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Note

On covering graphs by complete bipartite subgraphs

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ABSTRACT

We prove that, if a graph with e edges contains m vertex-disjoint edges, then m^2/e complete bipartite subgraphs are necessary to cover all its edges. Similar lower bounds are also proved for fractional covers. For sparse graphs, this improves the well-known fooling set lower bound in communication complexity. We also formulate several open problems about covering problems for graphs whose solution would have important consequences in the complexity theory of boolean functions.

1. Introduction

The *biclique covering number* $\mathrm{bc}(G)$ of a graph G is the smallest number of bicliques (complete bipartite subgraphs) of G such that every edge of G belongs to at least one of these bicliques. In the case when the bicliques are required to be edge-disjoint, the corresponding measure is known as the *biclique partition number* and is denoted by $\mathrm{bp}(G)$. Note that $\mathrm{bc}(G) \le \mathrm{bp}(G) \le n - 1$ holds for any graph G on n vertices, just because stars are bicliques.

These measures of graphs were considered by many authors. The classical result of Graham and Pollak [9] shows that $\mathrm{bp}(K_n) = n - 1$; here K_n is a complete graph on n vertices. On the other hand, we have $\mathrm{bc}(K_n) \le \lceil \log_2 n \rceil$: just encode the vertices of K_n by binary vectors of length $m = \lceil \log_2 n \rceil$ and define, for each $i = 1, \dots, m$, a biclique containing all edges, the codes of whose endpoints differ in the ith coordinate. Hence, the gap between the partition and the covering numbers may be exponential. Alon [2] generalized the Graham–Pollak theorem by showing that about $kn^{1/k}$ complete bipartite subgraphs are necessary and sufficient in order to cover K_n so that each edge belongs to at most k of the subgraphs.

One of the most important results in this direction is the following *degree bound* proved by Alon in [1]: If the complement graph of an n-vertex graph G has maximum degree D, then the edges of the graph G itself can be covered by $O(D^2 \ln n)$ complete subgraphs. For bipartite $n \times n$ graphs (i.e., graphs both parts of which contain n vertices) this can be improved to $\mathrm{bc}(G) = O(D \ln n)$ [12].

Concerning the maximum value $\mathrm{bc}(n)$ of $\mathrm{bc}(G)$ over all n-vertex graphs G, it is known that $n - c \log_2 n \le \mathrm{bc}(n) \le n - \lfloor \log_2 n \rfloor + 1$ for a constant $c > 0$; the upper bound is due to Tuza [22] and the lower bound to Rödl and Ruciński [19]. Chang [6] earlier proved that $\mathrm{bc}(n)/n$ tends to 1 as n tends to infinity.

In addition to being an important graph parameter, the biclique covering number also arises naturally when dealing with the communication complexity of boolean functions. The relation between bipartite $n \times n$ graphs with $n = 2^k$ and

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boolean functions is quite natural: each such graph G gives us a boolean function $f_G(z_1, \ldots, z_{2k})$ in $2k$ variables such that $f_G(\vec{x}_u, \vec{x}_v) = 1$ if and only if u and v are adjacent in G; here $\vec{x}_u \in \{0, 1\}^k$ is the binary code of the vertex u. Under this translation, $\log_2 \mathrm{bc}(G)$ is precisely the nondeterministic communication complexity of f_G, and $\log_2 \mathrm{bp}(G)$ is a lower bound on the deterministic communication complexity of f_G (see, e.g., the book [15]). Some questions about a relaxed version of the biclique covering number, related to some basic open problems in the computational complexity of boolean functions, are discussed in Section 4.

In this note we prove that, if a graph with e edges contains a matching of size m, then at least m^2/e complete bipartite subgraphs are necessary to cover all its edges. For sparse graphs, this improves the well-known fooling set lower bound in communication complexity. In Section 3 we obtain similar lower bounds for *fractional* covers.

A *bipartite* graph is a graph $(A \cup B, R)$ with $A \cap B = \emptyset$ and $R \subseteq A \times B$. Such a graph is *complete* if $R = A \times B$; with some abuse of notation, in this case we will identify the graph with the set $R = A \times B$ of its edges. A *biclique* of a graph $G = (V, E)$ is a complete bipartite graph $A \times B$ such that $A, B \subseteq V$, $A \cap B = \emptyset$ and $A \times B \subseteq E$.

2. Covering number

Given a graph $G = (V, E)$ and a set $S \subseteq E$ of its edges, let $w_G(S)$ denote the largest possible number of edges in S that can be covered by some biclique $R \subseteq E$ of G. (Note that R need not be contained in S.) Since no biclique of G can cover more than $w_G(S)$ edges of S, at least $|S|/w_G(S)$ bicliques are needed to cover G. Hence, the following *greedy covering number*,

$$\mu(G) = \max_{S \subseteq E} \frac{|S|}{w_G(S)},$$

is a lower bound on $\mathrm{bc}(G)$. Actually, this lower bound is already not very far from the truth.

Proposition 1 (*Folklore*). *For every graph $G = (V, E)$, we have* $\mathrm{bc}(G) \le \mu(G) \cdot \ln |E| + 1$.

Proof. Consider a greedy covering $R_1, \ldots, R_t$ of G by bicliques. That is, in the ith step we choose a biclique $R_i \subseteq E$ covering the largest number of all yet uncovered edges. Let $S_i = E \setminus \bigcup_{j=1}^{i} R_j$ be the set of all edges left uncovered after the ith step. Hence, $S_0 = E$ and $S_t = \emptyset$. Let $s_i = |S_i|$ and $w_i = w_G(S_i)$. Since, by the definition of $\mu = \mu(G)$, none of the fractions s_i/w_i can exceed μ, we have that $s_{i+1} = s_i - w_i \le s_i - s_i/\mu$. This yields $s_i \le s_0(1 - 1/\mu)^i \le |E| \cdot \mathrm{e}^{-i/\mu}$. For $i = t - 1$, we obtain $1 \le s_{t-1} \le |E| \cdot \mathrm{e}^{-(t-1)/\mu}$, and the desired upper bound $\mathrm{bc}(G) \le t \le \mu \ln |E| + 1$ follows. □

A natural choice for a set S of "difficult to cover" edges is to take a matching. Let $\nu(G)$ be the *matching number* of G, that is, the maximal number of vertex-disjoint edges of G, and let $\mathrm{cl}(G)$ be the largest number r such that G contains a complete bipartite $r \times r$ graph $K_{r,r}$. Since no complete bipartite $r \times s$ graph $K_{r,s}$ can contain more than $\min\{r, s\}$ edges of any matching, this yields

$$\mathrm{bc}(G) \ge \frac{\nu(G)}{\mathrm{cl}(G)}. \tag{1}$$

Although simple, the lower bound (1) – known as the *fooling set bound* – is one of the main tools for proving lower bounds on the nondeterministic communication complexity of boolean functions (see, e.g., [15]). Our first result improves the fooling set bound for sparse graphs.

Theorem 2. *For every non-empty graph $G = (V, E)$, we have*

$$\mathrm{bc}(G) \ge \frac{\nu(G)^2}{|E|}.$$

Proof. Let $M \subseteq E$ be an *m-matching*, that is, a set of m vertex-disjoint edges. Let $E = R_1 \cup \cdots \cup R_t$ be a covering of the edges of G by $t = \mathrm{bc}(G)$ (not necessarily disjoint) bicliques of G. Define a mapping $f : M \to \{1, \ldots, t\}$ by $f(e) = \min\{i \mid e \in R_i\}$, and let $M_i = \{e \in M \mid f(e) = i\}$ be the set of edges of M assigned to the ith biclique. That is, M_i consists of those edges of the matching M that are covered by the ith biclique R_i for the first time.

Let $F_i \subseteq R_i$ be a biclique spanned by the vertices of the matching M_i. That is, we leave in F_i only those edges of R_i such that *both* endpoints are incident with edges of M_i. Then $F = F_1 \cup \cdots \cup F_t$ is a union of vertex-disjoint bicliques satisfying $M \subseteq F \subseteq E$.

Note that each F_i is a complete bipartite $r_i \times r_i$ graph, where $r_i = |M_i|$ is the number of edges in the ith matching M_i. Since the bicliques $F_1, \ldots, F_t$ are vertex-disjoint, we have that

$$r_1 + \cdots + r_t = |M| = m$$

and

$$r_1^2 + \cdots + r_t^2 = |F|.$$

By the Cauchy–Schwarz inequality,

$$m^2 = (r_1 + \cdots + r_t)^2 \le t \cdot (r_1^2 + \cdots + r_t^2) = t \cdot |F|,$$

and the desired lower bound $t \ge m^2/|F| \ge m^2/|E|$ follows. □

Remark 3. For all graphs $G = (V, E)$ with $|E| < \nu(G) \cdot \mathrm{cl}(G)$, Theorem 2 yields better lower bounds than those given by the fooling set bound (1). If, for example, G consists of one n-matching and some constant number c of complete $r \times r$ graphs $K_{r,r}$ with $r = \sqrt{n}$, then Theorem 2 yields $\mathrm{bc}(G) \ge n^2/(cr^2 + n) = \Omega(n)$, whereas the fooling set bound (1) only yields $\mathrm{bc}(G) \ge n/r = \sqrt{n}$.

For bipartite graphs $G = (V_1 \cup V_2, E)$ with $|V_1| < |V_2|$, the bound of Theorem 2 can be slightly improved. Say that two matchings $M_1, M_2 \subseteq V_1 \times V_2$ are *dependent* if some two edges $e_1 \in M_1$ and $e_2 \in M_2$ have a common endpoint in V_2; otherwise the matchings are *independent*.

Corollary 4. *If a bipartite graph $G = (V_1 \cup V_2, E)$ contains k pairwise independent m-matchings, then*

$$\mathrm{bc}(G) \ge \frac{km^2}{|E|}.$$

Proof. If $M_1, \ldots, M_k \subseteq E$ are independent m-matchings contained in G, then we can consider k subgraphs $G_1, \ldots, G_k$ of G induced by their vertex sets. The independence of the matchings implies that the subgraphs G_i are edge-disjoint. For each of these subgraphs, Theorem 2 yields $\mathrm{bc}(G_i) \cdot |E_i| \ge m^2$, where E_i is the set of all edges of G_i. Since each G_i is an *induced* subgraph, its covering number is at most that of the whole graph G. Summing over all i, this yields $\mathrm{bc}(G) \cdot |E| \ge \mathrm{bc}(G)(|E_1| + \cdots + |E_k|) \ge km^2$. □

Remark 5. It may be interesting to compare the bound given in Theorem 2 with the following well-known lower bound on the rank proved, among other places, in [5,3]: For every real symmetric matrix A,

$$\mathrm{rk}(A) \ge \mathrm{tr}(A)^2/\mathrm{tr}(A^2), \quad (2)$$

where $\mathrm{rk}(A)$ is the rank of the matrix A over the reals, and $\mathrm{tr}(A)$ is its trace, that is, the sum of its diagonal elements. If A is the adjacency matrix of a bipartite $n \times n$ graph $G = (V_1 \cup V_2, E)$, then $\mathrm{tr}(A^2) = |E|$. Moreover, if G contains a matching with m edges, then $\mathrm{tr}(A) \ge m$. Hence, if additionally the adjacency matrix A of a graph G is symmetric, then (2) implies that $m^2/|E|$ is a lower bound for the rank $\mathrm{rk}(A)$ of A, and hence, for the biclique *partition* number $\mathrm{bp}(G)$ of G. Theorem 2 says that $m^2/|E|$ is even a lower bound for the clique *covering* number.

3. Fractional partition number

Our next result concerns the fractional version of the biclique partition number $\mathrm{bp}(G)$. If $\mathcal{R}$ is the set of all bicliques of a given n-vertex graph G, then each biclique covering of G can be described by a function $\phi : \mathcal{R} \to \{0, 1\}$ such that $\phi(R) = 1$ if and only if R belongs to the covering. Hence, $\mathrm{bc}(G)$ is the minimum of $\sum_{R\in\mathcal{R}} \phi(R)$ over all functions $\phi : \mathcal{R} \to \{0, 1\}$ such that

$$\sum_{R\in\mathcal{R}:e\in R} \phi(R) \ge 1 \quad \text{for each edge } e \text{ of } G. \quad (3)$$

The *fractional biclique covering number* $\mathrm{bc}^*(G)$ is the minimum of $\sum_{R\in\mathcal{R}} \phi(R)$ over all functions $\phi : \mathcal{R} \to [0, 1]$ satisfying (3). It is clear from the definition that $\mathrm{bc}^*(G)$ does not exceed $\mathrm{bc}(G)$. Scheinerman and Trenk [21] showed that $\mathrm{bc}^*(G) = \mathrm{bc}(G)$ if G contains no induced cycle on 4 or more vertices. On the other hand, Lovász [17] proved that, for *any* graph G, the gap can be at most logarithmic in the maximum degree D of G:

$$\mathrm{bc}^*(G) \ge \frac{\mathrm{bc}(G)}{1 + \ln D}. \quad (4)$$

Hence, Theorem 2 yields corresponding lower bounds for the fractional biclique covering number, as well.

Remark 6. As mentioned in Section 2, the greedy covering number $\mu(G)$ approximates $\mathrm{bc}(G)$ up to a logarithmic factor. In the case of fractional covering numbers the relation is even tighter: Karchmer, Kushilevitz and Nisan [13] showed that the following fractional version $\mu^*(G)$ of the greedy covering number just *coincides* with $\mathrm{bc}^*(G)$. To define $\mu^*(G)$ for a graph $G = (V, E)$, consider probability distributions $p : E \to [0, 1]$ on edges of G, and set $\mu^*(G) = \max_p 1/w_G(p)$, where the maximum is over all probability distributions, and $w_G(p)$ is the maximum probability $\sum_{e\in R} p(e)$ of a biclique R of G. Together with (4), this implies that the fractional greedy covering number $\mu^*(G)$ approximates the non-fractional biclique covering number $\mathrm{bc}(G)$, as well: $\mu^*(G) \le \mathrm{bc}(G) \le (1 + \ln D)\mu^*(G)$ (cf. Proposition 1). Moreover, we always have that $\mu^*(G) \ge \mu(G)$. Indeed, given a subset $S \subseteq E$ of edges, we can define $p(e) = 1/|S|$ for all $e \in S$ and $p(e) = 0$ for all $e \in E \setminus S$. Then $1/w_G(p) = |S|/w_G(S)$. Thus, using more sophisticated probability distributions, one may (apparently) obtain slightly larger lower bounds on $\mathrm{bc}(G)$ than those given by $\mu(G)$.

Our second result is an analog of Theorem 2 for the *fractional biclique partition number* $\mathrm{bp}^*(G)$ of G, that is, for the minimum of $\sum_{R\in\mathcal{R}} \phi(R)$ over all functions $\phi : \mathcal{R} \to [0,1]$ satisfying

$$\sum_{R\in\mathcal{R}:e\in R} \phi(R) = 1 \quad \text{for each edge } e \text{ of } G. \tag{5}$$

Theorem 7. *For every graph $G = (V, E)$,*

$$\mathrm{bp}^*(G) \geq \frac{\nu(G)^2}{4|E|}.$$

Proof. Let $G = (V, E)$ be a graph, and $\mathcal{R}$ be the set of all bicliques of G. By the duality theorem in linear programming (see, e.g., [7], Chapter 5), we have that $\mathrm{bp}^*(G) = \max_w \sum_{e\in E} w(e)$, where w ranges over all real-valued functions $w : E \to \mathbb{R}$ such that $\sum_{e\in R} w(e) \leq 1$ for all bicliques $R \in \mathcal{R}$.

Let $M \subseteq E$ be a matching with $m = \nu(G)$ edges contained in G. Define $w(e) = 1/p$ for all $e \in M$, and $w(e) = -1/p^2$ for all $e \in E \setminus M$, where $p > 0$ is a parameter to be specified soon. Since only edges of M have positive weights, the heaviest are bicliques of the form $R = A \times B$ with $|A| = |B| = |M \cap R| = k$ for some k. The weight of each such biclique is

$$\sum_{e\in R} w(e) = \frac{k}{p} - \frac{k(k-1)}{p^2} \leq \frac{k}{p}\left(1 - \frac{k-1}{p}\right) \leq 1.$$

Indeed, if $k \geq p + 1$ then the expression in the parenthesis is at most 0, and if $k \leq p$ then both the terms are at most 1. Hence, w is a legal weight function, and we obtain

$$\mathrm{bp}^*(G) \geq \sum_{e\in E} w(e) = \frac{m}{p} - \frac{|E| - m}{p^2} = \frac{m}{p}\left(1 - \frac{|E| - m}{pm}\right).$$

For $p = 2|E|/m$, the expression in the parenthesis is at least $1/2$, and the desired lower bound $\mathrm{bp}^*(G) \geq m^2/4|E|$ follows. □

4. Conclusion and open problems

We have shown that the biclique covering and partition numbers, as well as their fractional versions, are essentially bounded below by the fraction $\nu(G)^2/|E(G)|$. This yields lower bounds $\mathrm{bc}(G) = \Omega(n^\epsilon)$ for all n-vertex graphs $G = (V, E)$ containing a perfect matching and having at most $n^{2-\epsilon}$ edges. Moreover, for such graphs, the bound is easy to apply. It essentially says that $\mathrm{bc}(G)$ is always at least the matching number divided by the average degree d of G. Our lower bound is then $\nu(G)^2/|E| = (n/2)^2/(dn/2) = n/2d$. Much less, however, is known about the following relaxed version of $\mathrm{bc}(G)$.

Define the *resistance* $\rho(G)$ of a graph G as the smallest number r such that G can be written as an intersection of r graphs that each have biclique covering number at most r. That is, now we allow replacing up to the fraction $(r-1)/r$ of the non-edges by (new) edges in order to reduce the biclique covering number to r. For some graphs, this can drastically reduce the number of required bicliques in a cover: If, for example, M is a bipartite $n \times n$ graph consisting of n vertex-disjoint edges, then $\mathrm{bc}(M) = n$ but $\rho(M) = O(\ln n)$ [11].

Problem 8. Exhibit an explicit sequence of bipartite $n \times n$ graphs of resistance at least n^ϵ for a constant $\epsilon > 0$.

This would yield the first super-linear lower bound on the size of log-depth circuits, and hence, resolve a problem in boolean circuit complexity that has been open for more than 30 years (see, e.g., [11] on how this happens).

Using counting arguments it can be shown that bipartite $n \times n$ graphs of resistance $\Omega(\sqrt{n})$ *exist* [11]. However, no comparable lower bound is known for *explicit* graphs: the highest remains the lower bound $\Omega(\ln^{3/2} n)$ proved by Lokam in [16].

The following problem is about mere *existence* of graphs, and hence, may seem easier: Do there exist graphs whose resistance is much smaller than that of their complements? To be more specific, by the bipartite complement of a bipartite graph $G = (V_1 \cup V_2, E)$ we will mean the bipartite graph $\overline{G} = (V_1 \cup V_2, F)$ with $F = (V_1 \times V_2) \setminus E$.

Problem 9. Does there exist a sequence G_n of bipartite $n \times n$ graphs such that $\ln \rho(G_n) \leq (\ln \ln n)^c$ for a constant c, but $\ln \rho(\overline{G_n}) \geq (\ln \ln n)^\alpha$ for some α tending to infinity as $n \to \infty$?

If it does, this would resolve a problem in communication complexity that has been open for more than 20 years. Namely, this would separate the second level of the hierarchy of communication complexity classes introduced in [4] (Problem 3.1).

When trying to estimate the resistance of graphs, it would be interesting to understand what can be said about the covering number $\mathrm{bc}(G)$ of a graph, if we know that its complement is H-free, for some given graph H; as customary, a graph is *H-free* if it does not contain a copy of H as a (not necessarily induced) subgraph. Note that the degree upper bound from

[12], mentioned in the introduction, is of this form with $H = K_{1,D}$: If the bipartite complement $\overline{G}$ of a bipartite graph G is $K_{1,D}$-free, then $\mathrm{bc}(G) = O(D \ln n)$. What about other forbidden patterns H? In particular, what can be said about $\mathrm{bc}(G)$ if the complement of G is a dense enough $K_{r,r}$-free graph?

Problem 10. Do there exist constants $\epsilon > 0$ and $0 < \delta < 1/2$ such that, if H is a bipartite $K_{2,2}$-free graph of average degree at least n^{δ}, then $\mathrm{bc}(\overline{H}) \geq n^{\epsilon}$?

If true, this would resolve Problem 8. Indeed, we could take a bipartite $n \times n$ Erdős–Rényi graph G [8]. This graph is $K_{2,2}$-free and has minimum degree $d \geq \sqrt{n}/2$. Let $r = \rho(\overline{G})$ be the resistance of the bipartite complement $\overline{G}$ of G. By the definition of $\rho(\overline{G})$, the graph G itself can be written as a union of r bipartite graphs $H_1, \ldots, H_r$ such that $\mathrm{bc}(\overline{H_i}) \leq r$ for all $i = 1, \ldots, r$. In particular, each of the graphs H_i must be $K_{2,2}$-free. Hence, there exists a $K_{2,2}$-free graph $H \in \{H_1, \ldots, H_r\}$ such that H has average degree at least $d/r \geq \sqrt{n}/2r$ and $\mathrm{bc}(\overline{H}) \leq r$. If $\sqrt{n}/2r < n^{\delta}$, then $\rho(\overline{G}) = r > n^{1/2-\delta}/2$. If $\sqrt{n}/2r \geq n^{\delta}$, then a positive answer to Problem 10 would imply $r \geq \mathrm{bc}(\overline{H}) \geq n^{\epsilon}$. In both the cases we would have that the bipartite complement of the Erdős–Rényi graph has resistance at least n^{γ} for a constant $\gamma > 0$.

The following question about biclique *partitions* is related to the formula size of boolean functions. Consider bicliques $R = A \times B$ over the vertex set $V = \{0, 1\}^n$. Such a biclique is *monochromatic* if there is an $i (1 \leq i \leq n)$ such that all vectors of A differ from all vectors of B in the ith coordinate. For a subset $U \subseteq \{0, 1\}^n$, let $\kappa(U)$ be the smallest number t such that the biclique $U \times (\{0, 1\}^n \setminus U)$ can be decomposed into t edge-disjoint monochromatic bicliques.

It is well known (see, e.g., [20] or [18]) that $\kappa(U)$ is a lower bound on the size of any boolean formula with And, Or and Not gates computing the boolean function $f : \{0, 1\}^n \to \{0, 1\}$ given by $f(\vec{x}) = 1$ if and only if $\vec{x} \in U$. A classical result of Khrapchenko [14] shows that $\kappa(U) \geq n^2$ for the set U consisting of all vectors with an odd number of 1's.

Problem 11. Exhibit a subset $U \subseteq \{0, 1\}^n$ such that $\kappa(U) = \Omega(n^k)$ for a constant $k > 2$.

If done for $k \geq 3$, this would improve the best known lower bound on the formula size of boolean functions [10].

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