Discrete Mathematics 309 (2009) 3280–3291

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Minimum fractional dominating functions and maximum fractional packing functions

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ARTICLE INFO

Article history:
Received 8 December 2004
Received in revised form 8 July 2008
Accepted 17 September 2008
Available online 23 February 2009

Keywords:
Fractional domination
Fractional isomorphism
Complementary slackness

ABSTRACT

The fractional analogues of domination and 2-packing in a graph form an interesting pair of dual linear programmes in that the feasible solutions for both are functions from the vertices of the graph to the unit interval; efficient (fractional) domination is accomplished when one function simultaneously solves both LPs. We investigate some structural properties of the functions thus defined and classify some families of graphs according to how and whether the sets of functions intersect, developing tools that have proven useful in approaching problems in domination theory.

1. Introduction and terminology

Our notation follows that of Haynes, Hedetniemi, and Slater [8,9]. The *open neighbourhood* of a vertex $v \in V(G)$ is defined as $N(V) = \{u \in V \mid uv \in E\}$, the set of all vertices adjacent to v. The *closed neighbourhood* of a vertex $v \in V(G)$ is defined as $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, let $N(S) = \bigcup_{u \in S} N(u)$ and let $N[S] = \bigcup_{u \in S} N[u]$. The distance from any two vertices $u, v \in V(G)$, denoted by $\mathrm{dist}(u, v)$ is the length of a shortest path from u to v. We say that a vertex *dominates* itself and all of its neighbours. A set of vertices $S \subseteq V$ is called a *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to some element of S. That is, a set S is a dominating set if $|N[v] \cap S| \geq 1$ for all vertices $v \in V(G)$. The domination number $\gamma(G)$ is the size of a smallest dominating set.

A subset $S \subseteq V(G)$ is called a k-packing, if for any two distinct vertices u, v in S, we have $\mathrm{dist}(u, v) > k$. A set $S \subseteq V(G)$ is a *2-packing* if the minimum distance between any distinct vertices u, v of S is at least 3. A set S is a *closed neighbourhood packing* if for each $u, v \in S, u \neq v$ we have $N[u] \cap N[v] = \emptyset$. Alternatively, a set S is a closed neighbourhood packing if $|N[v] \cap S| \leq 1$ for all vertices $v \in V(G)$. A set S is a closed neighbourhood packing if and only if S is a 2-packing, since for any two distinct vertices u, v in a closed neighbourhood packing S, we have $\mathrm{dist}(u, v) > 2$. The packing number $\rho(G)$ is the size of a largest closed neighbourhood packing. A *fractional dominating function* (FDF) is a function $g : V \to [0, 1]$ such that $g(N[v]) = \sum_{u \in N[v]} g(u) \geq 1$ for all vertices $v \in V$. A *minimum fractional dominating function* (MFDF) is a fractional dominating function g such that the value $|g| = \sum_{v \in V} g(v)$ is as small as possible. This minimum value is the *fractional domination number* of G, denoted by $\gamma_f(G)$.

A function $h : V \to [0, 1]$ is a *fractional packing function* (FPF) provided that $h(N[v]) = \sum_{u \in N[v]} h(u) \leq 1$ for all $v \in V$. A *maximum fractional packing function* (MFPF) is a fractional packing function h such that the value attained by $|h| = \sum_{v \in V} h(v)$ is as large as possible. This maximum is the *fractional* (*closed neighbourhood*) *packing number* of G and is denoted by $\rho_f(G)$.

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0012-365X/$ – see front matter
doi:10.1016/j.disc.2008.09.049

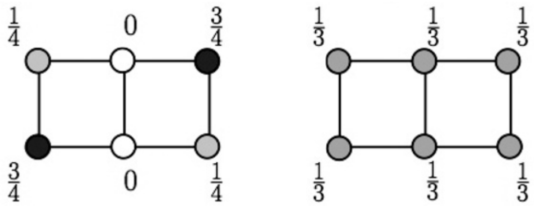

Fig. 1. (left) A fractional dominating-packing function (FDPF), and (right) a minimum fractional dominating function (MFDF) which is not a fractional packing function (FPF).

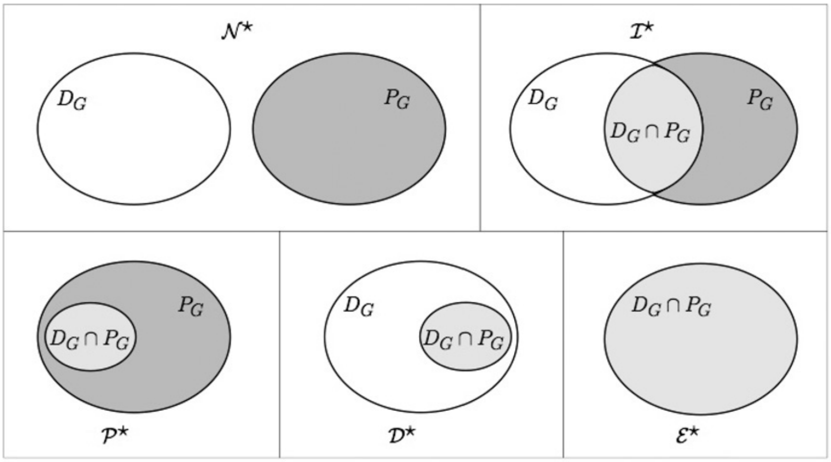

Fig. 2. The five classes as different intersections of the sets D_G (in white) and P_G (in grey); with $D_G \cap P_G$ in light grey.

Observation 1 (*Domke et al. [4]*). *For all graphs G, $\rho(G) \leq \rho_f(G) = \gamma_f(G) \leq \gamma(G)$.*

Bange, Barkauskas, and Slater [1] define a set S to be an *efficient dominating set* if $|N[v] \cap S| = 1$ for all vertices $v \in V(G)$, and they also introduced the following efficiency measure for a graph G. The *efficient domination number* of a graph, denoted $F(G)$, is the maximum number of vertices that can be dominated by a set S that dominates each vertex at most once. A graph G of order $n = |V(G)|$ has an efficient dominating set if and only if $F(G) = n$. A graph is efficient (or efficiently dominatable) if and only if there exists an efficient dominating set. Alternatively, a graph is efficient if and only if, there exists a set S which is both dominating and a closed neighborhood packing.

In this paper, we are interested in the in the fractional analogue of efficient domination also introduced by Grinstead and Slater [7]. A fractional dominating function $f : V \to [0, 1]$ is *efficient* if $f(N[v]) = \sum_{u \in N[v]} f(u) = 1$, for every $v \in V$. Note that any efficient FDF is necessarily an MFDF. The *fractional efficient domination number* of a graph, denoted $F_f(G)$, is the maximum of $\sum_{i=1}^{n}(1 + deg(v_i))X_i$, where X is the characteristic function of a maximum fractional packing function. A graph G of order $n = |V(G)|$ has an efficient fractional dominating function if and only if $F_f(G) = n$. A graph is efficiently fractional dominatable if and only if, there exists an efficient FDF. Alternatively, a graph is efficiently fractional dominatable if and only if, there exists a function which is both an FDF and an FPF.

As with efficiently fractional dominatable graphs, it is possible to have a function that is both an FPF and an FDF. We call such a function a *fractional dominating-packing function* (FDPF). A function which is both an FDF and FPF is necessarily an MFDF and an MFPF and is therefore an FDPF. We might also refer to such an object as a *(closed neighbourhood) fractional partition* on the vertices of G, as it forms a real-valued analogue of a partition of the vertex set of G into closed neighbourhoods (Fig. 1).

In this paper, our attention is turned to the sets of minimum fractional dominating functions and maximum fractional packing functions of a given graph, and specifically at how these two sets intersect. We divide all graphs into the following five classes based on the intersections of these sets: Let D_G be the set of all MFDFs on G and let P_G be the set of all MFPFs on G. Every finite simple graph G belongs to exactly one of the classes below:

- $G \in \mathcal{N}^\star$ (Null) if $D_G \cap P_G = \varnothing$.
- $G \in \mathcal{I}^\star$ (Intersection) if $D_G \cap P_G \neq \varnothing$, $D_G \not\subseteq P_G$ and $P_G \not\subseteq D_G$.
- $G \in \mathcal{P}^\star$ (Packing) if $D_G \subsetneq P_G$.
- $G \in \mathcal{D}^\star$ (Dominating) if $P_G \subsetneq D_G$.
- $G \in \mathcal{E}^\star$ (Equal or Efficient) if $D_G = P_G$ (Fig. 2).

Along the way, we shall survey results on fractional domination in general, and structural results on FDFs in particular. Our original intent was to provide a set of tools to better understand and classify fractional efficient domination, i.e. fractional dominating functions which are also fractional packings; for a graph to contain an efficient fractional dominating function it is necessary but not sufficient for the peripheral terms in the above inequality to be equal, and we hope that our results and constructions will result in further conditions for the existence or nonexistence of such sets. Our results have also shown themselves useful in attacking other problems concerning fractional domination, developing further "pure" results on MFDFs and MFPFs in [11,13] and applying the tools to studying fractional analogues of domination problems in [14,17,18,21].

In Section 2 we review the linear programming formulation of domination and fractional domination problems, and show how complementary slackness can be employed to help sort graphs into one of our five classes. Section 3 uses these tools to show that each of the five classes is infinite as we classify the members of several popular families of graphs; we also find that certain graph operations work naturally with fractional domination and packing, and show how they operate on the classes. In Section 4 we employ fractional isomorphism as an additional tool for reasoning with the classes, and the final section looks at other variations and future directions.

2. Integer and linear programming

Many problems in graph theory can be formulated as integer programmes. In fractional graph theory, many fractional parameters can be defined by the value of a relaxed integer programme. If the matrix and vectors of an LP all have rational entries, then the value will be rational, hence, the reason the term "fractional" instead of real in (2) (see [15] or [20]).

The problem of determining the domination number can be formulated as an integer program using the neighbourhood matrix $N = A + I_n$, where A is the adjacency matrix of the graph in question (under some ordering of the vertices); $\gamma(G)$ is the value of the integer program (1). From this, we can define fractional domination number as the value of the linear programming relaxation of the integer program (1); γ_f is the value of the linear programme (2). Determining $\rho_f(G)$ can be likewise formulated in LP terms, by taking the linear programming dual of (2). Determining the packing number can be formulated in IP terms, by adding the additional constraint to (3) that the optimal solutions need to be integer valued; $\rho(G)$ is the value of the integer programme (4). By duality we know that $\rho_f(G) = \gamma_f(G)$. Thus, for all graphs G, $\rho(G) \le \rho_f(G) = \gamma_f(G) \le \gamma(G)$, as shown in Domke et al. [4].

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \ge \mathbf{1}, \boldsymbol{y} \ge \mathbf{0}, y_i \in \mathbb{Z} \tag{1}$$

$$\text{minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to: } N\boldsymbol{y} \ge \mathbf{1}, \boldsymbol{y} \ge \mathbf{0} \tag{2}$$

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \le \mathbf{1}, \boldsymbol{x} \ge \mathbf{0} \tag{3}$$

$$\text{maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to: } N\boldsymbol{x} \le \mathbf{1}, \boldsymbol{x} \ge \mathbf{0}, x_i \in \mathbb{Z}. \tag{4}$$

Linear relaxation of the integer programs gives us that $\rho(G) \le \rho_f(G)$ and $\gamma_f(G) \le \gamma(G)$. Combining these inequalities with well-known duality results from linear programming yields the Observation 1 from above.

The principal tool that we shall use in our investigations is the Principle of Complementary Slackness, an important result in the duality theory of linear programming:

Theorem 2 (*Principle of Complementary Slackness*). *Let $\boldsymbol{x}'$ be any optimal solution to the linear programme: maximise $\boldsymbol{c}^T\boldsymbol{x}$ subject to $M\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{x} \ge \mathbf{0}$, and let $\boldsymbol{y}'$ be any optimal solution to the dual linear programme: minimise $\boldsymbol{b}^T\boldsymbol{y}$ subject to $M^T\boldsymbol{y} \ge \boldsymbol{c}$, $\boldsymbol{y} \ge \mathbf{0}$. Then:*

$$\boldsymbol{x}' \cdot (M^T\boldsymbol{y}' - \boldsymbol{c}) = \boldsymbol{y}' \cdot (M\boldsymbol{x}' - \boldsymbol{b}) = \mathbf{0}.$$

When applied to our LPs of interest, it takes the following form.

Proposition 3. *Let $\boldsymbol{x}'$ be an MFPF: that is, any optimal solution to the linear programme*

$$\text{Maximize } \mathbf{1}^T\boldsymbol{x} \text{ subject to } N\boldsymbol{x} \le \mathbf{1}, \boldsymbol{x} \ge \mathbf{0}$$

and let $\boldsymbol{y}'$ be an MFDF: that is, any optimal solution to the dual linear programme

$$\text{Minimize } \mathbf{1}^T\boldsymbol{y} \text{ subject to } N^T\boldsymbol{y} \ge \mathbf{1}, \boldsymbol{y} \ge \mathbf{0}.$$

Then

$$\boldsymbol{x}' \cdot (N^T\boldsymbol{y}' - \mathbf{1}) = \boldsymbol{y}' \cdot (N\boldsymbol{x}' - \mathbf{1}) = \mathbf{0}.$$

The consequence of this for our purposes is as follows:

Corollary 4. *If f is an MFDF and $v \in V(G)$ for which $f(v) > 0$ then $g(N[v]) = 1$ for any MFPF g. Likewise, if g is an MFPF and $v \in V(G)$ for which $g(v) > 0$, then $f(N[v]) = 1$ for any MFDF f.*

This in turn suffices to establish a some technical lemmas, which we now state and prove.

Lemma 5. (1) *If f is an MFDF with $f(v) > 0$ for every vertex $v \in V$, then every MFPF is an MFDF, and thus $P_G \subseteq D_G$.*
(2) *If we can find two functions f and g where f is an MFDF on G with $f(v) > 0$ for each vertex v, and where g is an MFDF on G which is not an FPF, then $G \in \mathcal{D}^\star$.*

Proof. For the first part, let f be an MFDF on G with $f(v) > 0$ for each vertex v. Then by Corollary 4, every MFPF g has the property that $g(N[v]) = 1$ for every vertex v. So g is an MFDF.

As for the second, the MFDF f gives us $P_G \subseteq D_G$ and the MFDF (non-packing) g gives us $P_G \subsetneq D_G$.

Dually, we have:

Lemma 6. (1) *If f is an MFPF with $f(v) > 0$ for each vertex $v \in V$, then every MFDF is an MFPF, and thus $D_G \subseteq P_G$.*
(2) *If we can find two functions f and g where f is an MFPF on G with $f(v) > 0$ for each vertex v, and where g is an MFPF on G which is not dominating, then $G \in \mathcal{P}^\star$.*

The results of Lemmas 5 and 6 also work if there is a single function which satisfy both properties of f and g simultaneously. This single function can be obtained by taking an appropriate convex combination of the two functions. If we can find a function which is both an MFDF and an MFPF which has positive weights on each vertex, then combining Lemmas 5 and 6 yields:

Corollary 7. *For a graph G, if there exists an FDPF f with $f(v) > 0$ for each vertex $v \in V$, then $G \in \mathcal{E}^\star$.*

3. A partial classification

With these preliminaries in place, we are ready to begin sorting families of graphs into our five classes.

3.1. Some basic graphs

Theorem 8. *Every regular graph is $\mathcal{E}^\star$.*

Proof. Let G be k-regular; then the function $f(v) = \frac{1}{k+1}$ for all $v \in V$ is an FDPF. Since f is nonzero at each vertex, Corollary 7 tells us that $G \in \mathcal{E}^\star$.

Theorem 9. *If $\Delta(G) = n - 1$ and $G \neq K_n$ then $G \in \mathcal{P}^\star$.*

Proof. Let S be the set of vertices of maximum degree $n - 1$. Since $\rho = \gamma = 1$, the constant function $f(v) = \frac{1}{n}$ is an MFPF. Since $f(N[v]) < 1$ for any $v \in V - S$, f is not dominating. Note that $V - S$ is non-empty since $G \neq K_n$. Thus, by Lemma 6, $G \in \mathcal{P}^\star$.

It would be nice if we could determine the class of the graph by induced subgraphs. From the above two theorems, we can see this does not work. The star $K_{1,2}$ is $\mathcal{P}^\star$ and K_2 is an induced subgraph, however, K_2 is regular and thus $\mathcal{E}^\star$.

Theorem 10. *Let G be the complete r-partite graph with parts of size $n_1, n_2, \ldots, n_r$, $r \geq 2$ and each $n_j \geq 2$. Then $G \in \mathcal{E}^\star$.*

Proof. As shown in [7] the function which assigns to each vertex in the jth part the positive weight of

$$\frac{1}{(n_j - 1)\left(\sum_{i=1}^{r} \frac{1}{n_i - 1} + r - 1\right)}$$

is an FDPF.

3.2. Paths and other trees

Theorem 11. *Let P_n be the path on n vertices for $n \geq 3$. Then:*

$$P_n \in \begin{cases} \mathcal{P}^\star, & n \equiv 0 \mod 3 \\ \mathcal{D}^\star, & n \equiv 1 \mod 3 \\ \mathcal{I}^\star, & n \equiv 2 \mod 3. \end{cases}$$

Proof. Let v_i represent the ith vertex of the path on n vertices. For any positive integer $k \geq 1$, it is easy to check that $\rho(P_{3k}) = \gamma(P_{3k}) = k$ and $\rho(P_{3k+i}) = \gamma(P_{3k+i}) = k + 1$ for $i = 1, 2$. In the following cases, the bracketed blocks of weights are repeated $k - 1$ times.

Case 1: $n = 3k$. Let f be the function which assigns the weight of $\frac{1}{3}$ to each vertex. Since $f(N[v_1]) = \frac{2}{3}$, f is not dominating. So by Lemma 6, we have $P_{3k} \in \mathcal{P}^\star$.

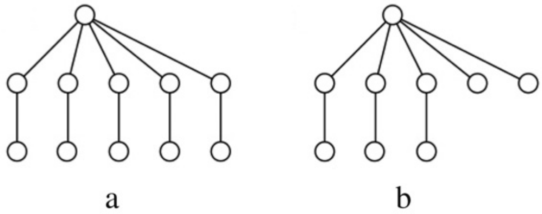

Fig. 3. (a) A healthy spider: $K^*_{1,6}$ and (b) a wounded spider.

Case 2: $n = 3k + 1$. The function $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}], \ldots, \frac{1}{2}, \frac{1}{2})^T$ is an MFDF with positive weights on every vertex. Since $f(N[v_2]) = \frac{4}{3}$, f is not packing. Therefore $P_{3k+1} \in \mathcal{D}^\star$ by Lemma 5.

Case 3: $n = 3k+2$. The function $\boldsymbol{f} = (\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, [\frac{1}{2}, 0, \frac{1}{2}], \ldots, \frac{1}{2})^T$ is an FDPF. The function $\boldsymbol{g} = (0, 1, 0, 1, [0, 0, 1], \ldots, 0)^T$ is an MFDF which is not packing (since $g(N[v_3]) = 2$). Lastly, the function $\boldsymbol{h} = (1, [0, 0, 1], \ldots, 0, 0, 0, 1)^T$ is an MFPF which is not dominating (since $h(N[v_{3k}]) = 0$). Therefore $P_{3k+2} \in \mathcal{I}^\star$.

Trees in general do not seem as easy to classify; however, certain classes of trees lend themselves easily to analysis. For instance, [8] defines a *healthy spider* $K^*_{1,t}$ as the result of subdividing each edge of a star $K_{1,t}$ into a path of length 3 (see Fig. 3). Exempting one or more (but not all) of the edges from this subdivision results in a *wounded spider* (see Fig. 6a). In both of these classes of graphs, the vertex of degree t is referred to as the *head* vertex, and those of degree one the *foot* vertices.

Theorem 12. $K^*_{1,t} \in \mathcal{I}^\star$.

Proof. The function which assigns the weight of 0 to the head vertex, $\frac{t-1}{t}$ to the foot vertices, and $\frac{1}{t}$ otherwise, is an FDPF. The function which assigns 1 to the foot vertices and 0 otherwise is an MFPF which is not dominating. Lastly, the function which assigns 1 to the vertices of degree two and 0 otherwise, is an MFDF which is not packing. Therefore, $G \in \mathcal{I}^\star$.

Note that for the healthy spider obtained from subdividing both edges of a $K_{1,2}$ we get P_5 which was already shown to be in $\mathcal{I}^\star$ by the preceding theorem.

Theorem 13. *Suppose that T is a tree and $T \in \mathcal{E}^\star$. Then $|V(T)| \leq 2$.*

Proof. Suppose that T is a $\mathcal{E}^\star$ tree with at least three vertices; then we can find two adjacent vertices x, y such that $d(x) = 1$ and $d(y) > 1$. Let f be an FDPF; then $f(N[x]) = f(x) + f(y) = 1$. We shall define two more functions, f_x and f_y, which are equal to f everywhere except on $N[x]$; we set $f_x(x) = 1$ and $f_x(y) = 0$; likewise, $f_y(x) = 0$ and $f_y(y) = 1$. Clearly f_x is an MFPF and f_y is an MFDF, but at least one of them is not a FDPF.

3.3. Graphs formed from cliques

If we take a finite collection of $q > 1$ disjoint cliques $\{K_{n_1}, \ldots, K_{n_q}\}$ and for each clique designate a vertex v_i to be adjacent to a vertex c outside of each clique, then we have a graph on $\sum n_i + 1$ vertices. We call c the *central vertex*, each of the vertices in the K_{n_i} which are not adjacent to the central vertex *peripheral vertices*, and the $\{v_i\}$ *juncture vertices*. The central vertex has degree q, the peripheral vertices have degrees n_i, and the juncture vertices have degrees $n_i + 1$ (Fig. 4).

Theorem 14. *Let G be constructed from a collection of $q > 1$ disjoint cliques as above. If $n_i \geq 2$ for all i, then $G \in \mathcal{I}^\star$.*

Proof. Clearly $\rho = \gamma = q$, so the function which assigns the weight of 1 to each of the juncture vertices and 0 otherwise is an MFDF which is non-packing. The function which assigns the weight of 1 to a single peripheral vertex in each clique and 0 otherwise is an MFPF which is non-dominating. Lastly, take the previous MFPF and move the weight of 1 from the peripheral vertex of just one clique to its juncture vertex. This is an FDPF. Therefore $G \in \mathcal{I}^\star$ (Fig. 5).

3.4. $\mathcal{N}^\star$-graphs

Up until now, we have seen infinite families of examples of graphs in every class except $\mathcal{N}^\star$, where no MFDF is an MFPF and no MFPF is an MFDF; in fact, no FDF is a FPF and no FPF is a FDF. There is an easy characterisation of $\mathcal{N}^\star$ graphs using neighbourhood matrices.

Proposition 15. *G is in $\mathcal{N}^\star$ if and only if the system $N\boldsymbol{x} = \boldsymbol{1}$ has no non-negative solutions.*

Proof. If $N\boldsymbol{x} = \boldsymbol{1}$ has a non-negative solution, then the vector $\boldsymbol{x}$ is an FDPF, thus G is not in $\mathcal{N}^\star$. Likewise, if G is not in $\mathcal{N}^\star$, then there exists a vector $\boldsymbol{x}$ satisfying $N\boldsymbol{x} = \boldsymbol{1}$, with $\boldsymbol{x} \geq \boldsymbol{0}$, that is each x_i is non-negative; thus the system has a non-negative solution.

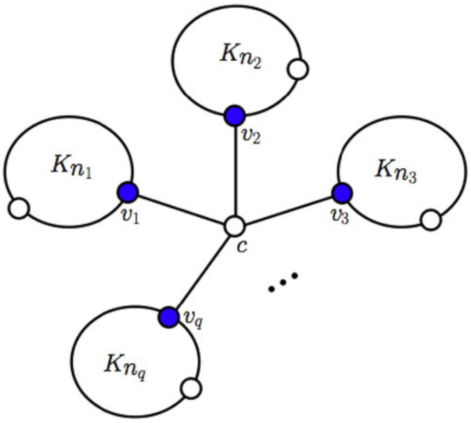

Fig. 4. A collection of disjoint cliques each connected to a central vertex.

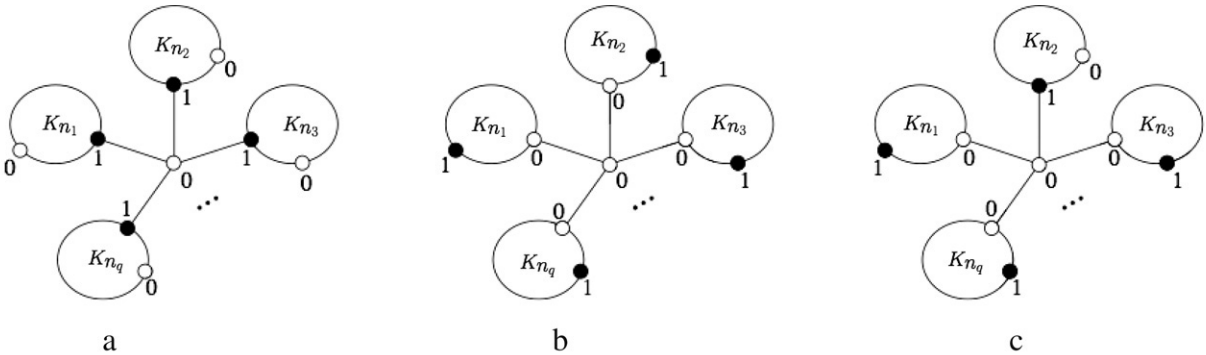

Fig. 5. (a) An MFDF (non-packing), (b) an MFPF (non-dominating), and (c) an FDPF.

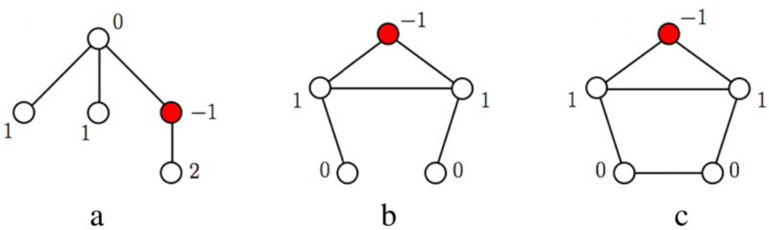

Fig. 6. Unique solutions to $N\boldsymbol{x} = \mathbf{1}$.

The smallest examples of graphs in $\mathcal{N}^\star$ are a wounded spider obtained from subdividing one edge of a $K_{1,3}$, a K_3 with two pendant edges, and C_5 with an added chord (depicted in Fig. 6). In each of these graphs, the red vertex is forced to have a negative weight when solving the system $N\boldsymbol{x} = \mathbf{1}$.

With the above wounded spider (depicted in Fig. 6a), upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is $\boldsymbol{x} = (2, -1, 0, 1, 1)^T$. By Proposition 15, the above wounded spider is in $\mathcal{N}^\star$.

The next $\mathcal{N}^\star$ graph is K_3 with two pendant edges (depicted in Fig. 6b). Upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is the function x which assigns a weight of -1 to the vertex of degree two, 1 to each vertex of degree three and 0 to each vertex of degree one. By Proposition 15, K_3 with two pendant edges is in $\mathcal{N}^\star$.

The last $\mathcal{N}^\star$ graph on five vertices is C_5 with a chord (see Fig. 6c). Upon solving $N\boldsymbol{x} = \mathbf{1}$, we find the unique solution is $\boldsymbol{x} = (-1, 1, 0, 0, 1)^T$. By Proposition 15, this graph is $\mathcal{N}^\star$. In [13], we describe all MFDFs of this graph, $\boldsymbol{f} = (0, \frac{1}{2}, t, \frac{1}{2} - t, \frac{1}{2})^T$ where $0 \le t \le \frac{1}{2}$. The unique MFPF is $\boldsymbol{g} = (\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0)^T$.

We showed above (in Theorems 8 and 11) that the other four classes are infinite. We shall now do the same for $\mathcal{N}^\star$ using some results from [13], which we restate here using our present terminology.

Lemma 16. *If f is an MFDF and $v \in V(G)$ for which $f(N[v]) > 1$, then every MFPF g satisfies $g(v) = 0$.*

Proof. Suppose g is an MFPF with $g(v) > 0$. Then by Corollary 4, every MFDF f satisfies $f(N[v]) = 1$, a contradiction.

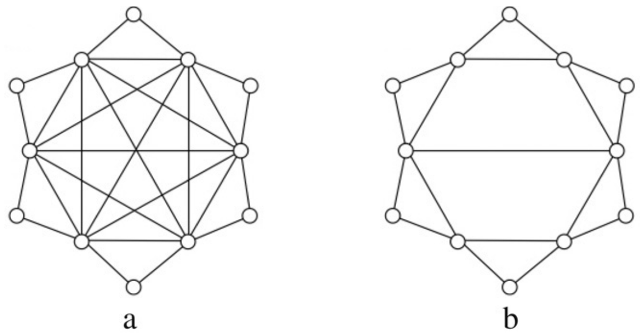

Fig. 7. (a) The trampoline on 12 vertices $T(K_6)$ and (b) the partial trampoline $T(P_2 \square P_3)$.

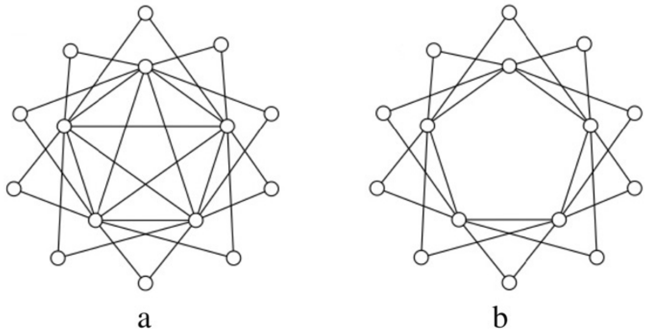

Fig. 8. (a) The generalized Hajós graph $[K_5]$ and (b) the partial generalized Hajós graph $[C_5]$.

Lemma 17. *If g is an MFPF and $v \in V(G)$ for which $g(N[v]) < 1$, then every MFDF f satisfies $f(v) = 0$.*

Proof. Suppose f is an MFDF with $f(v) > 0$. Then by Corollary 4, every MFPF g satisfies $g(N[v]) = 1$, a contradiction.

Motivated by [19,3], we define a *trampoline* $T(K_n)$ on $2n$ vertices ($n \geq 3$) as follows: begin with a complete graph on the vertices $\{v_1, \ldots, v_n\}$, add the vertices $\{u_1, \ldots, u_n\}$ and add the edges $u_i v_i$ and $u_i v_{i+1}$ (with $v_{n+1} = v_1$); see Fig. 7a. Trampolines are referred to as *n-suns* in [3]. A *partial trampoline* $T_H(G)$ is the graph on $2n$ vertices formed from any Hamiltonian graph G with Hamilton cycle $H = v_1 v_2 \ldots v_n$. This can be thought of as taking a trampoline and removing edges from "inside" the K_n (see Fig. 7b). When there is only one Hamiltonian cycle, the H will be omitted.

As in [2], the *generalized Hajós graph* is the graph $[K_n]$ on $n + \binom{n}{2}$ vertices formed by starting with a clique on three or more vertices, then adding a vertex u_{ij} for each pair of vertices v_i, v_j in K_n add the edges $u_{ij} v_i$ and $u_{ij} v_j$ (see Fig. 8a). As with partial trampolines we can start with any Hamiltonian graph G on three or more vertices and then apply the construction on G to obtain the *partial* generalized Hajós graph $[G]$ with $n + \binom{n}{2}$ vertices (see Fig. 8b).

Theorem 18. *Let G be Hamiltonian, then $T(G)$ is $\mathcal{N}^\star$.*

Proof. As noted in [13], the function f defined by $f(u_i) = \frac{1}{2}$ and $f(v_i) = 0$ (for all i) is an FPF; the function g defined by $g(v_i) = \frac{1}{2}$ and $g(u_i) = 0$ (for all i) is an FDF. Since $|f| = |g|$, f is a maximum FPF and g is a minimum FDF. Since $f(N[u_i]) < 1$ for each u_i, then by Lemma 17, every MFDF h satisfies $h(u_i) = 0$. Since $g(N[v_i]) = \frac{3}{2} > 1$ for each v_i, then by Lemma 16, every MFPF k satisfies $k(v_i) = 0$. Therefore, no MFPF can be an MFDF.

Note that the above proof does not depend on which Hamiltonian cycle H is chosen in the construction, hence, the H in $T_H(G)$ is omitted.

Corollary 19. *All trampolines are in $\mathcal{N}^\star$.*

Theorem 20. *For any Hamiltonian graph G, the partial generalised Hajós graph $[G]$ is $\mathcal{N}^\star$.*

Proof. The function $f(u_{ij}) = \frac{1}{n-1}$ (for all $1 \leq i < j \leq n$) and 0 otherwise is an MFPF. The function $g(v_i) = \frac{1}{2}$ (for all i) and 0 otherwise is an MFDF. Since $f(N[u_{ij}]) < 1$ for each u_{ij}, then by Lemma 17, every MFDF h satisfies $h(u_{ij}) = 0$. Since $g(N[v_i]) \geq \frac{3}{2} > 1$ for each v_i, then by Lemma 16, every MFPF k satisfies $k(v_i) = 0$. Therefore, no MFPF can be an MFDF.

Table 1
The class of the disjoint union of two graphs.

$\cup$	$\mathcal{D}^\star$	$\mathcal{E}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$
$\mathcal{D}^\star$	$\mathcal{D}^\star$	$\mathcal{D}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{I}^\star$
$\mathcal{E}^\star$	$\mathcal{D}^\star$	$\mathcal{E}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$
$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{I}^\star$
$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$
$\mathcal{P}^\star$	$\mathcal{I}^\star$	$\mathcal{P}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$

3.5. Disjoint unions

Domination and neighbourhood packing (in both their integer and fractional forms) interact quite simply with disjoint unions.

Lemma 21. *The function $f : V(G \cup H) \to [0, 1]$ is a fractional dominating (packing) function on $G \cup H$ if and only if $f|_G$ is dominating (packing) on G and $f|_H$ is dominating (packing) on H.*

Using this, we can determine the class of a disjoint union of two graphs, given the classes of the two starting graphs. The results are easy to check, and are summarised in Table 1.

3.6. Strong direct products

The *strong direct product* of G and H, denoted by $G \boxtimes H$, has vertex set $V(G) \times V(H)$, with vertices (u, w) and (v, x) being adjacent in $G \boxtimes H$ precisely when $u \in N_G[v]$ and $w \in N_H[x]$. The interaction between fractional domination and strong direct products is studied in [5]; the following facts are observed there, which we state as lemmas.

Lemma 22. $\gamma_f(G \boxtimes H) = \gamma_f(G)\gamma_f(H)$.

Lemma 23. *Let P be an $m \times k$ matrix, Q be an $s \times t$ matrix, x and z be k-vectors, y and w be t-vectors, and $\otimes$ denote the tensor product. Then:*

(1) $(P \otimes Q)(x \otimes y) = (Px) \otimes (Qy)$.
(2) *If $x \geq z \geq \mathbf{0}$ and $y \geq w \geq \mathbf{0}$, then $x \otimes y \geq z \otimes w$.*
(3) *Let G and H be graphs with adjacency matrices A_G and A_H, respectively; then the adjacency matrix of their product $A_{G\boxtimes H}$ is given by $A_G \otimes A_H$.*

Theorem 24. *Let x_1 and x_2 be MFDFs on G and H, respectively. Then $x^* = x_1 \otimes x_2$ is an MFDF on $G \boxtimes H$.*

Proof.

$$\begin{aligned} A_{G\boxtimes H}x^* &= (A_G \otimes A_H)(x_1 \otimes x_2) \\ &= (A_G x_1) \otimes (A_H x_2) \\ &\geq \mathbf{1}_{|V(G)|} \otimes \mathbf{1}_{|V(H)|} \\ &= \mathbf{1}_{|V(G\boxtimes H)|}. \end{aligned}$$

This shows $x_1 \otimes x_2$ is an FDF on $G \boxtimes H$; $x_1 \otimes x_2$ is an MFDF by Lemma 22.

An analogous proof gives us:

Theorem 25. *Let y_1 and y_2 be MFPFs on G and H, respectively. Then $y^* = y_1 \otimes y_2$ is an MFPF on $G \boxtimes H$.*

This shows that the properties of being dominating and packing are maintained in products; we can also show that the properties of being non-dominating and non-packing are likewise preserved.

Lemma 26. *If f_1 and f_2 are MFDFs on G and H, respectively, with at least one of f_1 and f_2 not packing; then $f_1 \otimes f_2$ is an MFDF on $G \boxtimes H$ which is not packing.*

Proof. From Theorem 24 we have that $f_1 \otimes f_2$ is an MFDF. Suppose f_1 is not a packing. To show that $f_1 \otimes f_2$ is not packing, let $u \in V(G)$ such that $f_1(N[u]) > 1$; such a vertex must exist, since otherwise f_1 would be an FPF. Since the weight of a vertex in the strong direct product equals the product of the weights on its component vertices, then by part 3 of Lemma 23, we can see that $(f_1 \otimes f_2)(N[(u, w)]) = f_1(N[u])f_2(N[w]) > 1$, and hence $f_1 \otimes f_2$ is not packing.

Lemma 27. *If f_1 and f_2 are MFPFs on G and H, respectively, with at least one of f_1 and f_2 not dominating; then $f_1 \otimes f_2$ is an MFPF on $G \boxtimes H$ which is not dominating.*

Table 2
The class of the strong direct product of two graphs.

$\boxtimes$	$\mathcal{D}^\star$	$\mathcal{E}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$
$\mathcal{D}^\star$	$\mathcal{D}^\star$	$\mathcal{D}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{I}^\star$
$\mathcal{E}^\star$	$\mathcal{D}^\star$	$\mathcal{E}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$
$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{I}^\star$
$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$	$\mathcal{N}^\star$
$\mathcal{P}^\star$	$\mathcal{I}^\star$	$\mathcal{P}^\star$	$\mathcal{I}^\star$	$\mathcal{N}^\star$	$\mathcal{P}^\star$

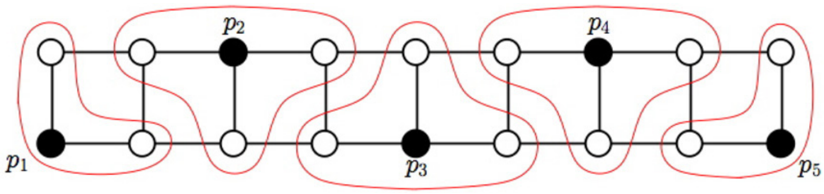

Fig. 9. A neighbourhood partition of $V(P_2\square P_9)$.

Proof. As above, with the inequalities reversed.

Together, these give:

Theorem 28. *The class of $G\boxtimes H$ is determined by the Table 2, where the first row is the class of G and the first column is the class of H.*

Proof. Theorems 24 and 25 show that the tensor product of MFDFs and MFPFs are themselves MFDFs and MFPFs of the product graph, and hence if f_1 and f_2 are FDPFs of G and H, respectively, then $f_1\otimes f_2$ is an FDPF of $G\boxtimes H$. Further, Lemma 27 can be used to find an MFPF which is not dominating if at least one of G and H is $\mathcal{P}^\star$ or $\mathcal{I}^\star$. Lemma 26 can be used to find an MFDF which is not packing, if at least one of G and H is $\mathcal{D}^\star$ or $\mathcal{I}^\star$. Thus, if one of G and H is $\mathcal{I}^\star$ and the other is not $\mathcal{N}^\star$, then $G\boxtimes H\in\mathcal{I}^\star$. If at least one of G, H is $\mathcal{N}^\star$, then $G\boxtimes H\in\mathcal{N}^\star$. The remaining cases are left to the reader.

Note that this table is identical to the one for disjoint unions.

Other graph products are not nearly as well-behaved with respect to our classification. As a demonstration, here is a classification of the ladders: graphs of the form $P_2\square P_k$.

Theorem 29. *Let G be the 2 by n grid graph $P_2\square P_n$. Then for $n>1$ we have:*

$$P_2\square P_n\in\begin{cases}\mathcal{E}^\star, & n\equiv 0 \mod 2\\ \mathcal{D}^\star, & n\equiv 1 \mod 2.\end{cases}$$

Proof. We consider odd and even values of n separately.

Case 1: $n=2k$. For $k=1$ we have C_4 which is regular. For $k>1$ order the vertices of $P_2\square P_{2k}$ as $\{v_{1,1},\dots,v_{1,2k};v_{2,1},\dots,v_{2,2k}\}$. The function

$$f(v_{i,j})=\begin{cases}\dfrac{j/2}{2k+1}, & j\equiv 0 \mod 2\\ \dfrac{k-((j-1)/2)}{2k+1}, & j\equiv 1 \mod 2\end{cases}$$

is an FDPF which has positive weights on each vertex so $P_2\square P_{2k}\in\mathcal{E}^\star$ by Corollary 7.

Case 2: $n=2k+1$. For $k\geq 1$ we can find a partition of vertices into closed neighbourhoods, that is we can find $k+1$ vertices $p_1,\dots,p_{k+1}$ so that each vertex of G is in precisely one closed neighbourhood. The vertices p_i are straightforward to find; Fig. 9 gives a depiction of such a partitioning of $V(P_2\square P_9)$ into the closed neighbourhoods $\{N[p_1],\dots,N[p_5]\}$, where the p_i are coloured black. In fact, there is a formula for finding the p_i based on the ordering used in case 1: $\{p_1,\dots,p_{k+1}\}=\{v_{1,1},v_{3,2},v_{5,1},\dots\}$, where p_{k+1} is $v_{2k+1,1}$ if k is even, or $v_{2k+1,2}$ if k is odd.

For $k=1$ we have a partition using the vertices p_1 and p_2. The function which assigns 1 to each p_i and 0 otherwise is an FDPF. Now consider the constant function which assigns the weight of $\frac{1}{3}$ to each vertex; this function is an MFDF which is not packing. Therefore by Lemma 5, $P_2\square P_3\in\mathcal{D}^\star$.

For $k\geq 2$ we have a partition using the vertices p_1, p_{k+1} of degree two and $p_2,\dots,p_k$ of degree three. The function which assigns 1 to each p_i and 0 otherwise is an FDPF. The function which assigns the weight of 0 to each of $\{p_2,\dots,p_k\}$ and $\frac{1}{3}$ otherwise is an MFDF which is not packing. Taking a convex combination of these two functions we have an MFDF with positive weights on each vertex. Therefore by Lemma 5, $P_2\square P_{2k+1}\in\mathcal{D}^\star$.

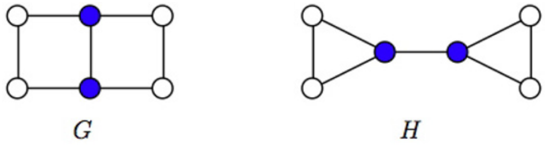

Fig. 10. G and H are fractionally isomorphic, with $G \in \mathcal{D}^\star$, thus $H \in \mathcal{D}^\star$.

4. Fractional isomorphisms and equitable partitions

Let G and H be two graphs with adjacency matrices A and B respectively. We say G and H are *fractionally isomorphic* if and only if there exists a doubly stochastic matrix S so that $AS = SB$; we denote this relationship by $G \cong_f H$.

Theorem 30. *If two graphs G and H are fractionally isomorphic, then they belong to the same class.*

Proof. We proceed by considering the action of the matrix S on a function f; specifically, we shall show that Sf has the property of being minimum fractional dominating (or maximum fractional packing) on G if f has that property on H. Suppose A and B are adjacency matrices of G and H respectively and S is a doubly stochastic matrix such that $AS = SB$. Suppose that f is an MFDF on H; then $(B + I)\boldsymbol{f} = \mathbf{1} + \epsilon$, where $\epsilon \geq \mathbf{0}$. Then:

$$\begin{aligned} N(S\boldsymbol{f}) &= (NS)\boldsymbol{f} \\ &= (AS + IS)\boldsymbol{f} \\ &= (SB + SI)\boldsymbol{f} \\ &= S((B + I)\boldsymbol{f}) \\ &= S(\mathbf{1} + \epsilon) \\ &= \mathbf{1} + S\epsilon. \end{aligned}$$

Since both S and ϵ are non-negative, so is their product. Therefore, Sf is an MFDF on G (note that Sf is minimum, since $|Sf| = |f|$ and $\gamma_f(G) = \gamma_f(H)$ as shown in Section 2). Further, $S\epsilon = \mathbf{0}$ if and only if $\epsilon = \mathbf{0}$. Hence, if f is an MFDF but not packing in H, then the same goes for Sf in G.

A similar demonstration will reveal that if f is a maximum fractional packing on H (and thus $(B + I)\boldsymbol{f} = \mathbf{1} - \epsilon$ for some nonnegative vector ϵ), then Sf is a maximum fractional packing on G, and likewise that the property of being non-dominating is preserved.

To complete the proof, note that fractional isomorphism is an equivalence relation, and hence symmetric; specifically, if $AS = SB$, then $BS^T = S^T A$. Hence, S sends D_H into D_G, P_H into P_G and $D_H \cap P_H$ into $D_G \cap P_G$. Further, S^T sends D_G into D_H, P_G into P_H and $D_G \cap P_G$ into $D_H \cap P_H$, hence the two graphs share a class (Fig. 10).

Although being in the same class is a necessary condition for two graphs to be fractionally isomorphic, it is not sufficient. Both $K_{2,3}$ and C_5 are in $\mathcal{E}^\star$, however, they are not fractionally isomorphic to each other (since their degree sequences are different).

Let $\mathcal{C} = \{V_1, \dots, V_r\}$ be an equitable partition of the vertices $v_1, \dots, v_n$ of G. Define the matrix $S^{(\mathcal{C})}$ by:

$$S^{(\mathcal{C})}_{i,j} = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are in different cells of } \mathcal{C} \\ |V_k|^{-1} & \text{if } v_i \text{ and } v_j \text{ are both in } V_k. \end{cases}$$

Theorem 31. *Let f be a fractional dominating (or packing) function on G. Then $f_{\mathcal{C}} = S^{(\mathcal{C})} f$ is a fractional dominating (packing) function on G with the property that, if v_i and v_j belong to the same cell of $\mathcal{C}$, then $f_{\mathcal{C}}(v_i) = f_{\mathcal{C}}(v_j)$.*

Proof. First, we show that $S^{(C)}$ is a fractional automorphism of G (with adjacency matrix A). To show that $S^{(C)}A = AS^{(C)}$, it suffices to show that either of these products is symmetric. Consider the element $(AS^{(\mathcal{C})})_{i,j} = \sum_k A_{i,k} S^{(\mathcal{C})}_{k,j}$ and its image under transposition. Let us say that $v_i \in V_a$ and $v_j \in V_b$; by the construction of the two matrices, it is clear that $(AS^{(\mathcal{C})})_{i,j} = \frac{d(v_i, V_b)}{|V_b|}$, and similarly $(AS^{(\mathcal{C})})_{j,i} = \frac{d(v_j, V_a)}{|V_a|}$. If $a = b$ then these two quantities are equal, since $G[V_a]$ is regular. If $a \neq b$, then we observe the two quantities to be equal from $d(v_i, V_b)|V_a| = d(v_j, V_a)|V_b|$; this equation results from counting the edges of the bipartite graph $G[V_i, V_j]$ two different ways. Therefore $S^{(\mathcal{C})}$ is a fractional automorphism of G.

It is proved in [12], that if S is a fractional automorphism of G and if f is a fractional dominating or packing function, then so is Sf. Thus, to complete the proof, we only need show that the product function is constant on each cell of the equitable partition. This follows from the observation that, if $V_i = \{v_{i_1}, \dots, v_{i_m}\}$, then for any k, $1 \leq k \leq m$ we have $S^{(\mathcal{C})} f(v_{i_k}) = \frac{1}{m} \sum_j f(v_{i_j})$.

Corollary 32. *Let $\mathcal{C}$ be an equitable partition of G. If G has an MFDF which is non-packing, an MFPF which is non-dominating, or an FDPF, then it has such a function which is constant on each cell of C.*

Suppose that f is a function on the vertex set of G which is constant on the cells of $\mathcal{C}$, and define a new function $f^{(\mathcal{C})}$ on the cells of $\mathcal{C}$ such that $f^{(\mathcal{C})}(V_i) = f(x_i)$ for $x_i \in V_i$. Then, clearly $(A^{(\mathcal{C})} + I)\boldsymbol{f}^{(\mathcal{C})} \geq \mathbf{1}$ if and only if $N\boldsymbol{f} \geq \mathbf{1}$, and likewise if f is a maximum fractional packing or an FDPF. Note that in the corollary below, the graphs G and H need not have the same order.

Corollary 33. *Suppose that G and H have identical cell adjacency matrices for some equitable partitions $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Then G and H belong to the same class.*

Thus, finding equitable partitions can make discovering fractional dominating and packing functions easier. It should be noted that the natural "reduced" linear program for fractional domination – Minimise $c^T x$ subject to $(A^{(\mathcal{C})} + I)x \geq \mathbf{1}$, $x \geq \mathbf{0}$ where c is the vector $(|V_1|, \ldots, |V_r|)^T$ – is no longer the dual to the corresponding "reduced" program for fractional packing since the cell-adjacency matrix need not be symmetric.

Corollary 33, also gives an alternative proof to Theorem 30, since two graphs are fractionally isomorphic if and only if they share some equitable partition (see [6]).

5. Quo vadimus

The material in this paper appeared in [16] in a fuller form, along with several further extrapolations and applications of our methods. For instance, one can consider *total domination*, where a vertex dominates the members of its open neighbourhood but not itself; the dual notion is *open neighbourhood packing*, and one can define the fractional variations and the classes of graphs analogously. Some of this is developed in the dissertation cited above; as one might expect, a graph's class with respect to fractional total domination is not necessarily the same as for ordinary fractional domination, although there are large families of graphs (i.e. regular graphs) for which congruency does hold.

There are numerous other variations on the theme of domination in [8,9]; many of our methods could be adapted to consider those parameters. One possible approach might be to consider the "natural" graph products associated with each parameter; for instance, the strong direct product acts naturally with respect to fractional domination, and it seems that the categorical product plays an analogous role with respect to fractional total domination.

[10] looks at the effects of small perturbations of graphs (the addition and deletion of single vertices or edges) on their domination numbers. We could ask similar questions in this setting: given a graph in a given class, what can we say about the class of the graph that results from deleting an edge or a vertex?

We are particularly interested in the above question for trees. While categorising all graphs into the five classes may be overly ambitious, we feel that there should be an accessible algorithmic method for determining the class of any tree. One approach which we have been pursuing is to examine which trees are in $\mathcal{N}^\star$, and devising measures for quantifying how far a $\mathcal{N}^\star$ tree is from being "partitionable".

The theory of efficient domination (see [2]), particularly the efficient fractional domination number, should be applicable here. Recall, that if there exists an efficient fractional dominating function on a graph G, then the efficient fractional domination number, $F_f(G) = n$. Any fractional efficient dominating function would also be a fractional packing, since by definition, the function g would satisfy $g(N[v]) = 1$ for all $v \in V(G)$. Thus, any graph G on n vertices in $\mathcal{I}^\star$, $\mathcal{D}^\star$, $\mathcal{P}^\star$, or $\mathcal{E}^\star$, would have $F_f(G) = n$; and if G is $\mathcal{N}^\star$, then $F_f(G) < n$.

Acknowledgements

We thank Pete Johnson and Pete Slater as well as the anonymous reviewers for their helpful comments and suggestions with this paper.

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