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On minor-minimally 3-connected binary matroids

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ABSTRACT

A matroid M is called *minor-minimally 3-connected* if M is 3-connected and, for each $e \in E(M)$, either $M\backslash e$ or M/e is not 3-connected. In this paper, we prove a chain theorem for the class of minor-minimally 3-connected binary matroids. As a consequence, we obtain a chain theorem for the class of minor-minimally 3-connected graphs.

1. Introduction

The terminology used in this paper will generally follow [3] with one exception: we will use $\text{si}(M)$ and $\text{co}(M)$ to denote the simplification and the cosimplification of a matroid M, respectively. A *triad* in a matroid is a 3-element cocircuit and a *triangle* is a 3-element circuit. A *quad* is a 4-element subset of $E(M)$ that is both a circuit and a cocircuit. We call $\{e_1, e_2, \ldots, e_n\}$ a *fan* if $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \ldots,$ and $\{e_{n-2}, e_{n-1}, e_n\}$ are alternatively triangles and triads. A 3-connected matroid M is *minor-minimally 3-connected* if, for every $e \in E(M)$, either $M\backslash e$ or M/e is not 3-connected. Wagner [5] conjectured the following.

Conjecture 1.1. *Let G be a minor-minimally 3-connected graph with at least four vertices. Then either*

(1) *G has a degree-three vertex u and an edge e incident with u such that $co(G\backslash e)$ is minor-minimally 3-connected,*
(2) *G has a triangle T and an edge $e \in T$ such that $si(G/e)$ is minor-minimally 3-connected,*
(3) *G has a degree-three vertex u such that $G \setminus u$ is minor-minimally 3-connected,*
(4) *G has a triangle T such that G/T is minor-minimally 3-connected, or*
(5) *$G = K_4$.*

It turns out that the above conjecture is almost right: one simply needs to add another reduction operation: deleting or contracting a 4-element fan. In this paper, we prove a chain-type theorem for minor-minimally 3-connected binary matroids, where deleting a vertex of degree three will be replaced by deleting a triad in the matroid. As an immediate consequence, we obtain a chain-type theorem for minor-minimally 3-connected graphs.

Let N be a 3-connected matroid. An element $e \in E(N)$ is called *violating* if both $N\backslash e$ and N/e are 3-connected. So N is minor-minimally 3-connected if and only if N has no violating element. Let $A = \{e_1, e_2, e_3, e_4\}$ be a 4-element fan in a minor-minimally 3-connected matroid M, where $T = \{e_1, e_2, e_3\}$ is a triangle and $T^* = \{e_2, e_3, e_4\}$ is a triad. We call A *strongly connected* if

(1) M has no fan of length 5 or more that contains A,

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(2) $M/e_2\backslash e_3, M/e_3\backslash e_2, M/T$ and $M\backslash T^*$ are all 3-connected,
(3) e_1 and e_4 are violating in $M/e_2\backslash e_3$ and $M/e_3\backslash e_2$, and
(4) e_1 is violating in $M\backslash T^*$ and e_4 is violating in M/T.

Note that, in the definition above, the fact that M/T and $M\backslash T^*$ are 3-connected follows from (3); we explicitly list it in (2) for ease of reference and understanding. The following is the main result of this paper.

Theorem 1.2. *Let M be a minor-minimally 3-connected binary matroid with at least four elements. Then either*

(1) *M has a triad T^* and $e \in T^*$ such that $co(M\backslash e)$ is minor-minimally 3-connected,*
(2) *M has a triangle T and $e \in T$ such that $si(M/e)$ is minor-minimally 3-connected,*
(3) *M has a triad T^* such that $M\backslash T^*$ is minor-minimally 3-connected,*
(4) *M has a triangle T such that M/T is minor-minimally 3-connected,*
(5) *M has a strongly connected 4-element fan A such that both $M\backslash A$ and M/A are minor-minimally 3-connected, or*
(6) *$M \in \{M(K_4), F_7, F_7^*\}$.*

By restricting to graphs, we obtain the following corollary.

Corollary 1.3. *Let G be a minor-minimally 3-connected graph with at least four vertices. Then either*

(1) *G has a degree-three vertex u and an edge e incident with u such that $co(G\backslash e)$ is minor-minimally 3-connected,*
(2) *G has a triangle T and an edge $e \in T$ such that $si(M/e)$ is minor-minimally 3-connected,*
(3) *G has a degree- 3 vertex u such that $G\backslash u$ is minor-minimally 3-connected,*
(4) *G has a triangle T such that G/T is minor-minimally 3-connected,*
(5) *G has a strongly connected 4-element fan A such that $G\backslash A$ and G/A are both minor-minimally 3-connected, or*
(6) *$G = K_4$.*

Chain theorems such as Tutte's Wheels and Whirls Theorem have proven to be of fundamental importance in inductive arguments. We believe that Theorem 1.2 will likewise be useful. Our theorem also gives a characterization of minor–minor minimally 3-connected matroids. A triangle T of a matroid M is *non-separating* if M/T is connected; and a triad T^* of M is *non-separating* if $M\backslash T^*$ is connected. It is straightforward to see that the triangles and triads in the statement of Theorem 1.2 are non-separating. Therefore, the following result is an immediate consequence of Theorem 1.2.

Corollary 1.4. *Let M be a minor-minimally 3-connected binary matroid with $|E(M)| \geq 4$. Then M has a non-separating triangle or a non-separating triad.*

Note that we insist that all matroids are binary; this is necessary as Anderson and Wu [1] gave a counterexample to Wagner's conjecture for general 3-connected matroids.

The paper is constructed as follows: Section 2 contains some preliminary results; in Section 3 we present the proof of the main result; and in Section 4 we show by examples why it is necessary to have the operation of deleting or contracting a 4-element fan.

2. Preliminaries

In this section we present some basic lemmas on separations.

Let $M = (E, r)$ be a matroid where r is the rank function. The connectivity function, λ_M, of M is defined by $\lambda_M(A) = r(A) + r(E \setminus A) - r(M)$ for all $A \subseteq E$. Tutte [4] proved that the connectivity function is *submodular*; that is, if $X, Y \subseteq E(M)$, then

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$

The following equivalent definition of λ_M shows that the connectivity function is invariant under duality.

$$\lambda_M(X) = r_M(X) + r_M^*(X) - |X|.$$

A set $A \subseteq E$ is said to be *k-separating* if $\lambda_M(A) \leq k - 1$; when equality holds we say that A is *exactly k-separating*. If A is k-separating and $|A|, |E \setminus A| \geq k$, then we say that $(A, E \setminus A)$ is a *k-separation* of M. A k-separation $(A, E\backslash A)$ of M is *minimal* if either $|A| = k$ or $|E\backslash A| = k$. The next lemma is an easy consequence of submodularity.

Lemma 2.1. *Let X and Y be k-separating sets of a matroid M. If $X \cap Y$ is not $(k-1)$-separating in M, then $X \cup Y$ is k-separating in M.*

A sequence $(e_1, e_2, \dots, e_i)$ of distinct elements in $E(M)$ is called a *fan* if $\{e_1, e_2, e_3\}, \{e_2, e_3, e_4\}, \dots, \{e_{i-2}, e_{i-1}, e_i\}$ are alternately triangles and triads. The *coclosure* of a set $X \subseteq E(M)$ is the closure of X in M^*. Clearly, an element $x \in E(M) \setminus X$ belongs to the coclosure of X if and only if x does not belong to the closure of $E(M) \setminus (X \cup \{x\})$. A set $X \subseteq E(M)$ is *coclosed* if the coclosure of X is the set X itself.

Let (A, B) be a k-separation of the matroid M. An element $x \in E(M)$ is in the *guts* of (A, B) if x belongs to the closure of both A and B. Dually, x is in the *coguts* of (A, B) if x belongs to the coclosure of both A and B. The next lemma follows easily from these definitions.

Lemma 2.2. *Let (A, B) be an exact k-separation of matroid M and let $x \in B$. Then*

- *$A \cup \{x\}$ is exactly k-separating if x belongs to either the guts or the coguts of (A, B), but not both.*
- *$A \cup \{x\}$ is exactly $(k-1)$-separating if x belongs to both the guts and the coguts of (A, B).*
- *$A \cup \{x\}$ is exactly $(k+1)$-separating if x belongs to neither the guts nor the coguts of (A, B).*

Let x be an element of the matroid M and let (A, B) be a k-separation of $M \setminus x$. Then x *blocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. Now let (A, B) be a k-separation of M/x. Then x *coblocks* (A, B) if neither $(A \cup \{x\}, B)$ nor $(A, B \cup \{x\})$ is a k-separation of M. The following lemma also follows easily from the definitions.

Lemma 2.3. *Let M be a matroid and let $\{A, B, \{x\}\}$ be a partition of $E(M)$. Then the following hold.*

- *If (A, B) is an exact k-separation of $M \setminus x$, then x blocks (A, B) if and only if x is not a coloop of M, $x \notin cl_M(A)$, and $x \notin cl_M(B)$.*
- *If (A, B) is an exact k-separation of M/x, then x coblocks (A, B) if and only if x is not a loop, $x \in cl_M(A)$, and $x \in cl_M(B)$.*

Bixby [2] proved the following lemma (see also [3, Proposition 8.4.6]).

Lemma 2.4. *Let M be a 3-connected matroid and e be an element of M. Then either $M \setminus e$ or M/e has no non-minimal 2-separations. Moreover, in the first case, $co(M \setminus e)$ is 3-connected, while in the second case, $si(M/e)$ is 3-connected.*

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We require the following two lemmas of Anderson and Wu [1]; we present alternate proofs.

Lemma 3.1. *Let N be a simple connected binary matroid with $|E(N)| \geq 2$. If N is not 3-connected but $si(N/e)$ is 3-connected, then e belongs to a series pair of N.*

Proof. Let (X, Y) be a 2-separation of N with $e \in Y$. Since N is simple, N/e is loopless. Since $\text{si}(N/e)$ is 3-connected, N/e is connected. By Lemma 2.3, $e \notin cl_N(X)$. Therefore, every triangle of N containing e meets X by at most one element. So we may assume that $X \subseteq E(\text{si}(N/e))$. Since $\text{si}(N/e)$ is 3-connected, $(X, Y \cap E(\text{si}(N/e)))$ is NOT a 2-separation of $\text{si}(N/e)$, and hence $|Y \cap E(\text{si}(N/e))| \leq 1$. Now note that $r_{\text{si}(N/e)}(Y \cap E(\text{si}(N/e))) \leq |Y \cap E(\text{si}(N/e))| \leq 1$. So $r_N(Y) \leq 2$. Since N is simple and $|Y| \geq 2$, $r_N(Y) = 2$. Since N is binary and simple, $|Y| \leq 3$. If $|Y| = 2$, then evidently Y is a series pair of N containing e. So we may assume that $|Y| = 3$. Now $\lambda_N(Y) = r_N(Y) + r^*_N(Y) - 3 = 1$. So $r_N(Y) + r^*_N(Y) = 4$, and hence $r^*_N(Y) = 2$ and Y is a 2-separating triangle. Since N is binary, Y contains a series pair S. Note that $e \in S$, as otherwise $e \notin cl^*_N(S) = cl^*_N(Y \backslash \{e\})$. Thus we have $e \in cl_N(X)$, a contradiction. □

Lemma 3.2. *Let M be a 3-connected binary matroid with $r(M) \geq 4$ and let f and g be distinct elements of $E(M)$. If M/f is not 3-connected and $si(M/g)/f$ is 3-connected, then there is a unique triangle T of M containing f and g; moreover, T is the only triangle containing f.*

Proof. Since $\text{si}(M/g)/f$ is 3-connected, no triangle of $\text{si}(M/g)$ contains f. Therefore, every triangle of M containing f must also contain g. Since M is binary, f is contained in at most one triangle of M. Now it suffices to show that there exists a triangle containing f.

Suppose that no triangle of M contains f. Then M/f is a simple matroid. Let (X, Y) be a 2-separation of M/f. Then $|X|, |Y| \geq 3$. Note that both $(X \cup \{f\}, Y)$ and $(X, Y \cup \{f\})$ are 3-separations of M, and f is in the guts of $(X \cup \{f\}, Y)$. Since M is binary and no triangle of M contains f, we deduce that $r_M(X), r_M(Y) \geq 4$. Let $M' = \text{si}(M/g)/f$. Note that $r_{M/f}(X), r_{M/f}(Y) \geq 3$, and hence $r_{M'}(X \cap E(M')), r_{M'}(Y \cap E(M')) \geq 2$. Therefore, we have that $|X \cap E(M')| \geq 2$ and $|Y \cap E(M')| \geq 2$. Note that $\lambda_{M'}(X \cap E(M')) \leq \lambda_{M/f}(X)$. Thus, $(X \cap E(M'), Y \cap E(M'))$ is a 2-separation of M', contrary to the assumption that $M' = \text{si}(M/g)/f$ is 3-connected. □

Propositions 3.3–3.5 are the key results for proving Theorem 1.2.

Proposition 3.3. *Let M be a minor-minimally 3-connected binary matroid with $|E(M)| \geq 8$. If M is not a wheel and M has a fan of length at least 4, then Theorem 1.2 holds.*

Proof. Choose a fan $(e_1, e_2, e_3, \ldots, e_k)$ of M with $k \geq 4$ such that k is as large as possible. Since M is not a wheel, $E(M) \neq \{e_1, e_2, \ldots, e_k\}$. First suppose that $k \geq 5$. In this case, we may assume that $\{e_1, e_2, e_3\}$ is a triangle by duality.

Claim 1: $\text{si}(M/e_2)$ is 3-connected.

Evidently $\{e_3, e_4, e_5\}$ is 2-separating in $M \backslash e_2$. By Lemma 2.4, M/e_2 has no non-minimal 2-separations, so $\text{si}(M/e_2)$ is 3-connected.

Claim 2: $\text{si}(M/e_2)$ is minor-minimally 3-connected.

Suppose this is not the case. Let $M' = \text{si}(M/e_2)$. Then there exists $y \in E(M')$ such that $M' \backslash y$ and M'/y are both 3-connected. Evidently $y \notin \{e_1, e_2, \ldots, e_k\}$. Since M is minor-minimally 3-connected, one of $M \backslash y$ and M/y is not 3-connected. First assume that M/y is not 3-connected. Then Lemma 3.2 implies that there exists a triangle T of M containing both e_2 and y. Since $\{e_2, e_3, e_4\}$ is a triad, either $e_3 \in T$ or $e_4 \in T$. Since M is binary, $T = \{e_2, e_4, y\}$. Now note that $\{e_1, e_5, y\}$ is a triangle of M'. So M'/y is not 3-connected, a contradiction. Next we assume that M/y is 3-connected but $M\backslash y$ is not 3-connected. Then there is no triangle containing both y and e_2, and hence $M'\backslash y \cong \mathrm{si}((M\backslash y)/e_2)$. By Lemma 3.1, e_2 belongs to a series pair of $M\backslash y$, and hence there exists a triad T^* of M containing both y and e_2. Since $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are both triangles, $T^* = \{e_1, e_2, y\}$. So $(y, e_1, e_2, \ldots, e_k)$ is a longer fan, contrary to our choice of k.

From now on we assume that $k = 4$ and M has no fan of length 5 or more. By possibly reordering the elements e_1, e_2, e_3, and e_4, we may assume that $T = \{e_1, e_2, e_3\}$ is a triangle and that $T^* = \{e_2, e_3, e_4\}$ is a triad. Note that no triangle of M contains e_4 and no triad of M contains e_1. So T is the only triangle containing e_2 or e_3, and T^* is the only triad containing e_2 or e_3.

By Lemma 2.4, one of $\mathrm{co}(M\backslash e_2)$ and $\mathrm{si}(M/e_2)$ is 3-connected. Note that $\mathrm{co}(M\backslash e_2) \cong \mathrm{si}(M/e_3)$. By the symmetry between e_2 and e_3, we may assume that $M' = \mathrm{co}(M\backslash e_2)$ is 3-connected. We are done if M' is minor-minimally 3-connected. So we may assume that there exists $h \in E(M')$ such that $M'\backslash h$ and M'/h are both 3-connected. Since M is minor-minimally 3-connected, one of $M\backslash h$ and M/h is not 3-connected. By possibly swapping e_2 and e_3 and taking the dual of M, the two cases are symmetric to each other. So we may assume that $M\backslash h$ is not 3-connected.

By the dual of Lemma 3.2, there exists a unique triad of M containing h and e_2. Since T^* is the only triad of M containing e_2, we have $h \in \{e_3, e_4\}$. Note that $M\backslash T^* \cong \mathrm{co}(M\backslash e_2)\backslash h$ is 3-connected. We are done if $M\backslash T^*$ is minor-minimally 3-connected. Hence we may assume that there exists $g \in E(M\backslash T^*)$ such that $M\backslash T^*\backslash g$ and $M\backslash T^*/g$ are both 3-connected.

Claim 3: $M\backslash g$ is 3-connected.

Note that $M\backslash e_1$ is 3-connected. So we may assume that $g \neq e_1$. Let (X, Y) be a 2-separation of $M\backslash g$. By symmetry, we may assume that $|X \cap T| \geq 2$. So $X \cup \{e_1, e_2, e_3, e_4\}$ is 2-separating in $M\backslash g$. So $(X\backslash T^*) \cup \{e_1\}$ is 2-separating in $M\backslash T^*\backslash g$. Since $M\backslash T^*\backslash g$ is 3-connected, either $|(X\backslash T^*) \cup \{e_1\}| \leq 1$ or $|Y\backslash\{e_1, e_2, e_3, e_4\}| \leq 1$. In the former case, we must have that $X \subseteq \{e_1, e_2, e_3, e_4\}$. Then since $g \in cl^*_M(X)$, $g \in cl^*_M(\{e_1, e_2, e_3, e_4\}) = cl^*_M(\{e_1, e_2, e_3\})$. Since M is binary, either $\{g, e_1, e_2, e_3, e_4\}$ or $\{g, e_1, e_3, e_2, e_4\}$ is a 5-element fan, a contradiction. So we have that $|Y\backslash\{e_1, e_2, e_3, e_4\}| \leq 1$, and hence $|Y| \leq 3$. If $|Y| = 2$, then Y is a series pair of $M\backslash g$. Since T is a triangle in $M\backslash g$, and $|X \cap T| \geq 2$, it follows that $Y \cap T = \emptyset$. Therefore, $e_4 \in Y$. Thus M has a triad containing both e_4 and g, contrary to the assumption that $M\backslash T^*$ is 3-connected. If $|Y| = 3$, then $|Y \cap \{e_1, e_2, e_3, e_4\}| = 2$. Since $|X \cap T| \geq 2$, $e_4 \in Y$. Since e_4 is not in any triangle of M or $M\backslash g$, we deduce that $r_{M\backslash g}(Y) = 3$. Since $\lambda_{M\backslash g}(Y) = 1$, we conclude that $r^*_{M\backslash g}(Y) = 1$, and every pair of elements of Y is a series pair of $M\backslash g$. Therefore, there exists a series pair of $M\backslash g$ that meets T by a single element, a contradiction.

Claim 4: $g = e_1$.

Suppose this is not the case. Since M is minor-minimally 3-connected, by Claim 3, M/g is not 3-connected. Let (X, Y) be a 2-separation of M/g. By symmetry, assume that $|X \cap T^*| \geq 2$. Then $X \cup \{e_1, e_2, e_3, e_4\}$ is 2-separating in M/g. So $(X\backslash T^*) \cup \{e_1\}$ is 2-separating in $M\backslash T^*/g$. Since $M\backslash T^*/g$ is 3-connected, either $|(X\backslash T^*) \cup \{e_1\}| \leq 1$ or $|Y\backslash\{e_1, e_2, e_3, e_4\}| \leq 1$. In the former case, we must have that $X \subseteq \{e_1, e_2, e_3, e_4\}$. Then since $g \in cl^*_M(X)$, $g \in cl^*_M(\{e_1, e_2, e_3, e_4\}) = cl^*_M(\{e_1, e_2, e_3\})$. Since M is binary, either $\{g, e_1, e_2, e_3, e_4\}$ or $\{g, e_1, e_3, e_2, e_4\}$ is a 5-element fan, a contradiction. So we have that $|Y\backslash\{e_1, e_2, e_3, e_4\}| \leq 1$, and hence $|Y| \leq 3$. If $|Y| = 2$, then Y is a parallel pair. Since T^* is a triad of M and M/g, and $|X \cap T^*| \geq 2$, we have that $T^* \cap Y = \emptyset$. Therefore, $e_1 \in Y$. So M has a triangle containing both e_1 and g, contrary to the assumption that $M\backslash T^*/g$ is 3-connected. If $|Y| = 3$, then $|Y \cap \{e_1, e_2, e_3, e_4\}| = 2$. Since $|X \cap T^*| \geq 2$, $e_1 \in Y$. Since e_1 is not in any triad of M or M/g, $r^*_{M/g}(Y) = 3$. Since $\lambda_{M/g}(Y) = 1$, we have $r_{M/g}(Y) = 1$ and every pair of elements of Y is a parallel pair in M/g. Therefore, there exists a parallel pair of M/g that meets T^* by a single element, a contradiction.

Now consider the matroid M' and the element e_1: either both $M'\backslash e_1$ and M'/e_1 are 3-connected, or one of them is not 3-connected. Therefore, we have the following three cases:

Case 1: $M'\backslash e_1 = \mathrm{co}(M\backslash e_2)\backslash e_1$ is not 3-connected.

Note that $M'\backslash e_1$ is a single-element extension by h of the 3-connected matroid $M\backslash T^*\backslash e_1$. (Recall that $h \in \{e_3, e_4\}$.) Since $M'\backslash e_1$ is connected but not 3-connected, $M'\backslash e_1$ has a parallel pair containing h and some $q \in E(M)\backslash\{e_1, e_2, e_3, e_4\}$. Therefore, $\{e_3, e_4, q\}$ is a triangle of M, contrary to the assumption that M has no fan of length 5.

Case 2: $M'/e_1 = \mathrm{co}(M\backslash e_2)/e_1$ is not 3-connected.

Note that M'/e_1 is a single-element extension by h of the 3-connected matroid $M\backslash T^*/e_1$. Since M'/e_1 is connected but not 3-connected, M'/e_1 has a parallel pair containing h and some $q \in E(M)\backslash\{e_1, e_2, e_3, e_4\}$. Therefore, $\{e_1, e_3, e_4, q\}$ is a circuit of M. Since M is binary, the symmetric difference of the two circuits $\{e_1, e_2, e_3\}$ and $\{e_1, e_3, e_4, q\}$ contains a circuit. As M is 3-connected, we deduce that $\{e_2, e_4, q\}$ is a triangle of M, contrary to the assumption that M has no fan of length 5.

Case 3: Both $M'\backslash e_1$ and M'/e_1 are 3-connected.

Let $A = \{e_1, e_2, e_3, e_4\}$. Let $M'' = \mathrm{co}(M\backslash e_3) = M\backslash e_3/e_2$. It is easily checked that M'' and M/T are 3-connected. Now assume that neither M'' nor M/T is minor-minimally 3-connected. Moreover, by applying the above argument to M^*, we deduce that e_1 and e_4 are violating in M'' and e_4 is violating in M/T. Thus, A is strongly connected. We now show that $M\backslash A$ and M/A are both minor-minimally 3-connected. By duality, it suffices to show that $M\backslash A$ is minor-minimally 3-connected. Note that $M\backslash A \cong M\backslash T^*\backslash e_1$ is 3-connected. Suppose that $M\backslash A$ is not minor-minimally 3-connected. Then there exists $h \in E(M)\backslash A$ such that both $M\backslash A\backslash h$ and $M\backslash A/h$ are 3-connected. Since M is minor-minimally 3-connected, one of $M\backslash h$ and M/h is not 3-connected.

First assume that $M\backslash h$ is not 3-connected. Let (X, Y) be a 2-separation of $M\backslash h$. By symmetry, assume that $|T \cap X| \geq 2$. Then $X \cup A$, or equivalently, $Y\backslash A$, is 2-separating in $M\backslash h$. So $(X\backslash A, Y\backslash A)$ is a 2-separating partition in $M\backslash A\backslash h$. Since $M\backslash A\backslash h$ is 3-connected, either $|X\backslash A| \leq 1$ or $|Y\backslash A| \leq 1$. In the former case, if $|X\backslash A| = 1$, then $X \cup A$ is 3-separating in M, which implies that $X \cup A$ is a 5-element fan, a contradiction; while if $|X \backslash A| = 0$, then $X \subseteq A$. Since M is binary and $h \in cl^*_M(X)$, $h \in cl^*_M(A)$, and hence $A \cup \{h\}$ is a 5-element fan, a contradiction. So we have $|Y \backslash A| \leq 1$, and hence $|Y| \leq 3$. If $|Y| = 2$, then Y is a series pair of $M\backslash h$ and $e_4 \in Y$. So M has a triad containing both e_4 and h, contrary to the fact that $M\backslash T*$ is 3-connected. If $|Y| = 3$, then since $|X \cap T| \geq 2$, $e_4 \in Y$ and $|Y \cap T| = 1$. Since no triangle of M contains e_4, $r_M(Y) = r_{M\backslash h}(Y) = 3$, and hence $r^*_{M\backslash h}(Y) = 1$. Therefore, every pair of elements of Y is a series pair in $M\backslash h$. This is not possible since $|Y \cap T| = 1$ and T is a triangle of M and $M\backslash h$.

Next assume that M/h is not 3-connected. Let (X, Y) be a 2-separation of M/h. By symmetry, assume that $|X \cap T^*| \geq 2$. Then $X \cup A$, or equivalently $Y\backslash A$, is 2-separating in $M\backslash h$. So $(X\backslash A, Y\backslash A)$ is a 2-separating partition in $M\backslash A/h$. Since $M\backslash A/h$ is 3-connected, either $|X\backslash A| \leq 1$ or $|Y\backslash A| \leq 1$. A similar argument as in the last paragraph shows that we cannot have $|X\backslash A| \leq 1$. So we have $|Y\backslash A| \leq 1$, and hence $|Y| \leq 3$. If $|Y| = 2$, then Y is a parallel pair of M/h. Since T^* is a triad of M and M/h, $T^* \cap Y = \emptyset$, and hence $e_1 \in Y$. Now, there exists a triangle of M containing h and e_1, contrary to the fact that M'/e_1 is 3-connected. If $|Y| = 3$, then, since $|X \cap T^*| \geq 2$, $e_1 \in Y$. Since no triad of M contains e_1, $r^*_M(Y) = r^*_{M/h}(Y) = 3$. Therefore, $r_{M/h}(Y) = 1$. So every pair of elements in Y is a parallel pair of M/h. This is not possible, since $|Y \cap T^*| = 1$ and T^* is a triad of M/h. □

Proposition 3.4. *Let M be a minor-minimally 3-connected binary matroid with $|E(M)| \geq 8$. If M contains no fan of length at least 4, then there exists $e \in E(M)$ such that*

- *$M\backslash e$ is not 3-connected, but $co(M\backslash e)$ is 3-connected; or*
- *M/e is not 3-connected, but $si(M/e)$ is 3-connected.*

Proof. Suppose that such an element does not exist. We first show that M has a non-minimal 3-separation. Let $e \in E(M)$. By duality we may assume that $M\backslash e$ is not 3-connected and $co(M\backslash e)$ is not 3-connected. Then $M\backslash e$ has a 2-separation (X_e, Y_e) with $|X_e|, |Y_e| \geq 3$. Since $|E(M)| \geq 8$, either $|X_e| \geq 4$ or $|Y_e| \geq 4$. By symmetry, assume that $|Y_e| \geq 4$. Now $(X_e \cup \{e\}, Y_e)$ is a non-minimal 3-separation of M.

Let (X, Y) be a non-minimal 3-separation of M such that $|X|$ is as small as possible. Let $f \in X$. Then there exists $M' \in \{M\backslash f, M/f\}$ such that M' is not 3-connected and M' has a non-minimal 2-separation (X_f, Y_f). Note that both X_f and $X_f \cup \{f\}$ are 3-separating in M.

Claim 1: Either $|X \cap X_f| \leq 2$ or $|Y \cap Y_f| \leq 1$.

Suppose that $|X \cap X_f| \geq 3$ and $|Y \cap Y_f| \geq 2$. Then by Lemma 2.1, both $X \cap X_f$ and $(X \cap X_f) \cup \{f\}$ are 3-separating in M. By our choice of X, $|X \cap X_f| = 3$ and $X \cap Y_f = \emptyset$. Now $(X \cap X_f) \cup \{f\}$ is a 4-element 3-separating set with f in the guts or coguts. So $(X \cap X_f) \cup \{f\}$ is a 4-element fan. This contradiction completes the proof for Claim 1.

Claim 2: $|X \cap X_f| \leq 2$ and $|X \cap Y_f| \leq 2$.

By symmetry, we need only show that $|X \cap X_f| \leq 2$. By Claim 1, we can assume that $|Y \cap Y_f| \leq 1$. Since $|Y| \geq 4$ and $|Y_f| \geq 3$, we deduce that $|X \cap Y_f| \geq 2$ and $|X_f \cap Y| \geq 3$. By Claim 1 and symmetry, we have $|X \cap Y_f| \leq 2$. Hence $|X \cap Y_f| = 2$. As $|Y_f| \geq 3$, we deduce that $|Y \cap Y_f| = 1$. Now $Y_f \cup \{f\}$ is a 4-element 3-separating set of M with f in the guts or coguts. So $Y_f \cup \{f\}$ is a 4-element fan, a contradiction. Thus Claim 2 holds.

Note that it follows from Claim 2 that $|X| \leq 5$. Since M is binary and contains no fan of length at least 4, we may assume by duality that either X is a quad Q or X contains a quad Q and the element in $X\backslash Q$ is in the guts of (X, Y). In the former case, $Q = X$ cannot be both closed and coclosed, since otherwise, for each $x \in Q$, $M\backslash x$ and M/x are both 3-connected, contrary to the fact that M is minor-minimally 3-connected. So there exists $y \in Y$ such that $Q \cup \{y\}$ is 3-separating in M. Now it is easy to check that in both cases each element in Q has the required property. □

Proposition 3.5. *Let M be a minor-minimally 3-connected matroid with $|E(M)| \geq 8$ and let $e \in E(M)$. If M/e is not 3-connected, and $si(M/e)$ is 3-connected, then Theorem 1.2 holds.*

Proof. By Proposition 3.3, we may assume that M has no fan of length at least 4. Note that e belongs to at least one triangle of M. We may assume that $\mathrm{si}(M/e)$ is not minor-minimally 3-connected. So there exists $f \in E(\mathrm{si}(M/e))$ such that $\mathrm{si}(M/e)\backslash f$ and $\mathrm{si}(M/e)/f$ are both 3-connected. Since M is minor-minimally 3-connected, one of $M\backslash f$ and M/f is not 3-connected. First assume that $M\backslash f$ is not 3-connected. Then by Lemma 3.1, e is in a series pair of $M\backslash f$, and hence e belongs to a triad of M, contrary to the assumption that M contains no fan of length at least 4. Hence, it must be the case that M/f is not 3-connected. By Lemma 3.2, there exists a unique triangle T containing f and e and no other triangle of M contains f. Let $T = \{e, f, g\}$. Note that $\{f, g\}$ is a parallel pair in M/e. By symmetry between f and g, T is the only triangle of M containing g.

Claim: $\mathrm{si}(M/f)$ and $\mathrm{si}(M/g)$ are both 3-connected.

By symmetry, it suffices to prove that $\mathrm{si}(M/f)$ is 3-connected. Suppose this is not the case. Let (X, Y) be a 2-separation of M/f with $|X|, |Y| \geq 3$. Note that neither e nor g is in the guts of (X, Y). By symmetry, we may assume that $\{e, g\} \subseteq X$. Let $x_1, x_2, \ldots, x_k$ be the elements that are removed in the simplification of M/e. Since e is not in the guts of (X, Y), we may assume that $x_i \in X$, $1 \leq i \leq k$. Now let $X' = X\backslash\{e, g, x_1, \ldots, x_k\}$. If $r_{\mathrm{si}(M/e)/f}(X') \geq 2$, then (X', Y) is a 2-separation of $\mathrm{si}(M/e)/f$, contradicting the fact that $\mathrm{si}(M/e)/f$ is 3-connected. Therefore, $r_{\mathrm{si}(M/e)/f}(X') \leq 1$. Since X is not a parallel class of M/f, $r_{M/f}(X) \geq 2$. So we deduce that $r_{\mathrm{si}(M/e)/f}(X') = 1$, which readily implies that $r_M(X) = 3$. Now X is 3-separating in

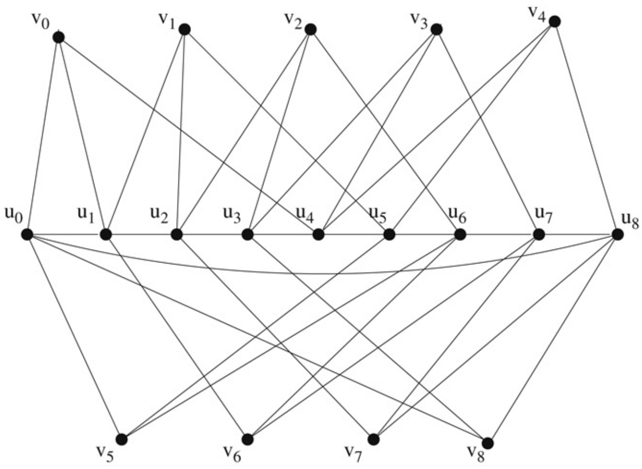

Fig. 1. The graph G_9.

M and f is in the guts of $(X, Y \cup f)$. Since f is only in one triangle, X contains no quad, and is therefore a triad. Now since M is binary, it is clear that $X \cup f$ contains a fan of length at least 4, contrary to our assumption.

Note that we now have symmetries between e, f, and g, and hence T is the only triangle containing e. Therefore $M/T = \text{si}(M/e)/f$ is 3-connected. We are left to show that M/T is minor-minimally 3-connected. Suppose this is not true. Then there exists $h \in E(M)\backslash T$ such that $M/T\backslash h$ and $M/T/h$ are both 3-connected. Note that no triangle or triad of M contains h; if this were so, then this triangle or triad could not intersect T, thus $M/T\backslash h$ or $M/T/h$ would not be 3-connected, a contradiction. Since M is minor-minimally 3-connected, there exists $M' \in \{M\backslash h, M/h\}$ such that M' has a non-minimal 2-separation (X, Y). By symmetry, we may assume that $|X \cap T| \geq 2$. Thus $(X \cup T, Y\backslash T)$ is a 2-separation of M'. Now $(X\backslash T, Y\backslash T)$ is a 2-separating partition of the 3-connected matroid M'/T. Hence, we have $|X\backslash T| \leq 1$. Thus $|X| \in \{3, 4\}$. Note that, if $|X| = 4$, then $T \subseteq X$; since X is a 4-element 3-separating set of M, X is a 4-element fan, a contradiction. If $|X| = 3$, then $X \cup \{h\}$ is a 4-element 3-separating set of M with h lying in the guts or coguts; so $X \cup \{h\}$ is a 4-element fan, a contradiction. □

Now we are ready to prove Theorem 1.2. There are exactly three 3-connected binary matroids of size at least 4 and at most 7; namely, $M(W_3)$, F_7, and F_7^*. Therefore the result holds if $|E(M)| \leq 7$. Moreover, the result clearly holds when M is a wheel. Therefore, Theorem 1.2 follows from Propositions 3.3–3.5 and its dual.

4. Examples

In this section, we give a class of examples in which we have to delete or contract a 4-element fan in Theorem 1.2. Since these examples are graphic, we define them as graphs. Note that, by taking the duals of their cycle matroids, we can obtain a class of non-graphic examples. (Of course, they are co-graphic.)

Let $n \geq 9$. We construct the graph G_n as follows: take a cycle $(u_0, u_1, \ldots, u_{n-1}, u_0)$ of length n, and then add n degree-3 vertices v_i $(0 \leq i \leq n-1)$ where, for each i, the neighbor set of v_i is $\{u_i, u_{i+1}, u_{i+4}\}$ (indices modulo n). Fig. 1 shows the graph G_9. Note that the automorphism group of G_n is acting transitively on the vertex set $\{u_i|0 \leq i \leq n-1\}$ and $\{v_i|0 \leq i \leq n-1\}$, respectively. To see this, one may check that a cyclic rotation of the cycle $(u_0, u_1, \ldots, u_{n-1}, u_0)$ gives rise to an automorphism of G_n. Therefore, all 4-element fans in G_n are symmetric to each other.

Note that every edge of G_n lies in a unique 4-element fan and G_n has no fan of length 5 or more. Choose the 4-element fan $A = \{e_1, e_2, e_3, e_4\}$, where $e_1 = u_0u_1$, $e_2 = v_0u_0$, $e_3 = v_0u_1$, and $e_4 = v_0u_4$. Let $T = \{e_1, e_2, e_3\}$ and $T^* = \{e_2, e_3, e_4\}$. It is straightforward to check that $\text{co}(G_n\backslash e_2) = G_n\backslash e_2/e_3$, $\text{co}(G_n\backslash e_3) = G_n\backslash e_3/e_2$, G_n/T, and $G_n\backslash T^*$ are all 3-connected; and, moreover, none of them is minor-minimally 3-connected, since either e_1 or e_4 is violating.

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References

[1] J. Anderson, H. Wu, Minimally 3-connected binary matroids, European J. Combin. 29 (7) (2008) 1755–1763.
[2] R.E. Bixby, A simple theorem on 3-connectivity, Linear Algebra Appl. (1982) 123–126.
[3] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[4] W.T. Tutte, Connectivity in matroids, Canad. J. Math. 18 (1966) 1301–1324.
[5] D.K. Wagner, Weakly 3-connected graphs, Combin. Probab. Comput. 15 (4) (2006) 605–617.