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Maximizing the number of spanning trees in K_n-complements of asteroidal graphs

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ABSTRACT

In this paper we introduce the class of graphs whose complements are asteroidal (star-like) graphs and derive closed formulas for the number of spanning trees of its members. The proposed results extend previous results for the classes of the multi-star and multi-complete/star graphs. Additionally, we prove maximization theorems that enable us to characterize the graphs whose complements are asteroidal graphs and possess a maximum number of spanning trees.

1. Introduction

The number of spanning trees of a graph G is an important, well-studied quantity in graph theory, and appears in a number of applications. Its most notable application is in the field of network reliability: in a network modeled by a graph, intercommunication between all nodes of the network implies that the graph must contain a spanning tree; thus, maximizing the number of spanning trees is a way of maximizing reliability [2,15,12,20]. Other application fields arise from enumerating certain chemical isomers [3], and counting the number of Eulerian circuits in a graph [10,11].

Thus, both for theoretical and for practical purposes, we are interested in deriving formulas for the number of spanning trees of a graph G, and also of the K_n-*complement* of G. For any subgraph H of the complete graph K_n, the K_n-*complement* of H, denoted by $K_n - H$, is defined as the graph obtained from K_n by removing the edges of H; note that if H has n vertices then $K_n - H$ coincides with the complement $\overline{H}$ of H. Many different types of graphs $K_n - H$ have been examined: for example, there exist closed formulas for the cases where H is a pairwise-disjoint set of edges [22], a chain of edges [13], a cycle [7], a star [19], a multi-star [18,23], a multi-complete/star graph [4], a labeled molecular graph [3], and more recently if H is a circulant graph [8,12,24], a quasi-threshold graph [17], and so on (see Berge [1] for an exposition of the main results).

A common approach for determining the number of spanning trees of a graph G relies on a classic result known as the *complement-spanning-tree matrix* theorem [21], which expresses the number of spanning trees of G as a function of the determinant of a matrix that can be easily constructed from the adjacency relation of G, i.e., adjacency matrix, adjacency lists, etc. Calculating the determinant of the complement-spanning-tree matrix seems to be a promising approach for computing the number of spanning trees of families of graphs of the form $K_n - H$, where H is a graph that exhibits symmetry (see [1,4,7,18,17,16,23,24]).

In this paper, we define two classes of graphs, namely, the *complete-planet* and the *star-planet* graphs, which generalize well-known classes of graphs; we call these two classes of graphs *asteroidal* graphs. It turns out that computing the number of

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spanning trees of these graphs is not difficult. However, computing the number of spanning trees of their K_n-complements is more interesting; we derive closed formulas for the number of spanning trees of (i) the K_n-complement of a complete-planet graph, and (ii) the K_n-complement of a star-planet graph. Our proofs are based on the complement-spanning-tree matrix theorem and use standard techniques from linear algebra and matrix theory. Our formulas generalize previous proposed formulas of classes of graphs such as complete graphs, star graphs, wheel graphs, gem graphs, multi-star graphs, multi-complete/star graphs, etc.

Although the problem of maximizing the number of spanning trees of a graph is difficult in general, it is possible to achieve an efficient solution for some non-trivial classes of graphs [4,9,18]. In this paper we also prove maximization results for the K_n-complements of asteroidal graphs. In particular, we characterize the graphs whose complements are asteroidal graphs and possess a maximum number of spanning trees. Our maximization results generalize and extend previous maximization results for the class of multi-star graphs [4].

2. Preliminaries

We consider finite undirected graphs with no loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. Let S be a subset of the vertex set of a graph G. Then, the subgraph of G *induced by* S is the graph $G[S] = (S, E')$, where $(u, v) \in E'$ if and only if $u, v \in S$ and $(u, v) \in E(G)$. Moreover, we denote by $G - S$ the subgraph $G[V(G) - S]$.

The *neighborhood* $N(x)$ of a vertex x is the set of all the vertices of G which are adjacent to x. The *closed neighborhood* of vertex x is defined as $N[x] = \{x\} \cup N(x)$. The *degree* of a vertex x in the graph G, denoted $d(x)$, is the number of edges incident on x; thus, $d(x) = |N(x)|$. If two vertices x and y are adjacent in G, we say that x *sees* y; otherwise we say that x *misses* y. We extend this notion to vertex sets: $V_i \subseteq V(G)$ sees (misses) $V_j \subseteq V(G)$ if and only if every vertex $x \in V_i$ sees (misses) every vertex $y \in V_j$.

By K_n we denote the complete graph on n vertices. Moreover, for symmetry, we denote by S_{n+1} a tree on $n+1$ vertices with one vertex having degree n and call it a *star graph* (it is commonly denoted by $S_{1,n}$); we call the vertex of S_{n+1} with degree n its *center vertex*. The chordless path (resp. cycle) on n vertices $v_1 v_2 \cdots v_n$ with edges $v_i v_{i+1}$ (resp. $v_i v_{i+1}$ and $v_1 v_n$), $1 \le i < n$, is denoted by P_n (resp. C_n).

Let G_1 and G_2 be two graphs. Their *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Their *join* is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ plus all edges joining $V(G_1)$ with $V(G_2)$. For any connected graph G, we write mG for the graph with m components, each isomorphic with G, $m \ge 2$. Thus, the graph mK_n (resp. mS_n, mP_n, mC_n) consists of m disjoint copies of K_n (resp. S_n, P_n, C_n). Note that, $S_{n+1} = K_1 + nK_1$. Throughout the paper, we refer to complete graphs, star graphs, path graphs, and cycle graphs as cliques, stars, paths, and cycles, respectively.

2.1. Asteroidal graphs

A graph G on n vertices is called a *complete-planet* (resp. *star-planet*) if its vertex set $V(G)$ admits a vertex-disjoint partition into sets A and B such that:

(S1) $A = \{v_1, v_2, \ldots, v_m\}$ and $G[A] = K_m$ (resp. $A = \{v_1, v_2, \ldots, v_m, c\}$ and $G[A] = S_{m+1}$), $m \ge 1$;

(S2) $B = B_1 \cup B_2 \cup \cdots \cup B_m$, and for each $i = 1, 2, \ldots, m$, $|B_i| \ge 0$ and B_i induces α_{ij}, β_{ij}, γ_{ij}, and δ_{ij} disjoint copies of cliques, stars, paths, and cycles, respectively, on j vertices; that is,

$$G_i \equiv G[B_i] = \bigcup_j \alpha_{ij} K_j \cup \beta_{ij} S_j \cup \gamma_{ij} P_j \cup \delta_{ij} C_j, \quad 1 \le i \le m;$$

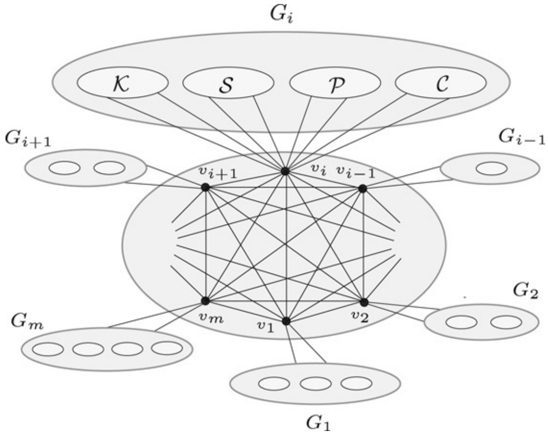
(S3) The vertex $v_i \in A$ sees all the vertices of G_i and misses all the vertices in $B - V(G_i)$, $1 \le i \le m$.

We collectively call the above defined graphs *asteroidal* graphs; Fig. 1 shows the general form of a complete-planet graph and a star-planet graph. Let G be an asteroidal graph and let A and B be the partition sets of $V(G)$ according to the above definition: we call the graph $G[A]$ the *sun-graph* of G, the graph $G[B]$ the *planet-graph*, and the graphs $G_1, G_2, \ldots, G_m$ the *planet-subgraphs*; by definition, $G[B] = \bigcup_{i=1}^{m} G_i$. A maximal connected subgraph of a planet-subgraph G_i is called *planet-component* of G_i. Let us denote by ℓ_i the cardinality of B_i. Then, the number of vertices of G_i is ℓ_i. Moreover, the definition of G_i implies that for each $i = 1, 2, \ldots, m$, it holds that

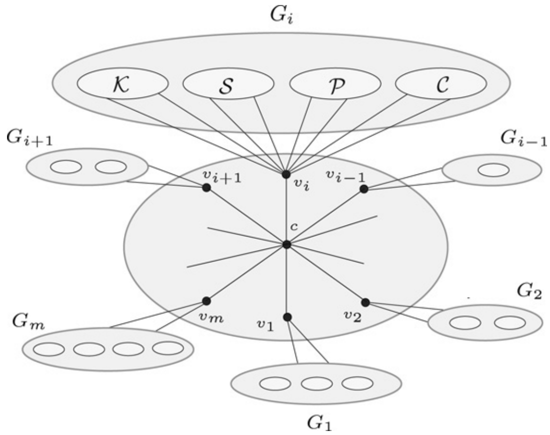
$$\sum_j \left(\alpha_{ij} + \beta_{ij} + \gamma_{ij} + \delta_{ij}\right) \cdot j = \ell_i. \tag{1}$$

Clearly, $|B| = \sum_{i=1}^{m} \ell_i$; we denote this sum by ℓ. Therefore, a complete-planet graph has exactly $m + \ell$ vertices, whereas a star-planet graph has $m + \ell + 1$ vertices. Hereafter, we call the vertices of the graphs $G[A]$ and $G[B]$, *sun-vertices* and *planet-vertices* respectively.

Throughout the paper, we use the following convention: any isolated vertex of a planet-subgraph is considered to be a K_1 (and not an S_1, P_1, or C_1); hence, for all i, $\beta_{i1} = \gamma_{i1} = \delta_{i1} = 0$. Similarly, $\beta_{i2} = \gamma_{i2} = \delta_{i2} = 0$, since $K_2 = S_2 = P_2 = C_2$.



(a) A complete-planet graph.



(b) A star-planet graph.

Fig. 1. Asteroidal graphs.

Table 1
Subclasses of complete-planet graphs

Parameters $m, \ell, \alpha, \beta, \gamma, \delta$	Complete-planet graph G_c	Reference
$m = k + 1, \ell = 0$ or $m = 1, \alpha(k) = 1, \alpha(i) = 0 \quad \forall i \neq k, \beta = \gamma = \delta = 0$	K_{k+1}	[1,19]
$m = 1, \beta(k) = 1, \beta(i) = 0 \quad \forall i \neq k, \alpha = \gamma = \delta = 0$	$K_2 + kK_1$	[1]
$m = 1, \gamma(k) = 1, \gamma(i) = 0 \quad \forall i \neq k, \alpha = \beta = \delta = 0$	$K_1 + P_k$ (gem, for $k = 4$)	[1]
$m = 1, \delta(k) = 1, \delta(i) = 0 \quad \forall i \neq k, \alpha = \beta = \gamma = 0$	Wheel graph W_{k+1}	[1]
$m \geq 1, \alpha(1) \neq 0, \alpha(i) = 0 \quad \forall i \neq 1, \beta = \gamma = \delta = 0$	Multi-star graph	[18,23]
$m \geq 1$, appropriate $\alpha \neq 0, \beta = \gamma = \delta = 0$	Multi-complete/star graph	[4]

Table 2
Subclasses of star-planet graphs

Parameters $m, \ell, \alpha, \beta, \gamma, \delta$	Star-planet graph G_s	Reference
$m \geq 1, \ell = 0$	$S_{1,m}$	[1,19]
$m \geq 1, \alpha(1) \neq 0, \alpha(i) = 0 \quad \forall i \neq 1, \beta = \gamma = \delta = 0$	Trees with diameter $d \leq 4$	[17]

Finally, $\gamma_{i3} = \delta_{i3} = 0$, since $K_3 = C_3$ and $S_3 = P_3$. Therefore, for a planet-component which is a star S_n, then $n \geq 3$, whereas if it is a path P_n or a cycle C_n then $n \geq 4$.

Let G be an asteroidal graph on m sun-vertices and ℓ planet-vertices. We denote by $\alpha(j)$ the number of the maximal cliques K_j of the planet-graph $G[B]$. Similarly, we denote by $\beta(j)$, $\gamma(j)$, and $\delta(j)$ the number of the maximal stars S_j, paths P_j, and cycles C_j, respectively, of $G[B]$. Thus, we have:

$$\alpha(j) = \sum_{i=1}^{m} \alpha_{ij}, \quad \beta(j) = \sum_{i=1}^{m} \beta_{ij}, \quad \gamma(j) = \sum_{i=1}^{m} \gamma_{ij}, \quad \text{and} \quad \delta(j) = \sum_{i=1}^{m} \delta_{ij}.$$

We define the vector $\alpha = [\alpha(1), \alpha(2), \ldots, \alpha(\ell)]$ on the planet-graph $G[B]$, and we call it the *clique vector* of the asteroidal graph G; in a similar manner, we define the vectors β, γ, and δ and we call them *star vector*, *path vector*, and *cycle vector*, respectively. Clearly, the vectors α, β, γ, and δ of an asteroidal graph G determine the number of the maximal cliques, stars, paths, and cycles in $G[B]$. Hereafter, we write $\alpha \neq 0$ to denote that there exists at least one j, $1 \leq j \leq \ell$, such that $\alpha(j) \neq 0$, i.e., $\alpha = 0$ is equivalent to $\alpha(1) = \alpha(2) = \cdots = \alpha(\ell) = 0$. We use a similar notation for the star vector, path vector, and cycle vector. For example, if $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$ and $\delta \neq 0$, then the planet-graph $G[B]$ contains only stars, paths, and cycles.

Many graphs can be derived as special cases from the asteroidal graphs, depending on the sun-graph and the values of the clique, star, path, and cycle vectors. For example, given a complete-planet graph with K_m and vectors α, β, γ, δ, and setting $m = 1$, $\gamma(5) = 1$, $\gamma(j) = 0$ for all $j \neq 5$, and $\alpha = \beta = \delta = 0$, we get the graph $K_1 + P_5$, and when setting $m = 1$, $\delta(k) = 1$, $\delta(j) = 0$ for all $j \neq k$, and $\alpha = \beta = \gamma = 0$, we get the *wheel graph* W_{k+1}, i.e., the graph obtained from a chordless cycle on k vertices by adding a vertex that sees every vertex of the cycle (see Fig. 2). A listing of such results is presented in Tables 1 and 2.

Computing the number of spanning trees of asteroidal graphs is not very interesting because it is fairly easy. Consider a complete-planet or a star-planet graph G; since the vertices V_{ij} of each clique, star, path, or cycle of a planet-subgraph G_i see vertex v_i and miss all vertices in $V(G) - (V_{ij} \cup \{v_i\})$, any spanning tree of G consists of a spanning tree of the sun-graph and

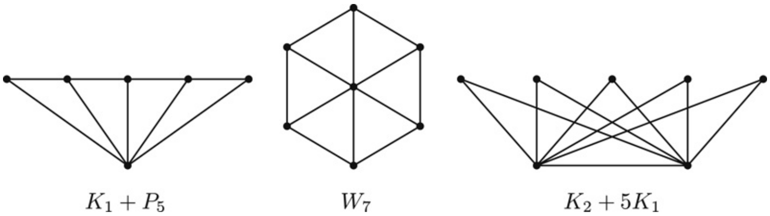


Fig. 2. Some simple asteroidal graphs.

a spanning tree of $G[V_{ij} \cup \{v_i\}]$ for each clique, star, path, or cycle of each G_i. Let $k_{1,j}$, $s_{1,j}$, $p_{1,j}$, and $c_{1,j}$ denote the numbers of spanning trees of $K_1 + K_j = K_{j+1}$, $K_1 + S_j$, $K_1 + P_j$, and $K_1 + C_j = W_{j+1}$ (i.e, the wheel graph on $j+1$ vertices), respectively; from K_{j+1} and W_{j+1} we have that $k_{1,j} = (j+1)^{j-1}$ and $c_{1,j} = Luc(2j) - 2$ [14], where $Luc(2j)$ denotes the $(2j)$th Lucas number,[1] while from combinatorial arguments we obtain $s_{1,j} = (j+2)\,2^{j-1}$ and $p_{1,j} = Fib(2j) \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2j}$ where $Fib(2j)$ denotes the $(2j)$th Fibonacci number.[2] Then, the numbers $\tau(G_c)$ and $\tau(G_s)$ of spanning trees of a complete-planet graph G_c and a star-planet graph G_s on m sun-vertices are equal to

$$\tau(G_c) = m^{m-2}\,\tau(G_s) \quad \text{and} \quad \tau(G_s) = \prod_j k_{1,j}^{\alpha(j)} \cdot s_{1,j}^{\beta(j)} \cdot p_{1,j}^{\gamma(j)} \cdot c_{1,j}^{\delta(j)};$$

note that a complete graph on m vertices has m^{m-2} spanning trees whereas a star graph has a single spanning tree.

In contrast, computing the number of spanning trees of K_n-complements of asteroidal graphs is not so easy. In order to facilitate the derivation of closed formulas for this number, we define the following ordering of the vertices of the graph G: for each planet-subgraph G_i of G in order, we place first the vertices that belong to the maximal cliques of G_i starting from the vertices of the smallest clique; the vertices that belong to each maximal star of G_i are placed next with the star's central vertex last; the vertices of the paths follow in the order they are met along the path, and after them, the vertices of the cycles in the order they are met around the cycle; in the end, we have the vertices that belong to the sun-graph of G in arbitrary order.

2.2. Complement-spanning-tree matrix

Let G be a graph on n vertices $v_1, v_2, \ldots, v_n$. The *complement-spanning-tree matrix* of the graph G is an $n \times n$ matrix A defined as follows:

$$A_{i,j} = \begin{cases} 1 - \dfrac{\overline{d}(v_i)}{n} & \text{if } i = j, \\ \dfrac{1}{n} & \text{if } i \neq j \text{ and } v_iv_j \text{ is an edge of } \overline{G}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{d}(v_i)$ is the degree of the vertex v_i in $\overline{G}$. It has been shown [21] (also known as the *complement-spanning-tree matrix theorem*) that the number of spanning trees $\tau(G)$ of G is given by

$$\tau(G) = n^{n-2} \cdot \det(A). \tag{2}$$

For $G = K_n$, we have that $A = I_n \implies \det(A) = 1$, and Eq. (2) implies *Cayley's tree formula* [10] which states that $\tau(K_n) = n^{n-2}$. Let us apply Eq. (2) on $G = K_n - H$ where $|V(H)| = p < n$; then, the complement-spanning-tree matrix A of G has the following form (empty entries in the matrix represent 0s):

$$A = \begin{bmatrix} I_{n-p} & \\ & M' \end{bmatrix},$$

where the submatrix M' is a $p \times p$ matrix which corresponds to the vertices in H. Note that the submatrix I_{n-p} corresponds to the $n-p$ remaining vertices which have degree $n-1$ in G, and, thus, they have degree 0 in $\overline{G}$. From the form of the matrix A, we see that $\det(A) = \det(M')$. Thus, we focus on the computation of the determinant of matrix M'.

The *degree matrix* of a graph H on p vertices is a $p \times p$ matrix D defined as follows: $D_{i,i} = d(v_i)$ and $D_{i,j} = 0$ for $i \neq j$, $1 \leq i, j \leq p$. Given the adjacency matrix B of H and the degree matrix D of H, we have $M' = I_p + \frac{1}{n}B - \frac{1}{n}D$. If we multiply each column (or row) of matrix M' by n, we get the $p \times p$ matrix M such that:

$$M = nI_p + B - D;$$

clearly, $\det(M') = n^{-p}\det(M)$. Concluding, we have the following result:

[1] The Lucas numbers satisfy the recurrence $Luc(n) = Luc(n-1) + Luc(n-2)$ with $Luc(1) = 1$ and $Luc(2) = 3$.

[2] The Fibonacci numbers satisfy the recurrence $Fib(n) = Fib(n-1) + Fib(n-2)$ with $Fib(1) = 1$ and $Fib(2) = 1$.

Corollary 2.1. *Let $G = K_n - H$ be a graph where $|V(H)| = p$, and let M be the $p \times p$ matrix of H as defined above. Then,*

$$\tau(G) = n^{n-p-2} \cdot \det(M).$$

Throughout the paper, empty entries in matrices represent 0s. Moreover we denote by $\mathbf{1}_p$ the vector of size p whose entries are all equal to 1.

3. The number of spanning trees

Before proving closed formulas for the number of spanning trees of graph $K_n - G$, where G is an asteroidal graph, let us consider the $j \times j$ matrices M_j^K, M_j^P, M_j^C, and M_j^S, which correspond to a complete graph K_j, a star S_j, a path P_j, and a cycle C_j on j vertices, respectively; that is,

$$M_j^K = \begin{bmatrix} n-j & 1 & \cdots & 1 & 1 \\ 1 & n-j & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & n-j & 1 \\ 1 & 1 & \cdots & 1 & n-j \end{bmatrix}, \quad M_j^S = \begin{bmatrix} n-2 & & & & & 1 \\ & n-2 & & & & 1 \\ & & \ddots & & & \vdots \\ & & & & n-2 & 1 \\ 1 & 1 & \cdots & 1 & 1 & n-j \end{bmatrix},$$

$$M_j^P = \begin{bmatrix} n-2 & 1 & & & \\ 1 & n-3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & n-3 & 1 \\ & & & 1 & n-2 \end{bmatrix}, \quad \text{and} \quad M_j^C = \begin{bmatrix} n-3 & 1 & & & 1 \\ 1 & n-3 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & n-3 & 1 \\ 1 & & & 1 & n-3 \end{bmatrix}.$$

Notice here that by the definition of the complement-spanning-tree matrix, the sum of the entries in each row and each column, in each of the three matrices, is $n - 1$. It is easy to derive a formula for the determinants of matrices M_j^K and M_j^S by subtracting the first row from all the other rows and by adding all the columns to the first column. We obtain:

$$\lambda(K_j) \equiv \det(M_j^K) = (n-1) \cdot (n-j-1)^{j-1}$$

$$\text{and} \quad \lambda(S_j) \equiv \det(M_j^S) = (n-2)^{j-1} \cdot \left(n - j - \frac{j-1}{n-2}\right).$$

For the matrices M_j^P and M_j^C, we define a recurrence which is solved using standard techniques (similar results can be found in [7]); for $n \geq 5$, we have:

$$\lambda(P_j) \equiv \det(M_j^P) = \frac{n-1}{r \cdot 2^j} \cdot \left((n-3+r)^j - (n-3-r)^j\right) = \frac{n-1}{2^{j-1}} \cdot \sum_{t=0}^{j-1} \left(r^t \cdot (n-3)^{j-t-1}\right)$$

$$\text{and} \quad \lambda(C_j) \equiv \det(M_j^C) = \frac{1}{2^j} \cdot \left((n-3+r)^j + (n-3-r)^j + (-2)^{j+1}\right)$$

where

$$r = \sqrt{(n-1) \cdot (n-5)}.$$

It is not difficult to see that the quantities $\lambda(K_j)$, $\lambda(S_j)$, $\lambda(P_j)$, and $\lambda(C_j)$ are all non-negative.

3.1. Complete-planet graphs

Let K_n be the complete graph on n vertices and G_c be a complete-planet graph on m sun-vertices $\{v_1, v_2, \ldots, v_m\}$ and ℓ planet-vertices such that $V(G_c) \subseteq V(K_n)$. We use Corollary 2.1 in order to derive a closed formula for the number of spanning trees of the graph $G = K_n - G_c$.

For each graph $G_i + v_i$, $1 \leq i \leq m$, where G_i is the ith planet-subgraph of G_c, we construct a matrix U_i which, based on our ordering scheme (see end of Section 2.1) has the following form:

$$U_i = \begin{bmatrix} M^K & & & & \mathbf{1}_{\alpha_i} \\ & M^S & & & \mathbf{1}_{\beta_i} \\ & & M^P & & \mathbf{1}_{\gamma_i} \\ & & & M^C & \mathbf{1}_{\delta_i} \\ \mathbf{1}_{\alpha_i}^{\mathrm{T}} & \mathbf{1}_{\beta_i}^{\mathrm{T}} & \mathbf{1}_{\gamma_i}^{\mathrm{T}} & \mathbf{1}_{\delta_i}^{\mathrm{T}} & n - d(v_i) \end{bmatrix},$$

where the submatrices M^K, M^S, M^P, and M^C correspond to the cliques, stars, paths, and cycles of G_i respectively, and $\alpha_i = \sum_j j\alpha_{ij}, \beta_i = \sum_j j\beta_{ij}, \gamma_i = \sum_j j\gamma_{ij}, \delta_i = \sum_j j\delta_{ij}$. The matrix M^K contains α_{i1} copies of matrix M_1^K, α_{i2} copies of matrix M_2^K, and so on; these copies are placed on the diagonals of M^K and thus M^K is a block diagonal matrix. More precisely, matrix M^K has exactly α_i rows and columns, each corresponding to a vertex of one of the α_{ij} complete graphs K_j of the graph G_i. The case is similar for the matrices M^S, M^P, and M^C. From its form shown above and Eq. (1), we conclude that the matrix U_i is of size $(\ell_i + 1) \times (\ell_i + 1)$.

In order to compute the determinant of matrix U_i, we add one more row and one more column at the top and left of the matrix U_i; the resulting $(\ell_i + 2) \times (\ell_i + 2)$ matrix U_i' has its $(1, 1)$-entry and $(1, \ell_i + 2)$-entry equal to 1 whereas all other positions of the first row and column are equal to 0. More precisely, matrix U_i' has the following form:

$$U_i' = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & U_i & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & & & & & 1 \\ & M^K & & & & \mathbf{1}_{\alpha_i} \\ & & M^S & & & \mathbf{1}_{\beta_i} \\ & & & M^P & & \mathbf{1}_{\gamma_i} \\ & & & & M^C & \mathbf{1}_{\delta_i} \\ & \mathbf{1}_{\alpha_i}^{\mathrm{T}} & \mathbf{1}_{\beta_i}^{\mathrm{T}} & \mathbf{1}_{\gamma_i}^{\mathrm{T}} & \mathbf{1}_{\delta_i}^{\mathrm{T}} & n - d(v_i) \end{bmatrix}.$$

By expanding with respect to the entries of the first column of matrix U_i', we have $\det(U_i') = \det(U_i)$. We subtract the first row of U_i' from all the rows of U_i', except the last row. Next, we multiply all the columns of U_i', except for the last column, by $1/(n-1)$ and add them to the first column. Recall that the sum of the elements of every column except the last one is equal to $n - 1$. Finally, we subtract the first column from the last column of matrix U_i'. Thus, substituting the value $d(v_i) = m + \ell_i - 1$ and since $\det(U_i') = \det(U_i)$, we obtain:

$$\det(U_i) = \begin{vmatrix} M^K & & & & \\ & M^S & & & \\ & & M^P & & \\ & & & M^C & \\ \mathbf{1}_{\alpha_i}^{\mathrm{T}} & \mathbf{1}_{\beta_i}^{\mathrm{T}} & \mathbf{1}_{\gamma_i}^{\mathrm{T}} & \mathbf{1}_{\delta_i}^{\mathrm{T}} & q_i \end{vmatrix} = q_i \cdot \prod_j \left(\lambda(K_j)^{\alpha_{ij}} \cdot \lambda(S_j)^{\beta_{ij}} \cdot \lambda(P_j)^{\gamma_{ij}} \cdot \lambda(C_j)^{\delta_{ij}} \right),$$

where $q_i = n - (m + \ell_i - 1) - \frac{\ell_i}{n-1}$.

Now we are ready to compute the number $\tau(G)$ of spanning trees for the graph $G = K_n - G_c$ using the complement-spanning-tree matrix theorem and Corollary 2.1. Thus we construct an $(m+\ell)\times(m+\ell)$ matrix $U(= M)$ for a complete-planet graph G_c, based on our vertex ordering scheme (end of Section 2.1). Then, we have:

$$\tau(G) = n^{n-m-\ell-2} \cdot \det(U) \tag{3}$$

where

$$U = \begin{bmatrix} U_{1,1} & & & & \mathbf{1}_{\ell_1} & & & \\ & U_{2,2} & & & & \mathbf{1}_{\ell_2} & & \\ & & \ddots & & & & \ddots & \\ & & & U_{m,m} & & & & \mathbf{1}_{\ell_m} \\ \mathbf{1}_{\ell_1}^{\mathrm{T}} & & & & n - d(v_1) & 1 & \cdots & 1 \\ & \mathbf{1}_{\ell_2}^{\mathrm{T}} & & & 1 & n - d(v_2) & \cdots & 1 \\ & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & & \mathbf{1}_{\ell_m}^{\mathrm{T}} & 1 & 1 & \cdots & n - d(v_m) \end{bmatrix}$$

is an $(m + \ell) \times (m + \ell)$ matrix and the submatrices $U_{i,i}$, $1 \le i \le m$, are obtained from U_i by deleting its last row and its last column (which correspond to vertex v_i). Note that U consists of two blocks: the first block corresponds to the vertices of the planet-subgraphs of G while the second block corresponds to the vertices of the sun-graph of G (it is easy to check the adjacencies). It now suffices to compute the determinant of matrix U. Following a procedure similar to the one we applied for the matrix U_i, we obtain:

$$\det(U) = \prod_{j=1}^{\ell} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)} \right) \cdot \det(D_c) \tag{4}$$

where

$$D_c = \begin{pmatrix} q_1 & 1 & \cdots & 1 \\ 1 & q_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & q_m \end{pmatrix}.$$

In order to compute the value $\det(D_c)$, we multiply the first row of matrix D_c by -1 and add it to the $m-1$ remaining rows. Then, we multiply column i by $\frac{q_1-1}{q_i-1}$, $2 \le i \le m$, and add it to the first column. The matrix D_c becomes upper triangular with its $(1, 1)$-entry equal to $q_1 + \sum_{i=2}^{m} \frac{q_1-1}{q_i-1}$ and each (i, i)-entry, $2 \le i \le m$, equal to $q_i - 1$. Thus,

$$\det(D_c) = \left(q_1 + \sum_{i=2}^{m} \frac{q_1 - 1}{q_i - 1}\right) \cdot \prod_{i=2}^{m} (q_i - 1) = \left(1 + \sum_{i=1}^{m} \frac{1}{q_i - 1}\right) \cdot \prod_{i=1}^{m} (q_i - 1) .$$

By using p_i to denote $p_i = q_i - 1$ and by substituting the value of $\det(D_c)$ into Eq. (4), we obtain the following theorem.

Theorem 3.1. *Let G_c be a complete-planet graph on m sun-vertices and ℓ planet-vertices. Then, the number of spanning trees of the graph $K_n - G_c$, where $n \ge m + \ell$, is equal to*

$$\tau(K_n - G_c) = n^{n-m-\ell-2} \cdot \prod_{j} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)}\right) \cdot \left(1 + \sum_{i=1}^{m} \frac{1}{p_i}\right) \cdot \prod_{i=1}^{m} p_i$$

where

$$p_i = n - m - \frac{n}{n-1} \cdot \ell_i \tag{5}$$

and $\alpha(j)$ ($\beta(j)$, $\gamma(j)$, $\delta(j)$, resp.) is the number of maximal cliques (stars, paths,cycles, resp.) on j vertices in the planet-graph of G_c.

For the quantities p_i, we obtain the following result. The proof is a straightforward derivation of the restrictions on the corresponding parameters of a complete-planet graph and, thus, it is omitted.

Lemma 3.1. *Let G_c be a complete-planet graph on m sun-vertices and ℓ planet-vertices. Then, for the quantities p_i, $i = 1, 2, \ldots, m$, we have:*

1. *if $m = 1$, then $p_1 \ge -1$ with equality holding iff $n = m + \ell = \ell + 1$;*
2. *if $m \ge 2$ and $\ell = 0$, then for all $i = 1, 2, \ldots, m$, $p_i = n - m \ge 0$, where equality holds iff $n = m + \ell = m$;*
3. *if $m \ge 2$, $\ell > 0$, and $\exists \ell_t = \ell$, then $p_t \ge -\frac{n-m}{n-1} > -1$ and for all $i \ne t$, $p_i = n - m > 0$;*
4. *if $m \ge 2$, $\ell > 0$, and $\forall i = 1, 2, \ldots, m$, $\ell_i \ne \ell$, then $p_i > 0$.*

It is important to note that although a p_i may be negative (see Cases (1) and (3) of Lemma 3.1), the number of spanning trees is never negative.

3.2. Star-planet graphs

Let G_s be a star-planet graph on $m + 1$ sun-vertices and ℓ planet-vertices such that $V(G_s) \subseteq V(K_n)$. We use the complement-spanning-tree matrix theorem in order to derive a closed formula for the number of spanning trees of the graph $G = K_n - G_s$.

In the previous section, for the graph $G = K_n - G_c$ we have formed the $(\ell_i+1) \times (\ell_i+1)$ matrix U_i, and we have computed its determinant. It is easy to see that the matrix U_i matches the corresponding matrix for the graph $K_n - G_s$ provided that we use $d(v_i) = \ell_i + 1$, $1 \le i \le m$. We compute the determinant of U_i using the same technique as the one we applied for the case $K_n - G_c$, and we obtain:

$$\det(U_i) = p_i \cdot \prod_{j} \left(\lambda(K_j)^{\alpha_{ij}} \cdot \lambda(S_j)^{\beta_{ij}} \cdot \lambda(P_j)^{\gamma_{ij}} \cdot \lambda(C_j)^{\delta_{ij}}\right),$$

where $p_i = n - (\ell_i + 1) - \frac{\ell_i}{n-1} = n - 1 - \frac{n}{n-1} \cdot \ell_i$.

Thus, by Corollary 2.1 we construct the $(m + \ell + 1) \times (m + \ell + 1)$ matrix $U(= M)$, according to our vertex ordering scheme for a star-planet graph G_s. For the number $\tau(G)$ of spanning trees of the graph $G = K_n - G_s$, we have the following formula:

$$\tau(G) = n^{n-m-\ell-3} \cdot \det(U) \tag{6}$$

where

$$U = \begin{bmatrix}
U_{1,1} & & & & \mathbf{1}_{\ell_1} & & & & \\
& U_{2,2} & & & & \mathbf{1}_{\ell_2} & & & \\
& & \ddots & & & & \ddots & & \\
& & & U_{m,m} & & & & \mathbf{1}_{\ell_m} & \\
\mathbf{1}_{\ell_1}^{\mathrm{T}} & & & & n - d(v_1) & & & & 1 \\
& \mathbf{1}_{\ell_2}^{\mathrm{T}} & & & & n - d(v_2) & & & 1 \\
& & \ddots & & & & \ddots & & \vdots \\
& & & \mathbf{1}_{\ell_m}^{\mathrm{T}} & & & & n - d(v_m) & 1 \\
& & & & 1 & 1 & \cdots & 1 & n - d(c)
\end{bmatrix}$$

is an $(m+\ell+1)\times(m+\ell+1)$ matrix. As in the case of the graph $K_n - G_c$, each submatrix $U_{i,i}$, $1 \le i \le m$, is obtained from U_i by deleting its last row and its last column. Note that

$$\det(U_{i,i}) = \prod_{j=1}^{\ell_i} \lambda(K_j)^{\alpha_{ij}} \cdot \lambda(S_j)^{\beta_{ij}} \cdot \lambda(P_j)^{\gamma_{ij}} \cdot \lambda(C_j)^{\delta_{ij}}.$$

Thus, for the determinant of matrix U, we have

$$\begin{aligned}\det(U) &= \prod_{j=1}^{\ell} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)}\right) \cdot \begin{vmatrix} p_1 & & & & 1 \\ & p_2 & & & 1 \\ & & \ddots & & \vdots \\ & & & p_m & 1 \\ 1 & 1 & \cdots & 1 & n-m \end{vmatrix} \\ &= \prod_{j=1}^{\ell} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)}\right) \cdot \det(D_s),\end{aligned}$$

where D_s is an $(m+1)\times(m+1)$ matrix. In order to compute the determinant of the matrix D_s, we work as in the case of the complete-planet graph: We multiply the first row of matrix D_s by -1 and add it to each of the next $m-1$ rows. Then, we multiply column i by $\frac{p_1}{p_i}$, $2 \le i \le m$, and add it to the first column. Finally, in order to make the matrix D_s upper triangular, we multiply the first column by $-\frac{1}{p_1}$ and add it to column $m+1$. Thus,

$$\det(D_s) = \left(n - m - \frac{1}{p_1} - \sum_{i=2}^{m} \frac{1}{p_i}\right) \cdot \prod_{i=1}^{m} p_i = \left(n - m - \sum_{i=1}^{m} \frac{1}{p_i}\right) \cdot \prod_{i=1}^{m} p_i.$$

We have the following theorem:

Theorem 3.2. *Let G_s be a star-planet graph on $m+1$ sun-vertices and ℓ planet-vertices. Then, the number of spanning trees of the graph $K_n - G_s$, where $n \ge m+\ell+1$, is equal to*

$$\tau(K_n - G_s) = n^{n-m-\ell-3} \cdot \prod_{j} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)}\right) \cdot \left(n - m - \sum_{i=1}^{m} \frac{1}{p_i}\right) \cdot \prod_{i=1}^{m} p_i$$

where

$$p_i = n - 1 - \frac{n}{n-1} \cdot \ell_i \tag{7}$$

and $\alpha(j)$ ($\beta(j)$, $\gamma(j)$, $\delta(j)$, resp.) is the number of maximal cliques (stars, paths, cycles, resp.) on j vertices in the planet-graph of G_s.

In this case, for the quantities p_i, $i = 1, 2, \ldots, m$, we have:

Lemma 3.2. *Let G_s be a complete-planet graph on $m+1$ sun-vertices and ℓ planet-vertices. Then,*

1. *if $\ell = 0$, then for all $i = 1, 2, \ldots, m$, $p_i = n-1 > 0$;*
2. *if $\ell > 0$ and $m = 1$, then $p_1 > 0$;*
3. *if $\ell > 0$, $m \ge 2$, and $\exists \ell_t = \ell$, then $p_t \ge \frac{1}{n-1} > 0$ and for all $i \ne t$, $p_i = n-1 > 0$;*
4. *if $\ell > 0$, $m \ge 2$, and $\forall i = 1, 2, \ldots, m$, $\ell_i < \ell$, then $p_i > m \ge 2$.*

Lemma 3.2 implies that the p_is are in all cases positive. Moreover, for case (3), Eq. (7) implies that $p_t = \frac{(n-1)^2 - n\ell}{n-1} = \frac{n(n-2-\ell)+1}{n-1} \ge \frac{1}{n-1}$.

4. Maximization results

As mentioned in the introduction, a uniformly-most reliable network [5,15] is defined to maximize the number of spanning trees. Thus, it is interesting to determine the types of graphs which have the maximum number of spanning trees for fixed numbers of vertices and edges. In this section, we provide maximization results for the number of spanning trees of $K_n - G$, where G is a complete-planet or a star-planet graph. In order to keep the number of vertices and edges fixed, we assume that:

- the clique K_n is fixed (i.e., n is fixed),
- the number m of vertices of the sun-graph is fixed, and
- the clique vector α, the star vector β, the path vector γ, and the cycle vector δ are fixed;

thus, our results are over the family of complete-planet graphs (resp., star-planet graphs) obtained by all possible combinations of connecting each clique, star, path, and cycle of each planet-subgraph to a sun-vertex.

4.1. Complete-planet graphs

Let G_c be a complete-planet graph on m sun-vertices and ℓ planet-vertices, where $\ell = \ell_1+\ell_2+\cdots+\ell_m$ and ℓ_i, $1 \le i \le m$, is the number of vertices of its planet-subgraphs $G_1, G_2, \ldots, G_m$. For notational convenience, we write the number $\tau(K_n-G_c)$ of spanning trees of the graph $K_n - G_c$ given by Theorem 3.1 as the product $\tau(K_n - G_c) = X(G_c)\cdot Y(G_c)$, where

$$X(G_c) = n^{n-m-\ell-2}\cdot \prod_{j=1}\left(\lambda(K_j)^{\alpha(j)}\cdot\lambda(P_j)^{\beta(j)}\cdot\lambda(C_j)^{\gamma(j)}\cdot\lambda(S_j)^{\delta(j)}\right),$$

$$Y(G_c) = \left(1+\sum_{i=1}^{m}\frac{1}{p_i}\right)\cdot\prod_{i=1}^{m}p_i;$$

recall that $p_i = n - m - \frac{n}{n-1}\cdot\ell_i$ (Eq. (5)). Since we are interested in maximizing the number of spanning trees when the parameters n and m, as well as the clique star, path, and cycle vectors are fixed, it suffices to maximize the factor $Y(G_c)$.

We will concentrate in the case where $m \ge 2$ and $\ell > 0$; if $m = 1$ or $\ell = 0$, we have no flexibility in changing the graph G_c. We note that

$$Y(G_c) = \left(1+\sum_{i=1}^{m}\frac{1}{p_i}\right)\cdot\prod_{i=1}^{m}p_i = \prod_{i=1}^{m}p_i + \sum_{i=1}^{m}\prod_{\substack{j=1\\ j\neq i}}^{m}p_j = p_m\prod_{i=1}^{m-1}p_i + p_m\sum_{i=1}^{m-1}\prod_{\substack{j=1\\ j\neq i}}^{m-1}p_j + \prod_{i=1}^{m-1}p_i$$

and if we substitute p_m by $S - \sum_{i=1}^{m-1}p_i$ in $Y(G_c)$, where $S = \sum_{i=1}^{m}p_i = (n-m)\,m - \frac{n}{n-1}\ell$ (which has a fixed value since n, m, ℓ are fixed), we get

$$Y(G_c) = \left(1+S-\sum_{i=1}^{m-1}p_i\right)\cdot\prod_{i=1}^{m-1}p_i + \left(S-\sum_{i=1}^{m-1}p_i\right)\cdot\sum_{i=1}^{m-1}\prod_{\substack{j=1\\ j\neq i}}^{m-1}p_j.$$

We compute the maximum by computing the partial derivative of $Y(G_c)$ with respect to any p_t, $1 \le t \le m-1$, and setting it equal to 0:

$$\frac{\partial Y(G_c)}{\partial p_t} = -\prod_{i=1}^{m-1}p_i + \left(1+S-\sum_{i=1}^{m-1}p_i\right)\cdot\prod_{\substack{i=1\\ i\neq t}}^{m-1}p_i - \sum_{i=1}^{m-1}\prod_{\substack{j=1\\ j\neq i}}^{m-1}p_j + \left(S-\sum_{i=1}^{m-1}p_i\right)\cdot\sum_{\substack{i=1\\ i\neq t}}^{m-1}\prod_{\substack{j=1\\ j\neq i,t}}^{m-1}p_j$$

which through standard algebraic manipulations is simplified to

$$\frac{\partial Y(G_c)}{\partial p_t} = \prod_{\substack{i=1\\ i\neq t}}^{m-1}p_i\cdot\left(1+\sum_{\substack{i=1\\ i\neq t}}^{m-1}\frac{1}{p_i}\right)\cdot\left(S-\sum_{i=1}^{m-1}p_i\ -p_t\right).$$

Then, we can show the following:

Lemma 4.1. *If $m \ge 2$ and $\ell > 0$, then for any $t \in \{1, 2, \ldots, m\}$*

$$\prod_{\substack{i=1\\ i\neq t}}^{m-1}p_i\cdot\left(1+\sum_{\substack{i=1\\ i\neq t}}^{m-1}\frac{1}{p_i}\right) > 0.$$

Proof. If for all $i = 1, 2, \ldots, m$, it holds that $\ell_i < \ell$, then $p_i > 0$ (Case (4) of Lemma 3.1) and the lemma clearly follows. Suppose now that $\ell_j = \ell$ for some j in $\{1, 2, \ldots, m\}$; then Case (3) of Lemma 3.1 applies and $p_j \ge -\frac{n-m}{n-1}$ and $p_i = n-m > 0$ for all $i \neq j$. Since $p_i > 0$ for all $i \neq j$, the lemma again readily follows if $t = j$, whereas if $t \neq j$ we need only show that $p_j\cdot\left(1+\sum_{\substack{i=1\\ i\neq t}}^{m-1}\frac{1}{p_i}\right) > 0$:

$$p_j\cdot\left(1+\sum_{\substack{i=1\\ i\neq t}}^{m-1}\frac{1}{p_i}\right) = p_j\cdot\left(1+\frac{m-2}{n-m}+\frac{1}{p_j}\right) = p_j\cdot\left(1+\frac{m-2}{n-m}\right)+1 \ge -\frac{n-2}{n-1}+1 = \frac{1}{n-1} > 0$$

since $p_j \ge -\frac{n-m}{n-1}$ and $n \ge m+\ell > m \ge 2$. ■

In light of Lemma 4.1, the partial derivative $\partial Y(G_c)/\partial p_t$ equals 0 if and only if $p_t = S - \sum_{i=1}^{m-1} p_i = p_m$. This equality holds for each $t = 1, 2, \ldots, m-1$; thus, the quantity $Y(G_c)$ reaches an extremum if and only if $p_1 = p_2 = \cdots = p_m$ or equivalently if $\ell_1 = \ell_2 = \cdots = \ell_m$. Moreover, since

$$\frac{\partial^2 Y(G_c)}{\partial^2 p_t} = \prod_{\substack{i=1\\ i\neq t}}^{m-1} p_i \cdot \left(1 + \sum_{\substack{i=1\\ i\neq t}}^{m-1} \frac{1}{p_i}\right) \cdot (-1-1) < 0,$$

we verify that the above extremum of $Y(G_c)$ is a maximum. Our result is stated in the following theorem.

Theorem 4.1. *Let G_c be a complete-planet graph with fixed clique, star, path, and cycle vectors, and $m + \ell$ vertices, where $\ell = \sum_{i=1}^{m} \ell_i$ and ℓ_i is the number of vertices of its ith planet-subgraph G_i. Then, the number of spanning trees of the graph $K_n - G_c$ is maximized when the ℓ_i s are all equal, if this is possible.*

It is worth noting that maximizing the number of spanning trees of $K_n - G_c$ is NP-complete; it follows from the well-known *Partition* problem [6]. In [4], a maximization theorem was provided for the graph $K_n - G$, where G is a multi-star graph, which follows as a consequence of Theorem 4.1; since the authors in [4] consider that the planet-components can only be single vertices, then if it is not possible to have $\ell_1 = \ell_2 = \cdots = \ell_m$, it is certainly feasible to ensure that any two of the ℓ_is differ by at most 1.

Since for given clique, star, path, and cycle vectors, achieving that $\ell_1 = \ell_2 = \cdots = \ell_m$, if possible, requires us to make a large number of combinations in general, below we give another result which when applied repeatedly helps us attain a maximum in this number of spanning trees, although this may not necessarily be the global maximum.

Let v_i and v_j be two arbitrary vertices of the sun-graph $G_c[A]$ and let G_i and G_j (on ℓ_i and ℓ_j vertices, respectively) be their corresponding planet-subgraphs. From G_c, we construct the complete-planet graph G'_c by moving planet-components between the planet-subgraphs G_i and G_j to obtain planet-subgraphs G'_i and G'_j on ℓ'_i and ℓ'_j vertices, respectively; then, the graphs G_c and G'_c have the same clique, star, path, and cycle vectors, and $\ell'_i + \ell'_j = \ell_i + \ell_j$.

Let us find the number of spanning trees of the graphs G'_c and G_c. First, the quantity $Y(G_c)$ can be written in terms of p_i and p_j as

$$Y(G_c) = p_i p_j \prod_{\substack{k=1\\ k\neq i,j}}^{m} p_k + p_i p_j \sum_{\substack{k'=1\\ k'\neq i,j}}^{m} \prod_{\substack{k=1\\ k\neq k',i,j}}^{m} p_k + (p_i + p_j) \prod_{\substack{k=1\\ k\neq i,j}}^{m} p_k = p_i p_j \cdot \Phi_1 + (p_i + p_j) \cdot \Phi_2$$

where $\Phi_1 = \prod_{\substack{k=1\\ k\neq i,j}}^{m} p_k \cdot \left(1 + \sum_{\substack{k'=1\\ k'\neq i,j}}^{m} \frac{1}{p_{k'}}\right)$ and $\Phi_2 = \prod_{\substack{k=1\\ k\neq i,j}}^{m} p_k$; note that Φ_1 and Φ_2 are independent of p_i and p_j. Similarly, for the graph G'_c we obtain $Y(G'_c) = p'_i p'_j \cdot \Phi_1 + (p'_i + p'_j) \cdot \Phi_2$. Since the graphs G_c and G'_c have the same clique, star, path, and cycle vectors, we have that $X(G'_c) = X(G_c)$. In order to compare the numbers of spanning trees of $K_n - G'_c$ and $K_n - G_c$, we examine their difference:

$$\begin{aligned}\tau(K_n - G'_c) - \tau(K_n - G_c) &= \big(Y(G'_c) - Y(G_c)\big) \cdot X(G_c)\\ &= \big((p'_i p'_j - p_i p_j) \cdot \Phi_1 + (p'_i + p'_j - p_i - p_j) \cdot \Phi_2\big) \cdot X(G_c).\end{aligned} \tag{8}$$

From Eq. (5) and the fact that $\ell'_i + \ell'_j = \ell_i + \ell_j$, it is easy to see that $p'_i + p'_j = p_i + p_j$ and $p'_i p'_j - p_i p_j = \left(\frac{n}{n-1}\right)^2 \left(\ell'_i \ell'_j - \ell_i \ell_j\right)$. Thus, Eq. (8) becomes

$$\tau(K_n - G'_c) - \tau(K_n - G_c) = \left(\frac{n}{n-1}\right)^2 \left(\ell'_i \ell'_j - \ell_i \ell_j\right) \cdot \Phi_1 \cdot X(G_c). \tag{9}$$

In a fashion similar to the one used in the proof of Lemma 4.1, we can show that $\Phi_1 > 0$. Additionally, $X(G_c) \geq 0$. Thus, in Eq. (9) we have to consider the value of $\ell'_i \ell'_j - \ell_i \ell_j$. We prove the following lemma.

Lemma 4.2. *If $|\ell'_i - \ell'_j| < |\ell_i - \ell_j|$, then $\ell'_i \ell'_j - \ell_i \ell_j > 0$.*

Proof. Without loss of generality, let $\ell_i \geq \ell_j$ and $\ell'_i \geq \ell'_j$; then, $|\ell'_i - \ell'_j| < |\ell_i - \ell_j| \iff \ell'_i - \ell'_j < \ell_i - \ell_j$. This inequality and the fact that $\ell'_i + \ell'_j = \ell_i + \ell_j$ imply that $\ell_i > \ell'_i$ and $\ell_i - \ell'_i = \ell'_j - \ell_j$. Let $\ell_i - \ell'_i = \ell'_j - \ell_j = r > 0$, that is, $\ell'_i = \ell_i - r$ and $\ell'_j = \ell_j + r$. Since $\ell'_i \geq \ell'_j$, it has to be that $\ell_i - r \geq \ell_j + r \Longrightarrow \ell_i - \ell_j \geq 2r$. Additionally, $\ell'_i \ell'_j = \ell_i \ell_j + r(\ell_i - \ell_j) - r^2$, and, thus, $\ell'_i \ell'_j - \ell_i \ell_j = r(\ell_i - \ell_j) - r^2 \geq r^2 > 0$. ∎

Therefore, we have the following theorem.

Theorem 4.2. *Let G_c be a complete-planet graph, v_i, v_j two vertices of its sun-graph and ℓ_i, ℓ_j the numbers of vertices of the planet-subgraphs G_i and G_j associated with v_i and v_j, respectively. If the cliques, stars, paths, and cycles of G_i and G_j are rearranged so that in the resulting graph G'_c the numbers of vertices of the planet-subgraphs associated with v_i and v_j are ℓ'_i and ℓ'_j with $|\ell'_i - \ell'_j| < |\ell_i - \ell_j|$, then the number of spanning trees of $K_n - G'_c$ is larger than that of $K_n - G_c$.*

Theorem 4.2 implies that we can pick a pair of vertices of the sun-graph of a complete-planet graph G_c and rearrange the planet-components of their planet-subgraphs so as to minimize the absolute value of the difference of their vertex numbers. We repeat the process for another pair of vertices of the sun-graph and so on so forth until no pair of vertices yields a larger number of spanning trees.

4.2. Star-planet graphs

We prove similar results for the number $\tau(K_n - G_s)$ of spanning trees of the graph $K_n - G_s$ for a star-planet graph G_s on $m + 1$ sun-vertices and ℓ planet-vertices where $\ell = \ell_1 + \ell_2 + \cdots + \ell_m$ and ℓ_i, $1 \le i \le m$, is the number of vertices of its planet-subgraph G_i. Then, from Theorem 3.2 we have that $\tau(K_n - G_s) = X(G_s) \cdot Y(G_s)$, where

$$X(G_s) = n^{n-m-\ell-3} \cdot \prod_{j=1} \left(\lambda(K_j)^{\alpha(j)} \cdot \lambda(S_j)^{\beta(j)} \cdot \lambda(P_j)^{\gamma(j)} \cdot \lambda(C_j)^{\delta(j)}\right),$$

$$Y(G_s) = \left(n - m - \sum_{i=1}^{m} \frac{1}{p_i}\right) \cdot \prod_{i=1}^{m} p_i,$$

and $p_i = n - 1 - \frac{n}{n-1} \cdot \ell_i$ (Eq. (7)). In order to maximize the number $\tau(K_n - G_s)$ of spanning trees under the assumption that the parameters n and m, and the clique, star, path, and cycle vectors are fixed, it suffices to maximize the factor $Y(G_s)$.

It is well known that the product of k positive numbers $a_1, a_2, \ldots, a_k$ whose sum A is constant is maximized when they are all equal; this is shown by replacing a_k by $A - \sum_{i=1}^{k-1} a_i$ in the product of the a_is and by working with the partial derivative of the resulting expression with respect to any of the a_is. In the same way, we can show that the sum $\sum_{i=1}^{k} \frac{1}{a_i}$ of k positive numbers $a_1, a_2, \ldots, a_k$ whose sum is constant is minimized when they are all equal. Thus, since the sum of the p_is is $(n-m)m - \frac{n\ell}{n-1}$ which is fixed, because n, m, ℓ are assumed to be fixed, both factors of $Y(G_s)$ are maximized when all the p_is are equal. Thus, if we take into account that any two p_i, p_j are equal if and only if $\ell_i = \ell_j$ (see Eq. (7)) we have a result similar to Theorem 4.1. Again observe that achieving a maximum number of spanning trees by making all ℓ_is equal is NP-complete [6].

Theorem 4.3. *Let G_s be a star-planet graph with fixed clique, star, path, and cycle vectors, and $m + \ell + 1$ vertices, where $\ell = \sum_{i=1}^{m} \ell_i$ and ℓ_i is the number of vertices of its ith planet-subgraph G_i. Then, the number of spanning trees of the graph $K_n - G_s$ is maximized when the ℓ_i s are all equal, if this is possible.*

Next, we consider a star-planet graph G_s and two arbitrary vertices v_i and v_j of its sun-graph $G_s[A]$ whose corresponding planet-subgraphs contain ℓ_i and ℓ_j vertices, respectively. We construct from G_s the star-planet graph G'_s by moving components between these subgraphs so that the resulting planet-subgraphs of v_i and v_j have ℓ'_i and ℓ'_j vertices, respectively. Then, $\ell'_i + \ell'_j = \ell_i + \ell_j$ and $X(G_s) = X(G'_s)$. The quantity $Y(G_s)$ can be written in terms of p_i and p_j as

$$Y(G_s) = (n-m)p_i p_j \prod_{\substack{k=1\\k\neq i,j}}^{m} p_k - p_i p_j \sum_{\substack{k'=1\\k'\neq i,j}}^{m} \prod_{\substack{k=1\\k\neq k',i,j}}^{m} p_k - (p_i + p_j) \prod_{\substack{k=1\\k\neq i,j}}^{m} p_k = p_i p_j \cdot \Psi_1 - (p_i + p_j) \cdot \Psi_2,$$

where $\Psi_1 = \left(n - m - \sum_{\substack{k'=1\\k'\neq i,j}}^{m} \frac{1}{p_{k'}}\right) \cdot \prod_{\substack{k=1\\k\neq i,j}}^{m} p_k$ and $\Psi_2 = \prod_{\substack{k=1\\k\neq i,j}}^{m} p_k$; again, Ψ_1, Ψ_2 are independent of p_i and p_j. A similar expression holds for $Y(G'_s)$ in terms of p'_i and p'_j. Since $\ell'_i + \ell'_j = \ell_i + \ell_j$, Eq. (7) implies that $p_i + p_j = p'_j + p'_j$ and $p'_i p'_j - p_i p_j = \left(\frac{n}{n-1}\right)^2 (\ell'_i \ell'_j - \ell_i \ell_j)$. Thus,

$$\tau(K_n - G'_s) - \tau(K_n - G_s) = \left(\frac{n}{n-1}\right)^2 \left(\ell'_i \ell'_j - \ell_i \ell_j\right) \cdot \Psi_1 \cdot X(G_s).$$

Lemma 4.3 establishes that for any i, j, $\Psi_1 > 0$.

Lemma 4.3. *If $m \ge 2$ and $\ell > 0$, then $\Psi_1 = \left(n - m - \sum_{\substack{k'=1\\k'\neq i,j}}^{m} \frac{1}{p_{k'}}\right) \cdot \prod_{\substack{k=1\\k\neq i,j}}^{m} p_k > 0$.*

Proof. If for all $k = 1, 2, \ldots, m$, $\ell_k \neq \ell$ then $p_k > m$ (see Case (4) of Lemma 3.2), and thus

$$\Psi_1 > \left(n - m - \frac{m-2}{m}\right) \cdot m^{m-2} > (n - m - 1) \cdot m^{m-2} > 0$$

since $n \geq m + 1 + \ell > m + 1$. Suppose now that there exists t such that $\ell_t = \ell$. Then, $p_t \geq \frac{1}{n-1} > 0$ and for all $k \neq t$, $p_k = n - 1$ (Case (3) of Lemma 3.2). If t is either i or j then

$$\Psi_1 = \left(n - m - \frac{m-2}{n-1}\right) \cdot (n-1)^{m-2} > (n - m - 1) \cdot (n-1)^{m-2} > 0$$

since $n - 1 > m - 2 \geq 0$. If t differs from both i and j then

$$\Psi_1 = \left(n - m - \left(\frac{m-3}{n-1} + \frac{1}{p_t}\right)\right) \cdot (n-1)^{m-3} \cdot p_t = (n-1)^{m-3} \cdot \left(\left(n - m - \frac{m-3}{n-1}\right) \cdot p_t - 1\right) > 0$$

because $n > m \implies -\frac{m-3}{n-1} > -1 \implies n - m - \frac{m-3}{n-1} > n - m - 1 \geq n - 1$ and $p_t \geq \frac{1}{n-1}$. ■

Additionally, $X(G_s) \geq 0$. Thus, from Lemmas 4.2 and 4.3, we have that:

Theorem 4.4. *Let G_s be a star-planet graph, v_i, v_j two vertices of its sun-graph and ℓ_i, ℓ_j the numbers of vertices of the planet-subgraphs G_i and G_j associated with v_i and v_j, respectively. If the cliques, stars, paths, and cycles of G_i and G_j are rearranged so that in the resulting graph G'_s the numbers of vertices of the planet-subgraphs associated with v_i and v_j are ℓ'_i and ℓ'_j with $|\ell'_i - \ell'_j| < |\ell_i - \ell_j|$, then the number of spanning trees of $K_n - G'_s$ is larger than that of $K_n - G_s$.*

Due to Theorem 4.4, a local minimum in the number of spanning trees can be obtained as in the case of complete-planet graphs.

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References

[1] C. Berge, Graphs and Hypergraphs, North-Holland, 1973.
[2] F.T. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, J. Graph Theory 10 (1986) 339–352.
[3] T.J.N. Brown, R.B. Mallion, P. Pollak, A. Roth, Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes, Discrete Appl. Math. 67 (1996) 51–66.
[4] K.-L. Chung, W.-M. Yan, On the number of spanning trees of a multi-complete/star related graph, Inform. Process. Lett. 76 (2000) 113–119.
[5] C.J. Colbourn, The Combinatorics of Network Reliability, Oxford University Press, New York, 1980.
[6] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, San Francisco, 1979.
[7] B. Gilbert, W. Myrvold, Maximizing spanning trees in almost complete graphs, Networks 30 (1997) 23–30.
[8] M.J. Golin, X. Yong, Y. Zhang, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298 (2005) 334–364.
[9] A.K. Kelmans, On graphs with the maximum number of spanning trees, Random Structures Algorithms 1–2 (1996) 177–192.
[10] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1969.
[11] Z. Huaxiao, Z. Fuji, H. Qiongxiang, On the number of spanning trees and Eulerian tours in iterated line digraphs, Discrete Appl. Math. 73 (1997) 59–67.
[12] Z. Lonc, K. Parol, J. Wojcieckowski, On the number of spanning trees in directed circulant graphs, Networks 36 (2001) 129–133.
[13] W. Moon, Enumerating labeled trees, in: F. Harary (Ed.), Graph Theory and Theoretical Physics, 1967, pp. 261–271.
[14] B.R. Myers, Number of spanning trees in a wheel, IEEE Trans. Circuit Theory 18 (1971) 280–282.
[15] W. Myrvold, K.H. Cheung, L.B. Page, J.E. Perry, Uniformly-most reliable networks do not always exist, Networks 21 (1991) 417–419.
[16] S.D. Nikolopoulos, C. Nomikos, P. Rondogiannis, A limit characterization for the number of spanning trees of graphs, Inform. Process. Lett. 90 (2004) 307–313.
[17] S.D. Nikolopoulos, C. Papadopoulos, The number of spanning trees in K_n-complements of quasi-threshold graphs, Graphs Combin. 20 (2004) 383–397.
[18] S.D. Nikolopoulos, P. Rondogiannis, On the number of spanning trees of multi-star related graphs, Inform. Process. Lett. 65 (1998) 183–188.
[19] P.V. O'Neil, The number of trees in a certain network, Notices Amer. Math. Soc. 10 (1963) 569.
[20] L. Petingi, F. Boesch, C. Suffel, On the characterization of graphs with maximum number of spanning trees, Discrete Math. 179 (1998) 155–166.
[21] H.N.V. Temperley, On the mutual cancellation of cluster integrals in Mayer's fugacity series, Proc. Phys. Soc. 83 (1964) 3–16.
[22] L. Weinberg, Number of trees in a graph, Proc. IRE 46 (1958) 1954–1955.
[23] W.-M. Yan, W. Myrvold, K.-L. Chung, A formula for the number of spanning trees of a multi-star related graph, Inform. Process. Lett. 68 (1998) 295–298.
[24] Y. Zhang, X. Yong, M.J. Golin, The number of spanning trees in circulant graphs, Discrete Math. 223 (2000) 337–350.