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More on almost self-complementary graphs

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ABSTRACT

A graph X is called almost self-complementary if it is isomorphic to one of its almost complements $X^c - \mathcal{I}$, where X^c denotes the complement of X and $\mathcal{I}$ a perfect matching (1-factor) in X^c. If $\mathcal{I}$ is a perfect matching in X^c and $\varphi : X \to X^c - \mathcal{I}$ is an isomorphism, then the graph X is said to be fairly almost self-complementary if φ preserves $\mathcal{I}$ setwise, and unfairly almost self-complementary if it does not.

In this paper we construct connected graphs of all possible orders that are fairly and unfairly almost self-complementary, fairly but not unfairly almost self-complementary, and unfairly but not fairly almost self-complementary, respectively, as well as regular graphs of all possible orders that are fairly and unfairly almost self-complementary.

Two perfect matchings $\mathcal{I}$ and $\mathcal{J}$ in X^c are said to be X-non-isomorphic if no isomorphism from $X + \mathcal{I}$ to $X + \mathcal{J}$ induces an automorphism of X. We give a constructive proof to show that there exists a graph X that is almost self-complementary with respect to two X-non-isomorphic perfect matchings for every even order greater than or equal to four.

1. Introduction

A graph X is said to be *self-complementary* if it is isomorphic to its complement X^c. Similarly, a graph X on an even number of vertices is said to be *almost self-complementary* if it is isomorphic to a graph (called an *almost complement of* X) obtained from X^c by removing the edges of a 1-factor of X^c. This definition was introduced by Alspach, who also proposed the determination of all possible orders of almost self-complementary circulant graphs. In [2], this problem was solved for a particularly "nice" subclass of almost self-complementary circulants (called cyclically almost self-complementary), while general almost self-complementary graphs, vertex-transitive almost self-complementary graphs, and almost self-complementary double covers were first studied in [3], [4], and [5], respectively. Almost self-complementary graphs on one hand represent a generalization of self-complementary graphs to graphs of even order (a generalization that is particularly suitable for regular graphs) and on the other hand one of the simplest examples of index-2 isomorphic factorizations of graphs that are not complete.

In this paper, we answer some of the open questions from [3]. In particular, in Section 3 we construct connected graphs of all possible orders that are fairly and unfairly almost self-complementary, fairly but not unfairly almost self-complementary, and unfairly but not fairly almost self-complementary, respectively, while in Section 4 we construct regular graphs of all possible orders that are fairly and unfairly almost self-complementary. In Section 2 we introduce the notation and terminology, and in Section 5 we construct, for every even order greater than or equal to four, a graph X that is almost self-complementary with respect to two X-non-isomorphic perfect matchings.

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2. Preliminaries

We begin by reviewing some basic terms from graph theory and setting up the notation before introducing terminology specific to almost self-complementary graphs. For terminology and concepts from the theory of permutation groups, the reader is referred to [1]. We note, however, that superscript notation will be used here for group action. That is, if G is a subgroup of the symmetric group Sym_Ω acting on a set Ω, then the image of a point $a \in \Omega$ by an element $\alpha \in G$ will be denoted a^α.

For a set V, let $V^{(2)}$ denote the set $\{\{u, v\} : u, v \in V, u \neq v\}$. All graphs in this paper are simple; that is, a *graph* is an ordered pair (V, E), where V is any finite non-empty set and E any subset of $V^{(2)}$. The *vertex set* V and *edge set* E of a graph $X = (V, E)$ will be denoted by $V(X)$ and $E(X)$, respectively. The *complement* of a graph $X = (V, E)$ is the graph $(V, V^{(2)} - E)$ denoted by X^c. If v is a vertex of a graph X, then the *neighbourhood* of v in X (that is, the set of all vertices adjacent to v in X) is denoted by $\mathrm{N}_X(v)$, and its size is the *degree* of v in X, denoted by $\deg_X(v)$. In these symbols, the subscript X may be omitted if it is clear from the context what the graph X is. A partition of a set V into subsets of size two is called a *perfect matching on* V. If $X = (V, E)$ is a graph and $\mathcal{I}$ a perfect matching on V with $\mathcal{I} \subseteq E$, then we say that $\mathcal{I}$ is a *perfect matching in* X.

As usual, the symbols K_n, P_n, and C_n will denote the *complete graph* with n vertices, the *path* with n edges, and the *cycle* with n vertices, respectively. We shall write $G + H$ to denote the union of the graphs G and H where it is assumed that the vertex sets of the two graphs are disjoint. For example, $P_2 + K_1$ denotes the graph with four vertices and two adjacent edges.

Let X be a graph with vertex set V of even size and let $\mathcal{I}$ be a perfect matching in the complement X^c. An *almost complement of X with respect to* $\mathcal{I}$, denoted by $\mathrm{AC}_\mathcal{I}(X)$, is the graph with vertex set V and edge set $V^{(2)} - (E(X) \cup \mathcal{I})$. A graph is called an *almost complement* of X, if it is the almost complement of X with respect to some perfect matching in X^c. A graph X is said to be *almost self-complementary with respect to a perfect matching* $\mathcal{I}$ in X^c if it is isomorphic to $\mathrm{AC}_\mathcal{I}(X)$. A graph X is called *almost self-complementary* if there exists a perfect matching $\mathcal{I}$ in X^c such that X is isomorphic to $\mathrm{AC}_\mathcal{I}(X)$.

An $\mathcal{I}$*-antimorphism* of a graph $X = (V, E)$ is any permutation $\varphi \in \mathrm{Sym}_V$ such that $V^{(2)}$ is a disjoint union of E, E^φ, and $\mathcal{I}$. By an *antimorphism* of an almost self-complementary graph X we mean an $\mathcal{I}$-antimorphism of X for some perfect matching $\mathcal{I}$ in X^c. An $\mathcal{I}$-antimorphism φ of a graph $X = (V, E)$ is called *fair* if $\mathcal{I}^\varphi = \mathcal{I}$, and *unfair* otherwise. Similarly, an automorphism α of X is called $\mathcal{I}$*-fair* if $\mathcal{I}^\alpha = \mathcal{I}$.

Let X be an almost self-complementary graph and $\mathcal{I}$ a perfect matching in X^c. Then X is called $\mathcal{I}$*-fairly* ($\mathcal{I}$*-unfairly*) *almost self-complementary* if it admits a fair (respectively, unfair) $\mathcal{I}$-antimorphism. A graph is called *fairly (unfairly) almost self-complementary* if it is $\mathcal{I}$-fairly (respectively, $\mathcal{I}$-unfairly) almost self-complementary for some perfect matching $\mathcal{I}$ in X^c.

We shall now describe two constructions, introduced in [3], that produce new almost self-complementary graphs from old. They will be used in Sections 3–5 to prove our results. In this paper, by the term *ordered bipartition* of a set V we mean an ordered pair (V_1, V_2) of subsets of V such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.

Let X and Y be graphs with disjoint vertex sets, and let (U_1, U_2) be an ordered bipartition of $V(X)$. The *skew join* of X and Y with respect to the ordered bipartition (U_1, U_2) of $V(X)$, denoted by $(X, (U_1, U_2)) \blacktriangleleft Y$, is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{xy : x \in U_1, y \in V(Y)\}$. We write shortly $X \blacktriangleleft Y$ for $(X, (U_1, U_2)) \blacktriangleleft Y$ if it is understood what the ordered bipartition is.

Let X and Y be graphs with disjoint vertex sets, and let (U_1, U_2) and (W_1, W_2) be ordered bipartitions of $V(X)$ and $V(Y)$, respectively. The *partial join* of X and Y with respect to ordered bipartitions (U_1, U_2) and (W_1, W_2), denoted by $(X, (U_1, U_2)) \blacklozenge (Y, (W_1, W_2))$, is the graph with vertex set $V(X) \cup V(Y)$ and edge set $E(X) \cup E(Y) \cup \{uw \mid u \in U_i, w \in W_i, i \in \{1, 2\}\}$. We write shortly $X \blacklozenge Y$ for $(X, (U_1, U_2)) \blacklozenge (Y, (W_1, W_2))$ if it is understood what the two ordered bipartitions are.

The following two lemmas from [3] explain the importance of skew joins and partial joins for constructing almost self-complementary graphs. The first can be easily obtained from [3, Lemma 3.2] and its proof, while the second was first published as [3, Lemma 3.5].

Lemma 2.1 ([3]). *Let X and Y be almost self-complementary graphs of orders* $2m$ *and* $2n$*, respectively. Let* (U_1, U_2) *be an ordered bipartition of* $V(X)$ *such that* $|U_1| = |U_2|$*, and suppose X admits an antimorphism* φ *with* $U_1^\varphi = U_2$*. Then:*

(i) *The skew join* $X \blacktriangleleft Y$ *with respect to the ordered bipartition* (U_1, U_2) *of* $V(X)$ *is an almost self-complementary graph. For any antimorphism* ψ *of Y, the permutation* (φ, ψ) *that acts as* φ *on* $V(X)$ *and as* ψ *on* $V(Y)$ *is an antimorphism of* $X \blacktriangleleft Y$*.*
(ii) *If* $\max\{\deg_X(x) : x \in U_2\} + \max\{\deg_Y(y) : y \in V(Y)\} < m + 2n - 2$*, then every automorphism of* $X \blacktriangleleft Y$ *induces an automorphism of Y.*

Lemma 2.2 ([3]). *Let X and Y be almost self-complementary graphs with disjoint vertex sets admitting ordered bipartitions* (U_1, U_2) *of* $V(X)$ *and* (W_1, W_2) *of* $V(Y)$*, and antimorphisms* φ *of X and* ψ *of Y such that* $U_1^\varphi = U_2$ *and* $W_1^\psi = W_1$*. Let* (φ, ψ) *denote the permutation on* $V(X) \cup V(Y)$ *that acts as* φ *on* $V(X)$ *and as* ψ *on* $V(Y)$*. Then:*

(i) *The partial join* $X \blacklozenge Y$ *with respect to the given ordered bipartitions is an almost self-complementary graph.*
(ii) *If both X and Y are regular graphs and* $|W_1| = |W_2|$*, then* $X \blacklozenge Y$ *is also regular.*
(iii) *If* φ *and* ψ *are fair antimorphisms of X and Y, respectively, then* (φ, ψ) *is a fair antimorphism of* $X \blacklozenge Y$*.*
(iv) *If at least one of* φ *and* ψ *is unfair, then* (φ, ψ) *is an unfair antimorphism of* $X \blacklozenge Y$*.*

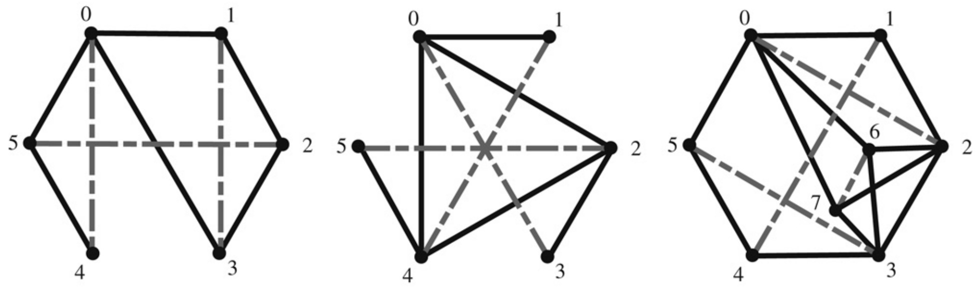

Fig. 1. Graphs D_6 (the "dipper"), W_6 (the "windmill"), and Q_8 (the "quasicube").

3. Constructing connected almost self-complementary graphs

In [3, Corollary 3.3], it is shown that fairly and unfairly, fairly but not unfairly, and unfairly but not fairly almost self-complementary graphs of order $2k$ exist if and only if $k \geq 1$, $k \geq 2$, and $k \geq 4$, respectively. However, most of the graphs constructed there to show existence have an isolated vertex. In this section, we constructively determine the necessary and sufficient conditions on k for there to exist fairly and unfairly, fairly but not unfairly, and unfairly but not fairly almost self-complementary connected graphs of order $2k$. The lemma below will be our main tool.

Lemma 3.1. *Let X be an almost self-complementary graph of order $2m$, (U_1, U_2) an ordered bipartition of $V(X)$, and φ an antimorphism of X with $U_1^{\varphi} = U_2$. Assume that $\deg_X(x) < m$ for all $x \in U_2$. Let Y be a graph isomorphic to ${K_2}^c$ with $V(Y) = \{y_1, y_2\}$ and $V(X) \cap \{y_1, y_2\} = \emptyset$. Furthermore, let $V_i = U_i \cup \{y_i\}$ for $i = 1, 2$. Then the following hold:*

(a) *$X◂Y$ admits an antimorphism ψ with $V_1^{\psi} = V_2$.*
(b) *$\deg_{X◂Y}(y) < m + 1$ for all $y \in V_2$.*
(c) *$X◂Y$ is a fairly (unfairly) almost self-complementary graph if and only if X is.*
(d) *If X is connected, then $X◂Y$ is connected.*

Proof. Statement (a) follows easily from Lemma 2.1 (i).

To see (b), note that $\deg_{X◂Y}(x) = \deg_X(x) < m$ for all $x \in U_2$, and $\deg_{X◂Y}(y_2) = m$.

Since $Y = {K_2}^c$ is fairly almost self-complementary, it is easy to see that a fair (unfair) antimorphism of X gives rise to a fair (unfair) antimorphism of $X◂Y$. Conversely, let Φ be an antimorphism of $X◂Y$. Then $\deg_{X◂Y}(y_i^{\Phi}) = (2m + 2) - 2 - \deg_{X◂Y}(y_i) = m$ for $i = 1, 2$. Since $\deg_{X◂Y}(x) < m$ for all $x \in U_2$ and $\deg_{X◂Y}(x) = \deg_X(x)+2 = (2m-2-\deg_X(x^{\varphi}))+2 > m$ for all $x \in U_1$, we have that $\{y_1, y_2\}^{\Phi} = \{y_1, y_2\}$. Hence Φ induces an antimorphism of X, which is fair (unfair) if and only if Φ is. Thus (c) follows.

Statement (d) is easy to see. ■

Construction 3.2. Let D_6 (Fig. 1, left) be the graph with vertex set $V(D_6) = \mathbb{Z}_6$ and edge set $E(D_6) = \{01, 03, 05, 12, 23, 45\}$. For $k \geq 3$, define the graph D_{2k+2} by

$$D_{2k+2} = D_{2k} ◂ {K_2}^c.$$

The skew join is taken with respect to the bipartition (U_{2k}^1, U_{2k}^2) defined recursively by

$$U_6^1 = \{0, 1, 2\} \quad \text{and} \quad U_6^2 = \{3, 4, 5\},$$

and

$$U_{2k+2}^i = U_{2k}^i \cup \{y_i\} \quad \text{for } i = 1, 2, \text{ and } k \geq 3,$$

where y_1 and y_2 are the vertices of the new copy of ${K_2}^c$ used to form $D_{2k+2} = D_{2k} ◂ {K_2}^c$.

Lemma 3.3. *The graphs D_{2k}, for $k \geq 3$, defined in Construction 3.2 are fairly but not unfairly almost self-complementary and connected.*

Proof. It is easy to check that D_6 admits a fair $\mathcal{I}$-antimorphism $\varphi = (0, 4)(1, 3)(2, 5)$ for $\mathcal{I} = \{04, 13, 25\}$, which interchanges the sets U_6^1 and U_6^2, and that it has no unfair antimorphisms. Moreover, $\deg_{D_6}(x) < 3$ for all $x \in U_6^2$, and so D_6 satisfies the conditions of Lemma 3.1. By induction, using Lemma 3.1, it now follows easily that each of the graphs D_{2k} satisfies the conditions of Lemma 3.1 and is fairly but not unfairly almost self-complementary, as well as connected. ■

Construction 3.4. Let W_6 (Fig. 1, centre) be the graph with vertex set $V(W_6) = \mathbb{Z}_6$ and edge set $E(W_6) = \{01, 02, 04, 23, 24, 45\}$. For $k \geq 3$, define the graph W_{2k+2} by

$$W_{2k+2} = W_{2k} \blacktriangleleft {K_2}^{c}.$$

The skew join is taken with respect to the bipartition (V_{2k}^1, V_{2k}^2) defined recursively by

$$V_6^1 = \{0, 2, 4\} \quad \text{and} \quad V_6^2 = \{1, 3, 5\}$$

and

$$V_{2k+2}^i = V_{2k}^i \cup \{y_i\} \quad \text{for } i = 1, 2, \text{ and } k \geq 3,$$

where y_1 and y_2 are the vertices of the new copy of ${K_2}^c$ used to form $W_{2k+2} = W_{2k} \blacktriangleleft {K_2}^c$.

Lemma 3.5. *The graphs W_{2k}, for $k \geq 3$, defined in Construction 3.4 are fairly and unfairly almost self-complementary and connected.*

Proof. It is easy to check that W_6 admits a fair $\mathcal{I}$-antimorphism $\varphi = (0, 1, 2, 3, 4, 5)$ and an unfair $\mathcal{I}$-antimorphism $\psi = (0, 3, 2, 1, 4, 5)$ for $\mathcal{I} = \{03, 14, 25\}$, each of which interchanges the sets V_6^1 and V_6^2. Moreover, $\deg_{W_6}(x) < 3$ for all $x \in V_6^2$, and so W_6 satisfies the conditions of Lemma 3.1. By induction, using Lemma 3.1, each of the graphs W_{2k} satisfies the conditions of Lemma 3.1 and is fairly and unfairly almost self-complementary, as well as connected. ■

Construction 3.6. Let Q_8 (Fig. 1, right) be the graph with vertex set $V(Q_8) = \mathbb{Z}_8$ and edge set $E(Q_8) = \{01, 05, 06, 07, 12, 23, 26, 27, 34, 36, 37, 45\}$. For $k \geq 4$, define the graph Q_{2k+2} by

$$Q_{2k+2} = Q_{2k} \blacktriangleleft {K_2}^c.$$

The skew join is taken with respect to the bipartition (S_{2k}^1, S_{2k}^2) defined recursively by

$$S_6^1 = \{0, 2, 3, 6\} \quad \text{and} \quad S_6^2 = \{1, 4, 5, 7\}$$

and

$$S_{2k+2}^i = S_{2k}^i \cup \{y_i\} \quad \text{for } i = 1, 2, \text{ and } k \geq 4,$$

where y_1 and y_2 are the vertices of the new copy of ${K_2}^c$ used to form $Q_{2k+2} = Q_{2k} \blacktriangleleft {K_2}^c$.

Lemma 3.7. *The graphs Q_{2k}, for $k \geq 4$, defined in Construction 3.6 are unfairly but not fairly almost self-complementary and connected.*

Proof. It is easy to check that Q_8 admits an unfair $\mathcal{I}$-antimorphism $\varphi = (0, 4, 3, 1, 2, 5)(6, 7)$ for $\mathcal{I} = \{02, 14, 35, 67\}$, which interchanges the sets S_8^1 and S_8^2, and it is proved in [3, Lemma 2.13] that it admits no fair antimorphisms. Moreover, $\deg_{Q_8}(x) < 4$ for all $x \in S_8^2$, and so Q_8 satisfies the conditions of Lemma 3.1. Using induction combined with Lemma 3.1, we can see that each of the graphs Q_{2k} satisfies the conditions of Lemma 3.1 and is unfairly but not fairly almost self-complementary, as well as connected. ■

Theorem 3.8. *There exists a connected fairly and unfairly (fairly but not unfairly) almost self-complementary graph of order $2k$ if and only if $k \geq 3$. There exists a connected unfairly but not fairly almost self-complementary graph of order $2k$ if and only if $k \geq 4$.*

Proof. The list of all almost self-complementary graphs of order at most six in [3, Figures 1,2, and 3] shows that there is no connected almost self-complementary graph of order less than six, and no unfairly but not fairly almost self-complementary graph of order less than eight. Hence the conditions on k stated in the theorem are necessary. Lemmas 3.3, 3.5, and 3.7 then prove sufficiency. ■

4. Constructing regular almost self-complementary graphs

In [3, Corollary 4.4] it is shown that there exist regular fairly almost self-complementary graphs of all even orders, and regular fairly and unfairly almost self-complementary graphs of all orders divisible by eight. In this section we show that there exist regular fairly and unfairly almost self-complementary graphs of all even orders.

We begin with yet another general construction for almost self-complementary graphs, which, in a special case, yields regular graphs with the desired property.

Construction 4.1. Let X be any graph. Denote by X_1 and X_4 two isomorphic copies of X, and by X_2 and X_3 two isomorphic copies of X^c, all pairwise vertex-disjoint. Let $\iota_i : X \to X_i$ for $i = 1, 4$, and $\iota_i : X \to {X_i}^c$ for $i = 2, 3$, be isomorphisms. Then for each $x \in V(X)$ and $i \in \{1, 2, 3, 4\}$, let $x_i = \iota_i(x)$. Now define the graph $\mathrm{U}(X) = (V, E)$ by

$$V = \bigcup_{i=1}^{4} V(X_i),$$

$$E = \bigcup_{i=1}^{4} E(X_i) \cup \{x_i y_{i+1} : x, y \in V(X), i = 1, 3\} \cup \{x_2 y_3 : x, y \in V(X), x \neq y\}.$$

Lemma 4.2. *For any graph X, the graph $\mathrm{U}(X)$ defined in Construction 4.1 admits a fair antimorphism $\varphi = \prod_{x \in V(X)} (x_1, x_3, x_4, x_2)$. If X has a non-trivial automorphism α, then $\mathrm{U}(X)$ also admits an unfair antimorphism $\psi = \prod_{x \in V(X)} (x_1, x_3, x_4, (x^\alpha)_2)$.*

Proof. Define a perfect matching $\mathcal{I}$ in $\mathrm{U}(X)^c$ by $\mathcal{I} = \{x_1 x_4 : x \in V(X)\} \cup \{x_2 x_3 : x \in V(X)\}$. It is then not difficult to verify that φ is a fair $\mathcal{I}$-antimorphism of $\mathrm{U}(X)$. Similarly, it can be verified that ψ is an $\mathcal{I}$-antimorphism of $\mathrm{U}(X)$. If $x \in V(X)$ is such that $x^\alpha = y \neq x$, then $(x_1 x_4)^\psi = x_3 y_2 \neq x_3 x_2$ so $(x_1 x_4)^\psi \notin \mathcal{I}$. Thus ψ is unfair. ■

Theorem 4.3. *There exists a regular fairly and unfairly almost self-complementary graph of order $2k$ if and only if $k \geq 2$.*

Proof. The list of all almost self-complementary graphs of order at most six in [3, Figures 1 and 2] shows that there is no fairly and unfairly almost self-complementary graph of order two, and that there exist regular fairly and unfairly almost self-complementary graphs of orders four and six (namely, $K_2 + K_2$ and C_6, respectively). It thus suffices to show existence of regular fairly and unfairly almost self-complementary graphs of all even orders greater than or equal to eight.

For any $n \geq 2$, the complete graph K_n admits a non-trivial automorphism, and so it follows directly from Lemma 4.2 that the graph $\mathrm{U}(K_n)$, as defined in Construction 4.1, is fairly and unfairly almost self-complementary. Moreover, it is not difficult to see that $\mathrm{U}(K_n)$ is regular of degree $2n - 1$. We thus have regular fairly and unfairly almost self-complementary graphs of all orders divisible by four.

Fix an integer $n \geq 2$ and let $X = K_n$. With the notation of Construction 4.1, let φ and ψ be the fair and unfair antimorphism of $\mathrm{U}(X)$ defined in Lemma 4.2. Let $U_1 = V(X_1) \cup V(X_4)$ and $U_2 = V(X_2) \cup V(X_3)$, and observe that both φ and ψ interchange the sets U_1 and U_2. Now let (W_1, W_2) be an ordered bipartition of ${K_2}^c$ into two singletons, and construct a graph $Y = \mathrm{U}(X) \blacklozenge {K_2}^c$ as the partial join of $\mathrm{U}(X)$ and ${K_2}^c$ with respect to the ordered bipartitions (U_1, U_2) and (W_1, W_2). Since the identity permutation on $V({K_2}^c)$ is a fair antimorphism of ${K_2}^c$, it follows from Lemma 2.2 that Y is a regular fairly and unfairly almost self-complementary graph of order $4n + 2$. This completes the proof. ■

Closely related to the result of Theorem 4.3 is the following open problem.

Problem 4.4. Determine the necessary and sufficient conditions on the order of a graph that is fairly but not unfairly (unfairly but not fairly, respectively) almost self-complementary.

Observe that Problem 4.4 is much more difficult than the result of Theorem 4.3 since showing that a graph X is fairly but not unfairly almost self-complementary, for example, requires finding a perfect matching $\mathcal{I}$ in X^c and a fair $\mathcal{I}$-antimorphism of X, as well as showing that no perfect matching $\mathcal{J}$ in X^c admits an unfair $\mathcal{J}$-antimorphism.

5. Graphs that are almost self-complementary with respect to two non-isomorphic perfect matchings

The list of all almost self-complementary graphs of order at most 6 in [3] shows that each of these graphs is almost self-complementary with respect to an "up-to-isomorphism unique" perfect matching. In this section we explain precisely what we mean by "up-to-isomorphism unique", and show that for each order greater than 6 there exists an almost self-complementary graph that admits two "non-isomorphic" perfect matchings.

If $X = (V, E)$ is a graph and $\mathcal{I}$ a perfect matching in the complement X^c, then we let $X + \mathcal{I}$ denote the graph $(V, E \cup \mathcal{I})$.

Definition 5.1. Let X be a graph that is almost self-complementary with respect to perfect matchings $\mathcal{I}$ and $\mathcal{J}$ in X^c. Then $\mathcal{I}$ and $\mathcal{J}$ are called X-*isomorphic* if there exists an isomorphism from $X + \mathcal{I}$ to $X + \mathcal{J}$ that induces an automorphism of the graph X. Otherwise, $\mathcal{I}$ and $\mathcal{J}$ are called X-*non-isomorphic*.

Below, we construct graphs X that are almost self-complementary with respect to two X-non-isomorphic perfect matchings (to be proved in Theorem 5.3).

Construction 5.2. Let Z (Fig. 2, left) be the graph with vertex set $V(Z) = \{x, y, z, w\}$, edge set $E(Z) = \{yz, zw\}$, and ordered bipartition (U_1, U_2) with $U_1 = \{x, y\}$ and $U_2 = \{z, w\}$. For every $k \geq 4$ we define a graph X_{2k} of order $2k$ as follows (see Fig. 2, centre and right). Let

$$V(X_8) = \mathbb{Z}_8,$$

$$E(X_8) = \{03, 05, 07, 12, 13, 14, 25, 26, 34, 46, 57, 67\},$$

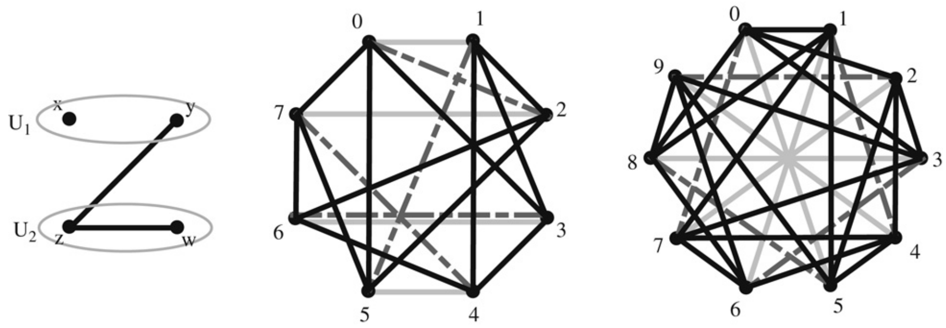

Fig. 2. Graphs Z, X_8, and X_{10}. The graph X_8 (X_{10}) admits two X_8-non-isomorphic (X_{10}-non-isomorphic, respectively) perfect matchings (solid gray and dashed edges).

and

$$V(X_{10}) = \mathbb{Z}_{10},$$
$$E(X_{10}) = \{01, 02, 03, 08, 15, 17, 18, 23, 24, 25, 37, 39, 45, 46, 47, 59, 67, 68, 69, 89\}.$$

Now, for $k \geq 4$ define

$$X_{2k+4} = Z \blacktriangleleft X_{2k},$$

where the skew join is taken with respect to the ordered bipartition (U_1, U_2) of Z.

Theorem 5.3. *For every integer $k \geq 4$ there exists a graph X of order $2k$ that is almost self-complementary with respect to two X-non-isomorphic perfect matchings.*

Proof. We show that the graphs X_{2k} (for $k \geq 4$) defined in Construction 5.2 are almost self-complementary with respect to two X_{2k}-non-isomorphic perfect matchings.

First consider the graph X_8. Then $\mathcal{I}_8 = \{01, 27, 36, 45\}$ and $\mathcal{J}_8 = \{02, 15, 36, 47\}$ are two perfect matchings in X^c, and it is not difficult to check that $\varphi_8 = (2,7)(3,6)(4,5)$ and $\psi_8 = (0,1,2,4,3,7,6,5)$ are an $\mathcal{I}_8$-antimorphism and $\mathcal{J}_8$-antimorphism of X_8, respectively. So X_8 is almost self-complementary with respect to both $\mathcal{I}_8$ and $\mathcal{J}_8$. Although $X_8 + \mathcal{I}_8$ and $X_8 + \mathcal{J}_8$ are isomorphic (since $\mathrm{AC}_{\mathcal{I}_8}(X_8) \cong X_8 \cong \mathrm{AC}_{\mathcal{J}_8}(X_8)$), we can show that $\mathcal{I}_8$ and $\mathcal{J}_8$ are X_8-non-isomorphic. Suppose $\Phi : X_8 + \mathcal{I}_8 \to X_8 + \mathcal{J}_8$ is an isomorphism that induces an automorphism of X_8. Then $\mathcal{I}_8^{\Phi} = \mathcal{J}_8$, and the subgraph of X_8 induced by the endpoints of the two edges 01 and 36 of $\mathcal{I}_8$ must be mapped to a subgraph of X_8 induced by the endpoints of two edges of $\mathcal{J}_8$. However, the subgraph of X_8 induced by the set $\{0, 1, 3, 6\}$ is isomorphic to $P_2 + K_1$, while it can be easily checked that no two edges of $\mathcal{J}_8$ have this property. Hence no isomorphism from $X_8 + \mathcal{I}_8$ to $X_8 + \mathcal{J}_8$ induces an automorphism of X_8, and $\mathcal{I}_8$ and $\mathcal{J}_8$ are X_8-non-isomorphic.

Next, consider X_{10} and the perfect matchings $\mathcal{I}_{10} = \{05, 16, 27, 38, 49\}$ and $\mathcal{J}_{10} = \{07, 14, 29, 36, 58\}$ in ${X_{10}}^c$. It is not difficult to check that $\varphi_{10} = (0,1,2,3,4,5,6,7,8,9)$ and $\psi_{10} = (1,5,3,9)(2,4,8,6)$ are an $\mathcal{I}_{10}$-antimorphism and $\mathcal{J}_{10}$-antimorphism of X_{10}, respectively. Suppose that there is an isomorphism $\Phi : X_{10} + \mathcal{I}_{10} \to X_{10} + \mathcal{J}_{10}$ with $\mathcal{I}_{10}^{\Phi} = \mathcal{J}_{10}$. As above, the subgraph of X_{10} induced by the endpoints of any two edges of $\mathcal{I}_{10}$ must be mapped to a subgraph of X_{10} induced by the endpoints of two edges of $\mathcal{J}_{10}$. However, it can be easily checked that the endpoints of any two edges of $\mathcal{I}_{10}$ induce a graph isomorphic to $K_2 + {K_2}^c$, while the endpoints of the edges 07 and 14 of $\mathcal{J}_{10}$, for example, induce P_3. Hence $\mathcal{I}_{10}$ and $\mathcal{J}_{10}$ are X_{10}-non-isomorphic.

Now, the graph Z defined in Construction 5.2 is almost self-complementary with an antimorphism $\varphi = (x,z)(y,w)$ that maps U_1 to U_2. Hence, if X is any almost self-complementary graph without isolated vertices (to satisfy the degree requirement), then by Lemma 2.1, $Z \blacktriangleleft X$ is almost self-complementary and any automorphism of $Z \blacktriangleleft X$ induces an automorphism of X. Moreover, $Z \blacktriangleleft X$ is connected. It follows that the graphs X_{2k} for $k \geq 4$ are all almost self-complementary and connected. We have seen above that X_{2k} for $k \in \{4, 5\}$ is almost self-complementary with respect to two X_{2k}-non-isomorphic perfect matchings $\mathcal{I}_{2k}$ and $\mathcal{J}_{2k}$. Suppose that this is the case for some $k \geq 4$. Since by Lemma 2.1 every antimorphism of X_{2k} gives rise to an antimorphism of $Z \blacktriangleleft X_{2k}$, the graph X_{2k+4} is also almost self-complementary with respect to two perfect matchings $\mathcal{I}_{2k+4}$ and $\mathcal{J}_{2k+4}$, where $\mathcal{I}_{2k} \subset \mathcal{I}_{2k+4}$ and $\mathcal{J}_{2k} \subset \mathcal{J}_{2k+4}$. Moreover, since every automorphism of $X_{2k+4} = Z \blacktriangleleft X_{2k}$ induces an automorphism of X_{2k}, no automorphism of X_{2k+4} can map $\mathcal{I}_{2k+4}$ to $\mathcal{J}_{2k+4}$. Hence the perfect matchings $\mathcal{I}_{2k+4}$ and $\mathcal{J}_{2k+4}$ are X_{2k+4}-non-isomorphic. By induction, we conclude that each of the graphs X_{2k} for $k \geq 4$ is almost self-complementary with respect to two X_{2k}-non-isomorphic perfect matchings. ■

Note that the graphs X_{2k} (for $k \geq 4$) from Construction 5.2 are all connected. Another (and simpler) way to construct graphs Y_{2k} (for $k \geq 4$) that are almost self-complementary with respect to two Y_{2k}-non-isomorphic perfect matchings is to take $Y_8 = X_8$ and $Y_{2k+2} = {K_2}^c \blacktriangleleft Y_{2k}$. Observe, however, that the graphs Y_{2k} for $k \geq 5$ all have an isolated vertex.

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