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β-neighborhood closures for graphs

Ahmed Ainouche

UAG-CEREGMIA, Campus de Schoelcher-B.P. 7209, 97275 Schoelcher Cedex. Martinique, France

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ABSTRACT

For any positive integer k and any $(2 + k - n)$-connected graph of order n, we define, following Bondy and Chvatàl, the k-neighborhood closure $NC_k(G)$ as the graph obtained from G by recursively joining pairs of nonadjacent vertices a, b satisfying the condition $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq k$, where $\delta_{ab} = \min\{d(x)|a, b \notin N(x) \cup \{x\}\}$ and ε_{ab} is a well defined binary variable. For many properties P of G, there exists a suitable k (depending on P and n) such that $NC_k(G)$ has property P if and only if G does.

1. Introduction

Let $G = (V, E)$ be a finite simple graph of order n, connectivity $\kappa(G)$, and let a, b be a pair of nonadjacent vertices satisfying the condition $d(a) + d(b) \geq n$. Bondy and Chvátal [7] observed that G is hamiltonian if and only if $G + ab$ is hamiltonian. This observation motivated the introduction of the concept of the k-closure $C_k(G)$ of G, for a given positive integer k. The graph $C_k(G)$ is the graph obtained from G by recursively joining pairs of nonadjacent vertices whose degree sum is at least k. This graph is unique and polynomially obtained from G. For a number of various properties of a graph G on n vertices, they showed that it is possible to find a suitable integer k, such that if G has property $P(k)$, so does $C_k(G)$.

In [3], we showed that the condition $d(a) + d(b) \geq k$ can be replaced by a better one: $d(a) + d(b) + |Q(G)| \geq k$, where $Q(G)$, depending on k, is a well defined subset of vertices nonadjacent to a, b. This closure condition is named "**β-dcc**" for ***β-degree closure condition***. The graph corresponding to $C_k(G)$ is denoted as $dC_k(G)$. The **β-dcc** condition is derived from a result obtained in [1] and improved in [2]

In this paper, we consider another condition, different from the ***β-degree closure condition***, still obtained as a relaxation of the result obtained in [1] and [2]. Our new condition, named "**β-ncc**" for *β-neighborhood closure condition*" consists in replacing the condition $d(a) + d(b) + |Q(G)| \geq k$ by $|N(a) \cup N(b)| + \delta_{ab} + \varepsilon_{ab} \geq k$, where $\delta_{ab} = \min\{d(x)|a, b \notin N(x) \cup \{x\}\}$ and ε_{ab} is a well defined binary variable, with the additional condition that G is $(2 + k - n)$-connected. The condition on connectedness is not a real constraint as it is in fact a necessary condition for each one of the properties considered. The corresponding graph closure will be denoted by $NC_k(G)$. Clearly the "**β-ncc**" condition can be checked in polynomial time and $NC_k(G)$ is unique. Faudree et al. [9] defined a neighborhood closure based on the condition $|N(a) \cup N(b)| \geq k'$. The corresponding closure graph is denoted as $N_{k'}(G)$. We would like to point out that the two graphs $NC_k(G)$ and $N_{k'}(G)$ are different. Given a property P of G, k has the same value in $NC_k(G)$ and $C_k(G)$ but is different from k', used to construct $N_{k'}(G)$.

For the particular case of the hamiltonicity property, the "***β-ncc***" was used to obtain a large number of extensions of known sufficient conditions (see [5,6,10]).

E-mail address: a.ainouche@martinique.univ-ag.fr.

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Throughout, we shall be concerned with properties which are preserved by addition of edges. In particular the complete graph has all these properties.

To state the different new conditions and to relate them to existing ones, we need some preliminary definitions and notation.

2. Definitions and notation

We use Bondy and Murty [8] for terminology and notation not defined here and consider simple graphs only. Let $G = (V, E)$ be a graph of order $n \geq 3$. The set of neighbors of a vertex $v \in V$ is denoted $N_G(v)$ and $d_G(v) = |N_G(v)|$ is the degree of v. Paths and cycles in $G = (V, E)$ are considered as subgraphs and for simplicity we use the same notation to mean a subgraph, its vertex set or its edge set. If A is a subset of V, $G[A]$ will denote the subgraph induced by A.

With any pair (a, b) of nonadjacent vertices and a positive integer k we associate:

(i) $\sigma_{ab}(G) := d_G(a) + d_G(b)$, $\gamma_{ab}(G) := |N_G(a) \cup N_G(b)|$.
(ii) $\lambda_{ab}(G) := |N_G(a) \cap N_G(b)|$, $T_{ab}(G) := \{x| \, a, b \notin N(x) \cup \{x\}\}$.
(iii) $Q_{ab}(G) := \{x \in T_{ab} \mid d_G(x) + \gamma_{ab}(G) + \varepsilon_{ab}(G) \geq k\}$ for $T_{ab} \neq \varnothing$.
(iv) $\delta_{ab}(G) := \min\{d(x) \mid x \in T_{ab}\}$ for $T_{ab} \neq \varnothing$ and $\delta^*_{ab}(G) := \delta_{ab}(G) + \varepsilon_{ab}(G)$,

where $\varepsilon_{ab}(G)$ is a binary variable such that $\varepsilon_{ab}(G) = 0$ iff the following two conditions are both satisfied:

- $d_G(x) + \gamma_{ab}(G) = k - 1$ holds for all $x \in T_{ab}$.
- T_{ab} is either an independent set or a clique.

If $T_{ab} = \varnothing$, we simply set $\delta^*_{ab}(G) = \kappa(G) \geq k + 2 - n$ and $Q_{ab} = \varnothing$.

For simplicity of notation we omit ab and/or G if no confusion can arise. In [1], we proved:

Theorem 1. *Let G be a 2-connected graph of order n and let $d^T_1 \leq \cdots \leq d^T_{|T|}$ be the degree sequence (in G) of the vertices of the set T. If*

$$d^T_i \geq 2 + |T| \text{ is true for all } i \text{ with } \max(1, \lambda_{ab} - 1) \leq i \leq t \text{ or } T = \varnothing \tag{1}$$

then G is hamiltonian if and only if G + ab is hamiltonian.

In [2], we improved Theorem 1 as follows:

Theorem 2. *Let G be a 2-connected graph of order n and let $d^T_1 \leq \cdots \leq d^T_{|T|}$ be the degree sequence (in G) of the vertices of the set T. If*

$$d^T_i + \varepsilon_{ab} \geq 2 + |T| \text{ is true for all } i \text{ with } \max(1, \lambda_{ab} - 1) \leq i \leq t \text{ or } T = \varnothing \tag{2}$$

then G is hamiltonian if and only if G + ab is hamiltonian.

This new condition, referred to as the "*β-cc*" for "*β-closure condition*" has two strong relaxations:

- A degree closure condition (*β-dcc*) involving the degree sum σ_{ab} of (a, b), corresponding to the case $\max(1, \lambda_{ab} - 1) = \lambda_{ab} - 1$. This is the condition considered in [3].
- A neighborhood closure condition (*β-ncc*), involving the neighborhood union γ_{ab} of (a, b) and corresponding to the case $\max(1, \lambda_{ab} - 1) = 1$, which is the subject of this paper.

As in [7], we use:

Definition 1. Let n, k be positive integers and let P be a property defined for all $(2 + k - n)$-connected graphs of order n. Let a, b be two nonadjacent vertices satisfying the condition

$$P(k) : \gamma_{ab}(G) + \delta^*_{ab} \geq k. \tag{$*$}$$

Then P is k-neighborhood *stable* if whenever $G + ab$ has property P and $P(k)$ holds then G itself has property P. We denote by $NC_k(G)$ the associated *k-neighborhood* closure.

The proposition below is an easy adaptation of Proposition 2.1 in [7].

Proposition 1. *If P is k-neighborhood stable and $NC_k(G)$ has property P then G itself has property P.*

3. Main results

In this section, we investigate the stability of a number of properties of graphs which remain in any supergraph of G (a graph obtained from G by addition of edges). Most of these properties are studied in [7] and a few in [9]. We also provide new properties. Throughout, (a, b) is pair of nonadjacent vertices of a graph G satisfying the condition $(*)$ for a given positive integer k. For each one of the properties P considered we fix k so that G has properties P whenever $G + ab$ does. For all properties considered in this paper and in [7], the parameter k for constructing $NC_k(G)$ and $C_k(G)$ is the same. It is also the same for constructing $dC_k(G)$, introduced in [3]. Throughout, $S \subset V$ denotes a subset with s vertices.

Theorem 3. *The property of being hamiltonian is n-neighborhood stable.*

Proof. Assume, to the contrary, that there exists a 2-connected graph G with two nonadjacent vertices a, b such that $\gamma_{ab}(G) + \delta^*_{ab} \geq n$, $G + ab$ is hamiltonian but G is not. So $T \neq \varnothing$ by Theorem 2. As clearly $2 + |T| + \gamma_{ab} = n$ we get $\gamma_{ab}(G) + \delta^*_{ab} \geq n \Rightarrow \delta^*_{ab} \geq 2 + |T|$. Thus (2) holds for any $x \in T$ and G is hamiltonian by Theorem 2, With this contradiction, Theorem 3 is proved. ■

Remark 1. For this property, the Faudree et al. closure is N_{n-2}. This is a very restrictive case as we must have $T = \varnothing$ in order to add ab to the graph N_{n-2}. Therefore $N_{n-2} \subset NC_n(G)$. In [9], it is proved that $C_n(G) \neq K_n \Rightarrow N_{n-2}(G) \neq K_n$. This is no longer the case if we compare $C_n(G)$ and $NC_n(G)$. For instance if $G = C_6$ then it is easy to check that $C_6(G) = C_6$ while $NC_6(G) = K_6$. The same conclusion is reached if $G = K_7 + e$ where e is any extra edge. As a last example, consider the graph $G' = G + K_1$ where G is the Petersen graph. By Theorem 3, G' is hamiltonian since for any pair (a, b) of nonadjacent vertices it is easy to check that $\gamma_{ab}(G') = 6$, $\delta_{ab} = 4$ and $\varepsilon_{ab}(G') = 1$. Thus $NC_{11}(G') = K_{11}$ while $C_{11}(G) = G'$.

Let $\mu(G)$ denote the minimum number of disjoint paths covering all vertices of G.

Corollary 1. *The property "$\mu(G) \leq p$" is $(n - p)$-stable.*

Proof. Consider the graph $G + pK_1$ and use Theorem 3. ■

Corollary 2. *The property of containing a hamiltonian path is $(n - 1)$-neighborhood stable.*

Remark 2. Let G be the Petersen graph. Clearly $NC_9(G) = K_{10}$ and hence it is traceable by Corollary 2.

The graph G is S-hamiltonian if it remains hamiltonian whenever a set $W \subseteq S$ of vertices of S are removed. We simply say that it is s-hamiltonian if we are only interested by the number s instead of the set of vertices. It is known that G must be $(2 + s)$-connected.

Theorem 4. *Let n, s be positive integers with $0 \leq s \leq \delta - 2$. The property of being S-hamiltonian is $(n + s)$-neighborhood stable.*

Proof. Consider a $[(2 + (n + s) - n) = (2 + s) \leq \delta]$-connected graph G and let $H := G - W$ for some set $W \subseteq S$. Clearly H is 2-connected. Suppose that a, b are two nonadjacent vertices such that $\gamma_{ab}(G) + \delta^*_{ab} \geq n + s$, $H + ab$ is hamiltonian but H is not. Put $\gamma_{ab}(H) = \gamma_{ab}(G) - |W| + \theta_1$ and $\delta_{ab}(H) = \delta_{ab}(G) - |W| + \theta_2$. Obviously θ_1 and θ_2 are nonnegative integers. By Theorem 2, $\gamma_{ab}(H) + \delta_{ab}(H) + \varepsilon_{ab}(H) < |H| = n - |W|$ since H is assumed nonhamiltonian. Therefore

$$\gamma_{ab}(G) + \delta_{ab}(G) - 2\,|W| + \theta_1 + \theta_2 + \varepsilon_{ab}(H) < n - |W|\,. \tag{3}$$

It follows that

$$s - \varepsilon_{ab}(G) + \theta_1 + \theta_2 + \varepsilon_{ab}(H) < |W|\,. \tag{4}$$

As $W \subseteq S$, this inequality implies

$$W = S, \qquad \theta_1 = \theta_2 = 0, \qquad \varepsilon_{ab}(H) = 0 \quad \text{and} \quad \varepsilon_{ab}(G) = 1. \tag{5}$$

Since $\theta_1 = 0$, then $\gamma_{ab}(H) = \gamma_{ab}(G) - |W|$, that is $W \subset N(a) \cup N(b)$. Thus $T(H) = T(G)$, in which case $\varepsilon_{ab}(H) = \varepsilon_{ab}(G)$. This contradicts (5). The proof is now complete. ■

Corollary 3. *Let n, s be positive integers with $0 \leq s \leq \delta - 2$. Then the property of being s-hamiltonian is $(n + s)$-neighborhood stable.*

We say that G is S-cyclable (resp., S-traceable) if it contains a cycle C (resp., a path) with all vertices of S.

Theorem 5. *The property "G is S-cyclable" is n-neighborhood stable.*

Proof. Suppose that G is 2-connected, $(G+ab)$ contains a cycle C such that $S \subset V(C)$ but G does not. Then a, b are connected by a path $\pi := a_1 \dots a_p$ with $a = a_1$, $b = a_p$, $n \geq p \geq s$. Without loss of generality, assume that π has a maximum length. If $|C| = n$, then G is hamiltonian by Theorem 3 and hence $S \subset V(C)$. For the sequel we may assume that $R := V \setminus V(\pi) \neq \emptyset$. For simplicity we also denote $G[R]$ by R. Note that there is no (a, b)-path with all internal vertices in R for otherwise we have a contradiction. Therefore $G[V(C)]$ is 2-connected whenever G is 2-connected. Without loss of generality, we may assume that $d(a_2) = 2$. If, for instance $d(a_2) > 2$, then we add a new vertex a' and join it to a and a_2. For this new graph G' we obviously have $\delta^*_{ab}(G) = \delta^*_{ab}(G')$ and $\gamma_{ab}(G) + \delta^*_{ab}(G) \geq n$ implies $\gamma_{ab}(G') + \delta^*_{ab}(G') \geq |G'|$. Moreover the segment $aa'a_2$ belongs to any hamiltonian cycle of G'. It suffices then to remove a' in order to get a hamiltonian cycle in G. Similarly we may assume $d(a_{p-1}) = 2$. For simplicity, we shall use the notation G, π even if we modify G. Moreover we denote by F the set of edges of type aa_i (resp., ba_i) with $a_i \in T$ and $N_R(a_i) \cap N_R(a) \neq \emptyset$ (resp., $N_R(a_i) \cap N_R(b) \neq \emptyset$) and we consider the graph $G' = G + F$. Again, we show that no edge of F belongs to any hamiltonian cycle of G'. Indeed suppose that a hamiltonian cycle C' such that $E(C') \cap F \neq \varnothing$ exists. Without loss of generality suppose aa_j, $j > 2$, is an edge of $E(C') \cap F$. Since $d_{G'}(a_2) = d_G(a_2) = 2$ then a is incident to at most one edge of F. Replacing aa_j with aua_j, $u \in N_R(a) \cap N_R(a_j)$, we get a cycle of greater length, a contradiction. Here again and for simplicity we still denote by G the original graph or the modified one. In this modified graph, we note that $N(x) \cap (N(a) \cup N(b)) \subset V(\pi)$ holds for any vertex $x \in T \cap V(\pi)$. Now $\gamma_{ab}(R) + |T \cap R| = (n - |H|)$. Moreover $\gamma_{ab}(H) = \gamma_{ab}(G) - \gamma_{ab}(R)$ and $\delta_{ab}(H) \geq \delta_{ab}(G) - |T \cap R|$. Thus

$$\gamma_{ab}(H) + \delta_{ab}(H) \geq \gamma_{ab}(G) + \delta_{ab}(G) - (\gamma_{ab}(R) + |T \cap R|) \tag{6}$$

$$\geq n - (n - |H|) - \varepsilon_{ab}(G) = |H| - \varepsilon_{ab}(G). \tag{7}$$

As H is assumed nonhamiltonian, we have $\gamma_{ab}(H) + \delta_{ab}(H) + \varepsilon_{ab}(H) < |H|$ by Theorem 3. Comparing the two inequalities, we get $\varepsilon_{ab}(H) = 0$ while $\varepsilon_{ab}(G) = 1$ and $\delta_{ab}(H) = \delta_{ab}(G) - |T \cap R|$. Clearly $|T \cap R| > 0$ for otherwise $T(G) = T(H)$ in which case $\varepsilon_{ab}(G) = 1 \Rightarrow \varepsilon_{ab}(H) = 1$. If $\varepsilon_{ab}(H) = 0$ then $d_G(x) - |T \cap R| = d_H(x) = \delta_{ab}(H)$ holds for all $x \in T(H)$. Moreover either $T(H)$ is an independent set or a clique. One can easily check that $T(H)$ cannot be a clique for otherwise $T(G)$ would be a clique and $\varepsilon_{ab}(G) = 0$. Thus $T(H)$ must be an independent set with at least two vertices. Consider now the graph $G' := H \cup \{x_1x_2\}$, where x_1, x_2 are vertices of $T(H)$. Now, $\varepsilon_{ab}(G') = 1$ and hence G' has a hamiltonian cycle C' by Theorem 3. Since H is assumed nonhamiltonian then $x_1x_2 \in E(C')$. But then replacing x_1x_2 by x_1yx_2 where y is any vertex of $T \cap R$, we see that $H \subset V(C')$ and $S \subset V(C')$. So, in either case G is S-cyclable, a contradiction which proves the theorem. ■

The S-circumference, denoted by $c_S(G)$, is the length of a cycle of G containing a maximum number of vertices of S. If $S = V$ then $c(G)$ denotes the circumference of G. The following corollaries are straightforward.

Corollary 4. *Let n, s be positive integers with $3 \leq s \leq n$. Then the property $c_S(G) \geq s$ is n-neighborhood stable.*

Corollary 5. *Let n, s be positive integers with $3 \leq s \leq n$. The property $c(G) \geq s$ is n-neighborhood stable.*

Corollary 6. *Let us have $S \subset V(G)$ with s vertices, $3 \leq s \leq n$. The property "G is S-traceable" is $(n-1)$-neighborhood stable.*

Proof. Consider the graph $H = G + K_1$. Then $\gamma_{ab}(H) + \delta_{ab}(H) + \varepsilon_{ab}(H) \geq n + 1 = |H|$. As clearly $\varepsilon_{ab}(H) \geq \varepsilon_{ab}(G)$ by construction, and $\gamma_{ab}(G) + \delta_{ab}(G) + \varepsilon_{ab}(G) \geq n - 1$ by hypothesis, we get

$$\gamma_{ab}(H) + \delta_{ab}(H) + \varepsilon_{ab}(H) \geq n - 1 + 2 = n + 1 = |H|. \tag{8}$$

The conclusion follows from Theorem 5. ■

Theorem 6. *Let n, s be positive integers such that $s + \delta \geq n + 2$. The property "$G[S]$ is hamiltonian" is $(2n - s)$-neighborhood stable.*

Proof. Set $H = G[S]$ and suppose H nonhamiltonian. Thus G cannot be $(V \setminus S)$-hamiltonian. By Theorem 4, the property is not $n + (n - s) = (2n - s)$-neighborhood stable, a contradiction to our hypothesis. ■

Corollary 7. *Let n, s be positive integers with $s + \delta \geq n + 2$. Then the property of containing C_s is $(2n - s)$-neighborhood stable.*

Proof. If G contains C_s then it must have the property "$G[S]$ is hamiltonian" where S is any set with s vertices. The conclusion follows. ■

Theorem 7. *Let n, s be positive integers with $s \leq \frac{n}{2}$. Then the property of containing sK_2 is $(2s - 1)$-stable.*

Proof. Suppose that G is $2 + (2s - 1) - n = (2s + 1 - n)$-connected, $G + ab$ contains an sK_2 but G does not. Then there exists an $(s - 1)$-matching $\{a_1b_1, \dots, a_{s-1}b_{s-1}\}$ in G and an s-matching in $G + ab$. For $i \in [1, s - 1]$ we set

$$\begin{cases} A := \{a_i\}, \quad B := \{b_i\}, \quad D := V \setminus (A \cup B \cup \{a, b\}) \\ M := \{a_ib_i | i \in [1, s-1]\}, \quad M_i = \{a_i, b_i\}. \end{cases} \tag{9}$$

An M-augmenting path is a path of even length with endpoints in $D \cup \{a, b\}$ and whose edges are alternately in $E - M$ and M. To avoid a contradiction, we obviously assume that G contains no M-augmenting path. Moreover $D \cup \{a, b\}$ must be an independent set for otherwise an s-matching would exist in G. We need two cases:

Case 1. $n > 2s$ (that is $D \neq \varnothing$).

As a first step we prove that

$$\begin{aligned} &\varepsilon_{ab}(G) = 1 \quad \text{and for all } x \in T \text{ and for all } i \in [1, s-1] \\ &\text{we have } \ |M_i \cap (N(a) \cup N(b))| + d_{M_i}(x) = 2. \end{aligned} \tag{10}$$

Choose any vertex $x \in D$. Clearly $|M_i \cap (N(a) \cup N(b))| \leq 2$ and $d_{M_i}(x) \leq 2$. If, for instance, $a_1x \in E$ then $b_1 \notin N(a) \cup N(b)$ for otherwise we have an even M-augmenting path with extremities a, x (or b, x). Therefore $|M_i \cap (N(a) \cup N(b))| + d_{M_i}(x) \leq 2$. By summing over all i we get $\gamma_{ab} + d(x) \leq 2(s-1)$. On the other hand $d_G(x) + \gamma_{ab} + \varepsilon_{ab}(G) \geq 2s - 1$ by hypothesis. Comparing these inequalities we see that (10) holds for any vertex of D. Next suppose that we have $y \in T \setminus D$, $y = b_1$ say. If $a_1 \notin N(a) \cup N(b)$ then by (10), $N(x) \supset \{a_1, b_1\}$. It is then clear that we can exchange x and y. If $b_1 \in N(a)$ for instance then by (10), $|N(x) \cap \{a_1, b_1\}| = 1$. If $xb_1 \in E$ then we have an even M-augmenting path with extremities a, x. If $xa_1 \in E$ then we exchange a_1b_1 with a_1x, in which case $y \in D$. Thus (10) holds for $x \in T$.

We complete the proof by induction on s. If $s = 1$ then $G = \overline{K_n}$ since G has no edge while $G + ab$ has one. Clearly $\sigma_{ab} = \gamma_{ab} = 0$. Pick any vertex x of T. Then $d(x) + \gamma_{ab} + \varepsilon_{ab} = \varepsilon_{ab} \geq 1$. Therefore $\varepsilon_{ab} = 1$. This is a contradiction since $G = \overline{K_n} \Rightarrow \varepsilon_{ab} = 0$ by definition of ε_{ab}. Suppose now that G has an sK_2 with $s > 1$. Assume first that, for some i and for some $x \in D$, we have either $M_i \subset N(a) \cup N(b)$ or $M_i \subset N(x)$. The graph $G' = G - M_i$ satisfies the hypothesis of the theorem since by (10), $\gamma_{ab}(G') + d_{G'}(y) = \gamma_{ab}(G) + d_G(y) - 2$ must be true for all $y \in T$. By induction hypothesis G' has an $(s-1)K_2$ and hence G has an sK_2. Suppose now that for all i, we have $d_{M_i}(x) = 1$ and $|M_i \cap (N(a) \cup N(b))| = 1$. Without loss of generality, suppose that $A = N(a) \cup N(b)$. Then necessarily $N(x) = A$ for all $x \in D$ since if for instance $xb_1 \in E$ then either xb_1a_1a or xb_1a_1b is an even M-augmenting path. Therefore $N(x) \cap B = \varnothing$. As clearly we can exchange any b_i with x, then $N(y) \cap B = \varnothing$ for any $y \in T$. Since $\gamma_{ab} = s - 1$ we get $d(y) = s - 1$. In conclusion $N(y) = A$ must be true for any $y \in T$. This means that T is an independent set and hence $\varepsilon_{ab} = 0$. This completes the proof of this Case.

Case 2. $n = 2s$ (that is $D = \varnothing$).

Now G must be $(k + 2 - n = 2s + 1 - n = 1)$-connected. For this case, we use part of the argument of Faudree et al. [9] in the proof of their Theorem 2. Assuming that $G + ab$ has a perfect matching but G does not, then $G + ab$ satisfies Tutte's Theorem [11] but G does not. Then there exists a subset $R \subset V$ with $r \geq 1$ vertices such that $G - R$ has $r + 2$ odd components $C_1, \ldots, C_{r+2}$ with $a \in V(C_{r+1})$ and $b \in V(C_{r+2})$. Clearly $T \supseteq V(C_1) \cup \cdots \cup V(C_r)$. Set $|V(C_j)| = 1 + \theta_j$ for $j = 1, \ldots, r$. As $n = 2s$ and clearly $\theta_j \geq 0$, we have

$$2s = 2 + \gamma_{ab} + |T \cap (V(C_{r+1}))| + |T \cap (V(C_{r+2}))| + |T \cap R| + r + \sum_{i=1}^{r} \theta_i. \tag{11}$$

Combining (11) with the hypothesis $\gamma_{ab} + \varepsilon_{ab} + \delta_{ab} \geq 2s - 1$, we get

$$\varepsilon_{ab} + \delta_{ab} \geq 1 + |T \cap (V(C_{r+1}))| + |T \cap (V(C_{r+2}))| + |T \cap R| + r + \sum_{i=1}^{r} \theta_i. \tag{12}$$

Suppose first that $\delta_{ab} \leq r$. Then from (12) we obtain $\varepsilon_{ab} = 1$, $T \cap (V(C_{r+1}) \cup V(C_{r+2}) \cup R) = \varnothing$ and $\sum_{i=1}^{r} \theta_i = 0$. This means that $N[a] \cup N[b] = V(C_{r+1}) \cup V(C_{r+2}) \cup R$, in other words $T = \cup_{i=1}^{r} V(C_i)$ and $|V(C_i)| = 1$ for $i = 1, \ldots, r$. This implies that T is an independent set, in which case $\varepsilon_{ab} = 0$. With this contradiction, we shall assume $\delta_{ab} > r$.

If $x_i \in V(C_i)$, then $d(x_i) \geq \delta_{ab}$, $N(x_i) \subseteq V(C_i) \cup R$ and hence $|V(C_i)| = 1 + \theta_i \geq 1 + \delta_{ab} - r$. Thus $\delta_{ab} - r \leq \theta_i$. Putting this inequality into (12), we get

$$\varepsilon_{ab} + (1 - r)(\delta_{ab} - r) \geq 1 + |T \cap (V(C_{r+1}))| + |T \cap (V(C_{r+2}))| + |T \cap R|. \tag{13}$$

As $\delta_{ab} > r \geq 1$, $\varepsilon_{ab} \leq 1$ we obtain $\varepsilon_{ab} = 1$, $r = 1$, $T \cap (V(C_2) \cup V(C_3) \cup R) = \varnothing$. Now $T = V(C_1)$. By (12) we derive $\delta_{ab} \geq 1 + \theta_1 = |V(C_1)|$. On the other hand $\delta_{ab} \leq |V(C_1)| - 1 + |R| = |V(C_1)|$. It follows that $\delta_{ab} = |V(C_1)|$ and hence T must be a clique, in which case $\varepsilon_{ab} = 0$. With this last contradiction, the proof is completed. ■

Theorem 7 is a slight improvement of both Theorems 2 and 3 in [9].

Remark 3. Consider again the Petersen graph G. Obviously, it has a perfect matching. By the Bondy–Chvátal result, we have $C_k(G) = K_{10}$ only for $s \leq 3$. By Theorem 7, $NC_k(G) = K_{10}$ for $s = 5$ since $\gamma_{ab} + \delta^*_{ab} \geq 9 = 2s - 1$ since $\gamma_{ab} = 5$ and $\varepsilon_{ab} = 1$. The same conclusion cannot be drawn from Theorem 3 in [9] since $N_{2s-1-\delta} = G$.

Theorem 8. *Let n, s be positive integers with $s + \delta \geq n$. Then the property "$\alpha(G) \leq s$" is $(2n - 2s - 1)$-neighborhood stable.*

Proof. Suppose that $\alpha(G+ab) \leq s$ while $\alpha(G) > s$. Then there must exist an independent set $W \subset V$ with $(s+1) \geq 3$ vertices including both a and b. Choose any vertex $x \in W \setminus \{a, b\}$. First we note that $\gamma_{ab}(G) \leq |V \setminus W| = n-(s+1)$. Thus we need the condition $s+\delta \geq n$ since by definition we must have $d_G(x)+\gamma_{ab}(G)+\varepsilon_{ab}(G) \geq (2n-2s-1)$. Set $\gamma_{ab}(G) = |V \setminus W|-\theta_1$ and $d_G(x) = |V \setminus W| - \theta_2$. Obviously θ_1, θ_2 are nonnegative integers. Then $2|V \setminus W| - (\theta_1+\theta_2) + \varepsilon_{ab}(G) \geq (2n-2s-1)$. As $|V \setminus W| = n-(s+1)$, we get $2\,(n-(s+1)) + \varepsilon_{ab}(G) \geq (\theta_1+\theta_2) + (2n-2s-1)$, that is $\varepsilon_{ab}(G) > \theta_1+\theta_2$. It follows that $\theta_1+\theta_2 = 0$ and $\varepsilon_{ab}(G) = 1$. Since $\theta_1 = 0$, we have $N(a) \cup N(b) = V \setminus W$ and hence $T(G) = W \setminus \{a, b\}$. Moreover all vertices of W have the same degree. It is now clear that $\varepsilon_{ab}(G) = 0$ since $T(G)$ is an independent set. With this contradiction, Theorem 8 is proved. ■

Remark 4. Consider again the Petersen graph G. We know that $\alpha(G) = 4$. By the Bondy–Chvátal result, we have $C_k(G) = K_{10}$, $k = 2n-2s-1$, only for $s \geq 7$. By our theorem, $NC_k(G) = K_{10}$ for $s = 5$.

If $G = C_6$, then obviously $\alpha(C_6) = 3$ and $NC_k(C_6) = K_6$ for $s = 3$ while $C_k(G) = K_6$ is obtained only for $s \geq 4$.

It may happen that, for some specific properties, one can find better closure conditions which allow the definition of a corresponding graph closure. This is the case for the independence and the connectivity properties considered in [4].

It is also possible to improve Corollary 7 if we adopt the following specific closure condition of Faudree et al. [9] for which we give a slightly different statement and proof.

Theorem 9. *Let G be a 2-connected graph and (a, b) be a pair on nonadjacent vertices satisfying the condition*

$$\gamma_{ab} = n-2 (\text{or equivalently } T = \emptyset). \tag{14}$$

Then G is $[6-n]$-pancyclic (or equivalently s-hamiltonian, $0 \leq s \leq n-6$) if and only if $G+ab$ is.

Proof. If G is $[6-n]$-pancyclic, $6 \leq s \leq n$, then obviously $G+ab$ has the same property. Conversely, suppose, by contradiction, that $G+ab$ is $[6-n]$-pancyclic but G is not. Then, for some s, $6 \leq s \leq n$, there exists a path $\pi := a_1a_2\ldots a_s$, where $a = a_1$ and $a_s = b$. If $G[V(\pi)]$ is 2-connected then it is hamiltonian by Theorem 1 since $T = \emptyset$. So, we assume that $G[V(\pi)]$ is not 2-connected and there exists at least one $a-b$ path with all internal vertices in $V \setminus V(\pi)$. As $N[a] \cup N[b] = V$, we may choose a shortest path whose length is at most 2. Suppose first that the length is 2 and let $auvb$ be this path. As $V(\pi) \subset N[a] \cup N[b]$, we cannot have $a_i \in N(a)$ and $a_{i-1} \in N(b)$ for some $i = 3, \ldots, s-1$ for otherwise $G[V(\pi)]$ would be 2-connected. If $a_4 \in N(a)$ then $a_2, a_3 \in N(a)$ and $G[V(\pi)] - \{a_2, a_3\} + \{u, v\}$ contains a hamiltonian cycle with length s. So $N(a) \cap V(\pi) \subset \{a_2, a_3\}$. Similarly $N(b) \cap V(\pi) \subset \{a_{s-1}, a_{s-2}\}$. If $N(a) \cap V(\pi) \supset \{a_2, a_3\}$ and $N(b) \cap V(\pi) \supset \{a_{s-1}, a_{s-2}\}$ then $G[V(\pi)] - \{a_2, a_{s-1}\} + \{u, v\}$ contains a hamiltonian cycle with length s. Therefore we may assume $N(a) \cap V(\pi) \subset \{a_2, a_3\}$ and $N(b) \cap V(\pi) \subset \{a_{s-1}\}$. But then $V(\pi) \subseteq \{a, a_2, a_3, a_4, a_5 = b\}$, a contradiction since $s \geq 6$ by hypothesis. The same arguments lead to a contradiction if a, b are connected by a path aub instead of $auvb$. Note that if G is $[6-n]$-pancyclic, it is also s-hamiltonian, $0 \leq s \leq n-6$. ■

Corollary 8. *Let G be a 2-connected graph. Then G is $[6-n]$-pancyclic (or equivalently s-hamiltonian, $0 \leq s \leq n-6$) if it contains an (a, b)-hamiltonian path such that $T = \emptyset$.*

4. Open problems

A *caterpillar* is a particular tree which results in a path when its leaves are removed. The *spine* of the caterpillar is the longest path of it. The graph G is called *S-caterpillar spannable* if it has a spanning tree that is a caterpillar, whose leaves are the vertices of S.

Problem 1. Let $S \subset V(G)$ with s vertices, $2 \leq s < n$. Then the property "G is S-caterpillar spannable" is $(n+s-1)$-neighborhood stable.

Let $F \subset E$ be a set of edges such that the components of the graph (V, F) are vertex disjoint paths. A graph G is Hamilton-connected if, given any two vertices x and y of G, there is a hamiltonian path in G with ends x and y. The graph G is defined to be $|F|$-Hamilton-connected if for each pair (x, y) of vertices there is a hamiltonian path with endpoints x, y that contains F.

Problem 2. The property "G is F-Hamilton-connected with $|F| \leq n-4$" is $(n+|F|+1)$-neighborhood stable.

An s-factor of a graph G is a spanning s-regular subgraph.

Problem 3. Let n, s be positive integers with $2 \leq s < n$. Then the property of having an s-factor is $(n+2s-4)$-neighborhood stable.

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