i}u_{2r+1} \notin E(G)$ or $u_{2i+1}u_{2r+2} \notin E(G)$.

Now we can calculate an upper bound for the sum of the degrees of u_{2r+1} and u_{2r+2}. If u_{2r+3} sends no edge to V_0, then

$$\begin{aligned} d(u_{2r+1}) + d(u_{2r+2}) &\le 2(\nu-1) - 2q - q - (t_1+t_2+t_4-1) - (t_1+t_3) - (m-2) \\ &= 2\nu - 3q - m - (t_1+t_2+t_3+t_4-1+t_1) \\ &= \nu + (2q+2m) - 3q - m - (m-1+t_1) \\ &= \nu - (q+t_1-1). \end{aligned}$$

If u_{2r+3} sends some edges to V_0, then

$$\begin{aligned} d(u_{2r+1}) + d(u_{2r+2}) &\le 2(\nu-1) - 2q - q - (t_1+t_2+t_4-1) - (t_1+t_3+t_4-1) - (m-2) \\ &= 2\nu - 3q - m - (t_1+t_2+t_3+t_4-2+t_1+t_4) \\ &= \nu + (2q+2m) - 3q - m - (m-2+t_1+t_4) \\ &= \nu - (q+t_1+t_4-2). \end{aligned}$$

Since $d(u_{2r+1}) + d(u_{2r+2}) \ge \nu$ we have $(q+t_1-1) \le 0$ or $(q+t_1+t_4-2) \le 0$. But since $q \ge 1$ and $t_1 = t_4 \ge 0$, in both cases we have $t_4 = t_1 = 0$. Therefore, for any $0 \le i \le m-1$, $\{u_{2i}, u_{2i+1}\} \in \mathscr{C}_2 \cup \mathscr{C}_3$, hence $|(N_C(V(G_1))) \cap \{u_{2i}, u_{2i+1}\}| = 1$. But then $|(N_C(V(G_1)))| = m \le \nu/2 - 1 < \nu/2$ and $N_C(V(G_1))$ is a cut set of G, contradicting $\kappa(G) \ge \nu/2$.

Case 2.2. $t_2 = t_3 = 0$. Then $t_4 = t_1 = m/2$. So m must be even.

Claim 2. For a segment $u_{2l}u_{2l+1}u_{2l+2}u_{2l+3}$ of C, if $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_1$ and $\{u_{2l+2}, u_{2l+3}\} \in \mathscr{C}_4$ ($\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_4$ and $\{u_{2l+2}, u_{2l+3}\} \in \mathscr{C}_1$), then the following statements hold.

(a) $|N_{G_1}(u_{2l})| = 1$ and $|N_{G_1}(u_{2l+3})| = 1$. The neighbors of u_{2l} and u_{2l+3} in G_1 are the endvertices of an edge in M.

(b) u_{2l+1} is adjacent to all vertices in V_0 (V_1) and u_{2l+2} is adjacent to all vertices in V_1 (V_0).

(c) u_{2l+1} is adjacent to all other vertices in $N_0(V_0)^+$ ($N_0(V_1)^+$) and u_{2l+2} is adjacent to all other vertices in $N_1(V_1)^-$ ($N_1(V_0)^-$).

We only prove the situation that $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_1$ and $\{u_{2l+2}, u_{2l+3}\} \in \mathscr{C}_4$, for the other situation the results follow similarly. Let $v_{2g} \in N_{G_1}(u_{2l})$ and $v_{2h+1} \in N_{G_1}(u_{2l+3})$, $0 \le g, h \le q-1$. By the condition of Case 2 there is a closed M-alternating path P_4 in G_1 connecting v_{2g} and v_{2h+1}. If $|P_4| > 1$, then the M-alternating cycle $u_{2l}v_{2g}P_4v_{2h+1}u_{2l+3}C^+u_{2l}$ is longer than C, a contradiction. So P_4 consists of exactly one edge in M and $g = h$. Since v_{2g} and v_{2h+1} is randomly chosen we have $|N_{G_1}(u_{2l})| = 1$ and $|N_{G_1}(u_{2l+3})| = 1$, thus (a) is proved.

Similar to Case 2.1.2 we count the sum of the degrees of u_{2l+1} and u_{2l+2}. Since $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_1$, u_{2l+1} cannot send any edge to V_1, so $|N_{G_1}(u_{2l+1})| \le q$. Similarly $|N_{G_1}(u_{2l+2})| \le q$. By Claim 1, u_{2l+1} cannot be adjacent to any vertex in $N_0(V_1)^+$, the number of which is $t_2 + t_4 = m/2$, and u_{2l+2} cannot be adjacent to any vertex in $N_1(V_0)^-$, the number of which is $t_1 + t_3 = m/2$. For any $\{u_{2i}, u_{2i+1}\}$ where $0 \le i \le m-1$, $i \ne l, l+1$, if $u_{2l+1}u_{2i} \in E(G)$ and $u_{2l+2}u_{2i+1} \in E(G)$, then the M-alternating cycle $u_{2l}v_{2g}v_{2g+1}u_{2l+3}C^+u_{2i}u_{2l+1}u_{2l+2}u_{2i+1}C^+u_{2l}$ is longer than C, a contradiction. Thus for any $\{u_{2i}, u_{2i+1}\}$, $0 \le i \le m-1$, $i \ne l, l+1$, $u_{2l+1}u_{2i} \notin E(G)$ or $u_{2l+2}u_{2i+1} \notin E(G)$. Therefore

$$d(u_{2l+1}) + d(u_{2l+2}) \le 2q + 2(2m-1) - (m/2 + m/2) - (m-2) = 2q + 2m = \nu.$$

But $d(u_{2l+1}) + d(u_{2l+2}) \ge \nu/2 + \nu/2 = \nu$, thus all equalities must hold. Hence $|N_{G_1}(u_{2l+1})| = q$ and $|N_{G_1}(u_{2l+2})| = q$ and (b) holds. Meanwhile, except those we excluded above, u_{2l+1} must be adjacent to all other vertices. Therefore u_{2l+1} must be adjacent to all other vertices in $N_0(V_0)^+$. Similarly u_{2l+2} must be adjacent to all other vertices in $N_1(V_1)^+$ and (c) holds. The proof of Claim 2 is complete. $\square$

Case 2.2.1. There exists an integer r, $0 \le r \le m-1$, such that $\{u_{2r}, u_{2r+1}\}, \{u_{2r+2}, u_{2r+3}\} \in \mathscr{C}_1$.

We can choose r so that $\{u_{2r}, u_{2r+1}\}, \{u_{2r+2}, u_{2r+3}\} \in \mathscr{C}_1$ and $\{u_{2r+4}, u_{2r+5}\} \in \mathscr{C}_4$. By Claim 2 (c) and (a), $u_{2r+1}u_{2r+3} \in E(G)$ and $|N_{G_1}(u_{2r+2})| = 1$. Let v_{2g}, v_{2h_1+1} and v_{2h_2+1} be the neighbors of u_{2r}, u_{2r+4} and u_{2r+5} in G_1. By the condition of Case 2, there is a closed M-alternating path P_5 in G_1 connecting v_{2g} and v_{2h_1+1}, and a closed M-alternating path P_6 in G_1 connecting v_{2g} and v_{2h_2+1}.

If $u_{2r+2}u_{2r+5} \in E(G)$, then the M-alternating cycle

$$u_{2r+2}u_{2r+1}u_{2r+3}u_{2r+4}v_{2h_1+1}P_5v_{2g}u_{2r}C^-u_{2r+5}u_{2r+2}$$

is longer than C, a contradiction. So $u_{2r+2}u_{2r+5} \notin E(G)$. By Claim 1, we have $u_{2r+2}u_{2r+4} \notin E(G)$.

If there exists an integer l, $0 \le l \le m-1$, $l \ne r+2$, such that $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_4$ and $u_{2r+2}u_{2l+1} \in E(G)$. By Claim 2, $u_{2r+4}u_{2l} \in E(G)$. Then the M-alternating cycle

$$u_{2r+2}u_{2r+1}u_{2r+3}u_{2r+4}u_{2l}C^-u_{2r+5}v_{2h_2+1}P_6v_{2g}u_{2r}C^-u_{2l+1}u_{2r+2}$$

is longer than C, a contradiction. Thus for all $\{u_{2l}, u_{2l+1}\} \in \mathscr{C}_4$, $u_{2r+2}u_{2l+1} \notin E(G)$. But since $u_{2l} \in N_1(V_1)^-$ and $u_{2r+2} \in N_1(V_0)^-$, by Claim 1, we also have $u_{2r+2}u_{2l} \notin E(G)$. Therefore u_{2r+2} has at most $2m-1-m = m-1$ neighbors on C. Thus $d(u_{2r+2}) \le m-1+1 = m < \nu/2$, contradicting $d(u_{2r+2}) \ge \kappa \ge \nu/2$.

Case 2.2.2. There does not exist any integer i, $0 \le i \le m-1$, such that

$$\{u_{2i}, u_{2i+1}\}, \{u_{2i+2}, u_{2i+3}\} \in \mathscr{C}_1.$$

Since $t_1 = t_4 = m/2$, there can neither be any j, $0 \le j \le m-1$, such that

$$\{u_{2j}, u_{2j+1}\}, \{u_{2j+2}, u_{2j+3}\} \in \mathscr{C}_4.$$

Thus the sets $\{u_{2i}, u_{2i+1}\}$, $0 \le i \le m-1$ belong to $\mathscr{C}_1$ and $\mathscr{C}_4$ alternatively. Without loss of generality suppose $\{u_0, u_1\} \in \mathscr{C}_1$, then $\{u_{4i}, u_{4i+1}\} \in \mathscr{C}_1$ and $\{u_{4i+2}, u_{4i+3}\} \in \mathscr{C}_4$, for $0 \le i \le m/2-1$. Consider the segment $u_{4i}u_{4i+1}u_{4i+2}u_{4i+3}$. By Claim 2 (b), u_{4i+2} is adjacent to all vertices in V_1. Consider the segment $u_{4i+2}u_{4i+3}u_{4i+4}u_{4i+5}$. By Claim 2 (a), u_{4i+2} can have only one neighbor in G_1. Thus we have $|G_1| = 2$. G_1 consists of the edge $v_0v_1 \in M$ only. $N_C(v_0) = \{u_{4i}, u_{4i+1} : 0 \le i \le m/2-1\}$ and $N_C(v_1) = \{u_{4i+2}, u_{4i+3} : 0 \le i \le m/2-1\}$.

For any segment $u_{4i}u_{4i+1}u_{4i+2}u_{4i+3}$ of C, we obtain another longest M-alternating cycle

$$C' = u_{4i}v_0v_1u_{4i+3}C^+u_{4i}.$$

Let $G'_1 = G - C'$, which consists of the edge $u_{4i+1}u_{4i+2}$ only. Note that when we get here, we have dismissed all other cases. Therefore, C' and G'_1 must have structures similar to C and G_1, as we have stated in this case. Hence the vertices in the sets $\{u_{4i}, v_0\}$, $\{v_1, u_{4i+3}\}$ and $\{u_{2j}, u_{2j+1}\}$, $0 \le j \le m-1, j \ne 2i, 2i+1$, are adjacent to u_{4i+1} and u_{4i+2} alternatively, according to their orders on C'. Thus we have $N(u_{4i+1}) = \{u_{4i+2}, u_{4i}, v_0\} \cup \{u_{4j}, u_{4j+1} : 0 \le j \le m/2-1, j \ne i\}$ and $N(u_{4i+2}) = \{u_{4i+1}, u_{4i+3}, v_1\} \cup \{u_{4j+2}, u_{4j+3} : 0 \le j \le m/2-1, j \ne i\}$. Analogous discussion on any segment $u_{4i-2}u_{4i-1}u_{4i}u_{4i+1}$ and $u_{4i+2}u_{4i+3}u_{4i+4}u_{4i+5}$ leads to the conclusion that $N(u_{4i}) = \{u_{4i-1}, u_{4i+1}, v_0\} \cup \{u_{4j}, u_{4j+1} : 0 \le j \le m/2-1, j \ne i\}$ and $N(u_{4i+3}) = \{u_{4i+4}, u_{4i+2}, v_1\} \cup \{u_{4j+2}, u_{4j+3} : 0 \le j \le m/2-1, j \ne i\}$.

By the arbitrariness of i, we conclude that all vertices u_{4i} and u_{4i+1}, $0 \le i \le m/2-1$, are adjacent to each other. They, together with v_0, form a complete graph K_{m+1}. Similarly, vertices u_{4i+2} and u_{4i+3}, $0 \le i \le m/2-1$, with v_1, form a complete graph K_{m+1}. These two complete graphs, together with the edges in M, constitute G. Since m is even, let $m = 2n$ then $|G| = 4n+2$. Therefore $G \in \mathcal{G}_1$ and M is exactly the jointing matching. □

Corollary 4.3. *Let G be a k-extendable graph with $k \ge \nu/4$, and M a perfect matching of G. Then G has an M-alternating Hamilton cycle.*

Proof. By Theorem 1.2, either G is bipartite or $\kappa \ge 2k$. If G is bipartite, then by Theorem 1.1, $\delta \ge \kappa \ge k+1 \ge \nu/4+1$. Hence, for any two vertices x and y in different parts of G, $d(x)+d(y) \ge \nu/2+2$. By Theorem 2.1, G has an M-alternating Hamilton cycle. If $\kappa \ge 2k \ge \nu/2$, then by Theorem 4.2, G has an M-alternating Hamilton cycle or $G \in \mathcal{G}_1$. If $G \in \mathcal{G}_1$, then $|G| = 4n+2$, $n \ge 1$, so $k \ge n+1$. Thus $\kappa \ge 2k \ge 2n+2$. But G is regular with degree $2n+1$, a contradiction. So G has an M-alternating Hamilton cycle. □

5. Final remark

Theorem 4.2 is a special case of the following conjecture.

Conjecture 5.1 (*Lovász–Woodall*). *Let L be a set of k independent edges in a k-connected graph G, if k is even or G–L is connected, then G has a cycle containing all the edges of L.*

Professor Kawarabayashi has published [5], which is the first step towards a solution for the conjecture. He is still working towards a whole proof of the conjecture as we finish the current paper.

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References

[1] R.E.L. Aldred, D.A. Holton, D. Lou, A. Saito, M-alternating paths in n-extendable bipartite graphs, Discrete Math. 269 (2003) 1–11.
[2] R.E.L. Aldred, D.A. Holton, D. Lou, N. Zhong, Characterizing $2k$-critical graphs and n-extendable graphs, Discrete Math. 287 (2004) 135–139.
[3] C. Berge, Two theorems in graph theory, Proc. Natl. Acad. Sci. USA 43 (1957) 842–844.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan Press, London, 1976.
[5] K. Kawarabayashi, One or two disjoint cycles cover independent edges: Lovász–Woodall Conjecture, J. Combin. Theory Ser. B 84 (2002) 1–44.
[6] D. Lou, Q. Yu, Connectivity of k-extendable graphs with large k, Discrete Appl. Math. 136 (2004) 55–61.
[7] M.D. Plummer, On n-extendable graphs, Discrete Math. 31 (1980) 201–210.