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On the existence of graphs of diameter two and defect two

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ABSTRACT

In the context of the degree/diameter problem, the 'defect' of a graph represents the difference between the corresponding Moore bound and its order. Thus, a graph with maximum degree d and diameter two has defect two if its order is $n = d^2 - 1$. Only four extremal graphs of this type, referred to as $(d, 2, 2)$-graphs, are known at present: two of degree $d = 3$ and one of degree $d = 4$ and 5, respectively. In this paper we prove, by using algebraic and spectral techniques, that for all values of the degree d within a certain range, $(d, 2, 2)$-graphs do not exist.

The enumeration of $(d, 2, 2)$-graphs is equivalent to the search of binary symmetric matrices A fulfilling that $AJ_n = dJ_n$ and $A^2 + A + (1 - d)I_n = J_n + B$, where J_n denotes the all-one matrix and B is the adjacency matrix of a union of graph cycles. In order to get the factorization of the characteristic polynomial of A in $\mathbb{Q}[x]$, we consider the polynomials $F_{i,d}(x) = f_i(x^2 + x + 1 - d)$, where $f_i(x)$ denotes the minimal polynomial of the Gauss period $\zeta_i + \overline{\zeta_i}$, being ζ_i a primitive ith root of unity. We formulate a conjecture on the irreducibility of $F_{i,d}(x)$ in $\mathbb{Q}[x]$ and we show that its proof would imply the nonexistence of $(d, 2, 2)$-graphs for any degree $d > 5$.

1. Introduction and preliminaries

The modelling of interconnection networks by graphs motivated the study of the optimization problem known as the *degree/diameter problem* (for a survey of it see [12]). In this context, given the values of the maximum degree d and the diameter k of a graph, there is a natural upper bound for its number of vertices n, called *Moore bound* $M_{d,k}$,

$$n \le M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}.$$

Graphs attaining such a bound are referred to as *Moore graphs*. In the case of diameter $k = 2$, Hoffman and Singleton [10] proved that Moore graphs exist for $d = 2, 3, 7$ (being unique) and possibly 57 but for no other degrees. They also showed that for diameter $k = 3$ and degree $d > 2$ Moore graphs do not exist. The enumeration of Moore graphs of diameter $k > 3$ was concluded by Damerell [3], who used the theory of distance-regularity to prove their nonexistence unless $d = 2$, which corresponds to the cycle graph of order $2k + 1$ (an independent proof of it was given by Bannai and Ito [1]).

The fact that there are very few Moore graphs suggested the relaxation of some of the constraints implied by the Moore bound. This led to the study of graphs with order n 'close' to the Moore bound; that is, $n = M_{d,k} - \delta$, where δ is called the *defect*. Such extremal graphs, called (d, k, δ)-graphs for short, must be regular, if $\delta < M_{d,k-1}$. In the case of diameter $k = 2$ and defect $\delta = 1$, Erdös, Fatjlowicz and Hoffman [5] proved that $(d, 2, 1)$-graphs do not exist unless $d = 2$, which corresponds to the cycle graph of order 4. Subsequently, Bannai and Ito [2] extended such a result for any diameter $k > 2$. For larger defect, $\delta \ge 2$, the problem of the existence (d, k, δ)-graphs is widely open (see [12]).

This paper concentrates upon the case of $(d, 2, 2)$-graphs; that is, graphs of degree $d > 2$, diameter $k = 2$ and order $n = M_{d,2} - 2 = d^2 - 1$. Only four $(d, 2, 2)$-graphs are known at present: two of degree $d = 3$ and one of degree $d = 4$ and 5,

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respectively (the last two graphs were found by Elspas [4]). All these constructions turn out to be unique (see [13]). Nguyen and Miller [13] found a number of structural properties of $(d, 2, 2)$-graphs and showed the nonexistence of such graphs for some degrees. In particular, they proved the nonexistence of $(d, 2, 2)$-graphs for $d = 6, 8$ and for infinitely many values of odd d.

1.1. Preliminaries

Let G be a $(d, 2, 2)$-graph. Since its diameter and defect are two, for every vertex v of G there is a multiset of vertices $r(v)$ of cardinality two, $r(v) = \{r_1(v), r_2(v)\}$ (where $r_1(v)$ and $r_2(v)$ may be equal), such that there is one 'extra' $v - r_i(v)$ path of length ≤ 2 to each vertex $r_i(v)$ (in the case $r_1(v) = r_2(v)$ two 'extra' paths are counted). Vertices of $r(v)$ are referred to as the *repeats* of v (if $r_1(v) = r_2(v)$ then $r_1(v)$ is called a *double repeat* of v). Notice that $v \notin r(v)$, since otherwise G would have loops or multiple edges and, consequently, its defect would be at least $1 + d$. Let $R(G)$ be the (multi)graph defined on the same vertex set as G and with an edge between u and v if and only if v is a repeat of u (such an edge becomes double whenever v is a double repeat of u). Notice that $R(G)$ is a union of vertex disjoint cycles of lengths ≥ 2. Such cycles are referred to as *repeat cycles* of G.

Let A and B be the adjacency matrices of G and $R(G)$, respectively. Then,

$$A^2 + A + (1 - d)I_n = J_n + B, \tag{1}$$

where J_n denotes the all-one matrix (see [13]). Fajtlowicz [6] considered the case where B is the adjacency matrix of the n-cycle (G has cyclic defect) and proved that Eq. (1) has no solution unless $d = 3$, which corresponds to the Möbius ladder of order 8. In the general case, as Fajtlowicz pointed out, since J_n commutes with A and B, and therefore A commutes with B, all three matrices can be simultaneously diagonalized. So, the spectrum of A is closely related with the spectrum of B, which only depends on the number m_i of cycles of each length i in which $R(G)$ decomposes, $i = 1, \ldots, n$. The vector $(m_1, \ldots, m_n)$, which represents a partition of n with m_i parts equal to i, will be referred to as the *repeat cycle structure of* G.

We remark that instead of working with the eigenvalues of A, as it is usually done in spectral graph theory, we will collect them into irreducible factors of the characteristic polynomial of A (see Section 2). Such a polynomial approach has also been used in the literature (see, for instance, [10,11]). Then, we will compute spectral invariants like the trace of A (number of loops of G, which is 0) and the trace of A^3 (six times the number of triangles of G, which is known from the work of Nguyen and Miller [13]). As a result, for all values of the degree d within a certain range, a contradiction on some algebraic multiplicities of the eigenvalues of A will be derived and, therefore, the nonexistence of the corresponding $(d, 2, 2)$-graphs will be concluded (see Section 3).

2. The characteristic polynomial of a graph of diameter two and defect two

Let G be a $(d, 2, 2)$-graph with repeat cycle structure $(m_1, \ldots, m_n)$. Let A and B be the adjacency matrices of G and $R(G)$, respectively. Since $R(G)$ is a union of graph cycles, we first derive the factorization in $\mathbb{Q}[x]$ of the characteristic polynomial of the n-cycle graph. If C_n is the adjacency matrix of it then

$$\det(xI_n - C_n) = \prod_{l=1}^{n} (x - (\zeta_n^l + \overline{\zeta_n^l})), \quad \text{where } \zeta_n = e^{2\pi i/n},$$

since C_n is a circulant matrix with Hall polynomial $x + x^{n-1}$ and $\zeta_n^{-1} = \overline{\zeta_n}$. We recall that each nth root of unity ζ_n^l has order a divisor i of n and, consequently, it can be expressed as ζ_i^j, where j is relatively prime with i. By classifying the nth roots of unity according to their order, we have

$$\det(xI_n - C_n) = \prod_{i|n} \prod_{\substack{\gcd(i,j)=1 \\ 1\le j\le i}} (x - (\zeta_i^j + \overline{\zeta_i^j})).$$

Taking into account that $\zeta_1 = 1$ and $\zeta_2 = -1$, and using the fact that $\gcd(i, j) = \gcd(i, i - j)$ and $\zeta_i^{i-j} = \overline{\zeta_i^j}$, we have

$$\det(xI_n - C_n) = \begin{cases} (x-2)(x+2) \prod_{\substack{i|n \\ i\ge 3}} f_i(x)^2, & \text{if } n \text{ is even} \\ (x-2) \prod_{\substack{i|n \\ i\ge 3}} f_i(x)^2, & \text{if } n \text{ is odd,} \end{cases} \tag{2}$$

where

$$f_i(x) = \prod_{\substack{\gcd(i,j)=1 \\ 1\le j<i/2}} (x - (\zeta_i^j + \overline{\zeta_i^j})) \quad (i \ge 3).$$

Notice that $f_i(x)$ is a monic polynomial of degree $\varphi(i)/2$, where $\varphi(i)$ stands for Euler's phi function. It is known that $f_i(x)$ has rational coefficients and, moreover, it is an irreducible polynomial in $\mathbb{Q}[x]$ (see [8]). In fact, $f_i(x)$ is the minimal polynomial of the Gauss periods

$$\theta_v = \sum_{x\in H} \zeta_i^{vx} \quad (v \in \mathbb{Z}_i^*/H),$$

corresponding to the congruence subgroup $H = \{\pm 1\}$. Gurak [9] obtained an explicit formula for the coefficients of

$$f_i(x) = x^{\varphi(i)/2} + \sum_{j=0}^{\varphi(i)/2-1} c_j x^j$$

in terms of the coefficients of the cyclotomic polynomial $\Phi_i(x)$. In particular,

$$c_{\frac{\varphi(i)}{2}-1} = -\sum_{\substack{\gcd(i,j)=1\\ 1\le j<i/2}} (\zeta_i^j + \overline{\zeta_i^j}) = -\sum_{\substack{\gcd(i,j)=1\\ 1\le j<i}} \zeta_i^j = -\mu(i),$$

where $\mu(i)$ denotes the Möbius function.

Now, we obtain the factorization of the characteristic polynomial of B. From (2),

$$\det(xI_n - B) = \prod_{i=1}^{n} \det(xI_i - C_i)^{m_i} = (x-2)^{m(1)}(x+2)^{m(2)} \prod_{\substack{i|n\\ i\ge 3}} f_i(x)^{2m(i)},$$

where $m(i) = \sum_{i|l} m_l$ represents the number of repeat cycles of G of length multiple of i. In particular, $m(1)$ and $m(2)$ correspond to the total number of cycles and even cycles, respectively.

Furthermore, since B and J_n share the eigenvector $(1,\ldots,1)$, with respectively eigenvalues 2 and n, we have

$$\det(xI_n - (J_n + B)) = (x-(n+2))(x-2)^{m(1)-1}(x+2)^{m(2)} \prod_{i=3}^{n} f_i(x)^{2m(i)}.$$

Then, from Eq. (1), the following known results on the characteristic polynomial of G, $\phi(G,x) = \det(xI_n - A)$, are derived:

(P1) Since G is a connected d-regular graph, $x-d$ is a linear factor of $\phi(G,x)$, which corresponds to the factor $x-(n+2)$ of $\det(xI_n - (J_n+B))$.

(P2) If the equation $x^2+x+1-d = 2$ has no rational roots, which is equivalent to saying that its discriminant $4d+5$ is not a square integer, then its two roots are eigenvalues of G with the same multiplicity, $(m(1)-1)/2$; in such a case, $m(1)$ is odd. In other words, if $x^2+x-1-d$ is irreducible in $\mathbb{Q}[x]$ then it is a factor of $\phi(G,x)$ with multiplicity $(m(1)-1)/2$.

(P3) If the equation $x^2+x+1-d = -2$ has no rational roots, which is equivalent to saying that $4d-11$ is not a square integer, then $(x^2+x+3-d)^{m(2)/2}$ is a factor of $\phi(G,x)$; in such a case, $m(2)$ is even.

So, if the polynomials $x^2+x-1-d$ and $x^2+x+3-d$ are both irreducible in $\mathbb{Q}[x]$ then $m(1)$ must be odd and $m(2)$ even. But this parity difference cannot occur when d is odd, since $n = d^2-1$ is even and $n \equiv m(1)-m(2) \pmod 2$. Such argument was given by Nguyen and Miller in [13] to prove the nonexistence of $(d,2,2)$-graphs for infinitely many odd degrees d such that neither $4d+5$ nor $4d-11$ are squares. It turns out that $4d+5$ $[4d-11]$ is a square if and only if $d = l^2+l-1$ $[d = l^2+l+3]$, where l is a nonnegative integer. Clearly, the expressions l^2+l-1 and l^2+l+3 are always odd integers. Notice that if $d = l_1^2+l_1-1 = l_2^2+l_2+3$, with $0 \le l_2 < l_1$, then $(l_1-l_2)(l_1+l_2+1) = 4$, whence $l_1 = 2$ and $l_2 = 1$; that is, $d = 5$.

The above results can be summarized as follows:

Theorem 1 (*Nguyen and Miller [13]*). *Let G be a $(d,2,2)$-graph and let $(m_1,\ldots,m_n)$ be its repeat cycle structure.*

(i) *If d is odd then $d = l^2+l-1$ or $d = l^2+l+3$, in which case $m(1)$ and $m(2)$ have the same parity.*
  (i1) *If $d = l^2+l+3$ and $l>1$ then $(x^2+x-1-d)^{(m(1)-1)/2}$ is a factor of $\phi(G,x)$; in particular $m(1)$ is odd.*
  (i2) *If $d = l^2+l-1$ and $l>2$ then $(x^2+x+3-d)^{m(2)/2}$ is a factor of $\phi(G,x)$; in particular $m(2)$ is even.*

(ii) *If d is even then $(x^2+x-1-d)^{(m(1)-1)/2}$ and $(x^2+x+3-d)^{m(2)/2}$ are both factors of $\phi(G,x)$; in particular, $m(1)$ is odd and $m(2)$ is even.*

Next, we show how the study of the irreducibility in $\mathbb{Q}[x]$ of the polynomials

$$F_{i,d}(x) = f_i(x^2+x+1-d)$$

is related with the factorization of $\phi(G,x)$.

Lemma 1. *Let G be a $(d,2,2)$-graph, with repeat cycle structure $(m_1,\ldots,m_n)$, and let $3 \le i \le n$. If $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$ then $F_{i,d}(x)$ is a factor of $\phi(G,x)$ and its multiplicity is $m(i)$.*

Proof. Since $f_i(x)$ is an irreducible factor of $\det(xI_n - (J_n + B))$ with multiplicity $2m(i)$, for each of its roots $\mu_{i,k}$ there are $2m(i)$ eigenvalues of G (counting multiplicities) that satisfy the equation $x^2 + x + 1 - d = \mu_{i,k}$. So, all these eigenvalues are roots of the polynomial

$$\prod_{k=1}^{\varphi(i)/2} (x^2 + x + 1 - d - \mu_{i,k}) = F_{i,d}(x).$$

Therefore, if $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, and $m(i) \geq 1$, then it must be a factor of $\phi(G, x)$, since $\phi(G, x) \in \mathbb{Q}[x]$ and $\gcd(\phi(G, x), F_{i,d}(x)) > 1$. In addition, since the sum of the multiplicities of the eigenvalues of G that are roots of $F_{i,d}(x)$ is equal to $m(i)\varphi(i)$, the multiplicity of $F_{i,d}(x)$ as a factor of $\phi(G, x)$ is equal to $m(i)$. □

Notice that $F_{i,d}(x)$ has degree two if and only if $\varphi(i) = 2$; that is, $i = 3, 4, 6$. In these three cases the irreducibility of the polynomials $F_{i,d}(x)$ is easily determined.

Lemma 2. *Let $d \geq 3$ be an integer.*

(i) *The polynomial $F_{3,d}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $d = l^2 + l + 2$, where $l \geq 1$.*
(ii) *The polynomial $F_{4,d}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $d = l^2 + l + 1$, where $l \geq 1$.*
(iii) *The polynomial $F_{6,d}(x)$ is reducible in $\mathbb{Q}[x]$ if and only if $d = l^2 + l$, where $l \geq 2$.*

Proof. We know that $f_3(x) = x + 1$, $f_4(x) = x$ and $f_6(x) = x - 1$. Therefore, $F_{3,d}(x) = x^2 + x + 2 - d$ is reducible in $\mathbb{Q}[x]$ if and only if $4d - 7$ is a square; that is, $d = l^2 + l + 2$. Analogously, $F_{4,d}(x) = x^2 + x + 1 - d$ $[F_{6,d}(x) = x^2 + x - d]$ is reducible in $\mathbb{Q}[x]$ if and only if $4d - 3$ $[4d + 1]$ is a square; that is, $d = l^2 + l + 1$ $[d = l^2 + l]$. □

In order to 'find out' what happens with the irreducibility of the polynomials $F_{i,d}(x)$, when $i \neq 3, 4, 6$ and $3 \leq i \leq d^2 - 1$, we have carried out some explorations using the open source mathematics software PARI ([14]). All the computations performed suggest that, for even $d > 6$, $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$.

Conjecture 1. *Let $d > 6$ be an even integer and let i be an integer such that $3 \leq i \leq d^2 - 1$.*

(i) *If $d = l^2 + l + 2$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$ unless $i = 3$.*
(ii) *If $d = l^2 + l$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$ unless $i = 6$.*
(iiii) *If $d \neq l^2 + l$ and $d \neq l^2 + l + 2$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$.*

For odd d we have the following conjecture:

Conjecture 2. *Let $d > 3$ be an odd integer such that $d = l^2 + l - 1$ or $d = l^2 + l + 3$. Then, $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$ for every i, $3 \leq i \leq d^2 - 1$.*

3. Nonexistence of graphs of diameter two and defect two

3.1. Case of odd degree

As we have already mentioned, $(d, 2, 2)$-graphs of odd degree d do not exist unless $d = l^2 + l - 1$ or $d = l^2 + l + 3$.

Theorem 2. *Let $d > 5$ be an odd integer such that either $d = l^2 + l - 1$ or $d = l^2 + l + 3$. If the polynomial $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for every $i = 3, \ldots, d^2 - 1$, then no $(d, 2, 2)$-graph exists.*

Proof. Let G be a $(d, 2, 2)$-graph, with order n, and let $(m_1, \ldots, m_n)$ be its repeat cycle structure.

First, let us consider the case $d = l^2 + l - 1$, with $l > 2$. Let us assume that $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for $i = 3, \ldots, d^2 - 1$. Then, in order to obtain the characteristic polynomial of G, we apply Theorem 1 and Lemma 1. Thus, taking into account that $x^2 + x - 1 - d = (x - l)(x + l + 1)$,

$$\phi(G, x) = (x - d)(x - l)^{\alpha}(x + l + 1)^{m(1)-1-\alpha}(x^2 + x + 3 - d)^{m(2)/2} \prod_{i=3}^{n} F_{i,d}(x)^{m(i)},$$

where $0 \leq \alpha \leq m(1) - 1$. Now, we can compute the spectral invariants of G in terms of its repeat cycle structure. In particular, we obtain the trace of the adjacency matrix A of G from the traces of the factors of $\phi(G, x)$. We recall that if $a(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$ is a monic polynomial of degree $n \geq 1$, its trace $\operatorname{tr} a(x)$ is defined as the sum of all its roots; that is, $\operatorname{tr} a(x) = -a_{n-1}$. Obviously, $\operatorname{tr} a(x)b(x) = \operatorname{tr} a(x) + \operatorname{tr} b(x)$ for all pairs of polynomials. Since

$$\operatorname{tr} F_{i,d}(x) = -\frac{\varphi(i)}{2},$$

we have

$$\operatorname{tr} A = d + \alpha l - (l+1)(m(1) - 1 - \alpha) - \frac{m(2)}{2} - \frac{1}{2}\sum_{i=3}^{n} m(i)\varphi(i).$$

Then, taking into account the identity $\sum_{i=1}^{n} m(i)\varphi(i) = n$ (see [7]) and since $d = l^2 + l - 1$ and $n = d^2 - 1$, it follows that

$$\operatorname{tr} A = -\frac{1}{2}l(l^3 + 2l^2 - 3l - 6) + (2l+1)\left(\alpha - \frac{m(1)}{2}\right).$$

By imposing the condition $\operatorname{tr} A = 0$, we get

$$\alpha = \frac{m(1)}{2} + \frac{1}{32}\left(8l^3 + 12l^2 - 30l - 33 + \frac{33}{2l+1}\right). \tag{3}$$

Since α must be an integer, $(2l+1)|33$; that is, $l = 5, 16$. It can be checked that for these two particular cases a 'feasible' value for α is obtained, since $m(1)$ is even. Let us derive another constraint by using the trace of a power of A. Notice that the relation $\operatorname{tr} A^2 = n \cdot d$ is implied by the condition $\operatorname{tr} A = 0$, since $A^2 + A + (1-d)I_n = J_n + B$ and $\operatorname{tr} B = 0$. So, we proceed with the computation of $\operatorname{tr} A^3$.

Given a monic polynomial $a(x) = x^n + \sum_{j=0}^{n-1} a_j x^j$, let $\operatorname{tr}^{(3)}(a(x))$ be the sum of the cubes of all its roots. Such a sum can be expressed in terms of the coefficients of $a(x)$, by means of Newton's formulas. Thus,

$$\operatorname{tr}^{(3)}(a(x)) = -a_{n-1}^3 + 3a_{n-1}a_{n-2} - 3a_{n-3}. \tag{4}$$

In particular, taking into account that

$$F_{i,d}(x) = (x^2 + x + 1 - d)^{\varphi(i)/2} - \mu(i)(x^2 + x + 1 - d)^{\varphi(i)/2-1} + \cdots,$$

we obtain

$$\operatorname{tr}^{(3)}(F_{i,d}(x)) = -\frac{1}{2}(3d-2)\varphi(i) - 3\mu(i).$$

By applying (4) to each factor of $\phi(G, x)$, we get

$$\begin{aligned}\operatorname{tr} A^3 &= d^3 + l^3\alpha - (l+1)^3(m(1) - 1 - \alpha) + (8 - 3d)\frac{m(2)}{2} \\ &\quad - \frac{1}{2}(3d-2)\sum_{i=3}^{n} m(i)\varphi(i) - 3\sum_{i=3}^{n} m(i)\mu(i).\end{aligned}$$

Then, using the identity $\sum_{i=1}^{n} m(i)\mu(i) = m_1$ (see [7]) and since $m_1 = 0$, it follows that

$$\operatorname{tr} A^3 = -\frac{1}{2}l(l+2)(l^4 + l^3 - 4l^2 - 3l - 1) + (2l+1)(l^2 + l + 1)\left(\alpha - \frac{m(1)}{2}\right).$$

By substituting α in the previous expression for (3), we obtain

$$\operatorname{tr} A^3 = (l-1)l(l+1)(l+2).$$

Since the number of triangles of G must be either 0 or 3 (see Nguyen and Miller [13, Theorem 4]), $\operatorname{tr} A^3 = 0, 18$, which is impossible.

Now, let us take $d = l^2 + l + 3$, with $l > 1$, and let us assume that $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for $i = 3, \ldots, d^2 - 1$. Then,

$$\phi(G, x) = (x - d)(x^2 + x - 1 - d)^{(m(1)-1)/2}(x - l)^{\alpha}(x + l + 1)^{m(2)-\alpha}\prod_{i=3}^{n} F_{i,d}(x)^{m(i)},$$

where $0 \le \alpha \le m(2)$. As a consequence,

$$\operatorname{tr} A = -\frac{1}{2}(l+1)(l^3 + l^2 + 4l + 2) + (2l+1)\alpha - \frac{1}{2}(2l+1)(m(2) - 1).$$

So, the condition $\operatorname{tr} A = 0$ implies that

$$\alpha = \frac{m(2) - 1}{2} + \frac{1}{32}\left(8l^3 + 12l^2 + 34l + 31 + \frac{1}{2l+1}\right).$$

Since α must be an integer, $2l + 1 = \pm 1$, which is impossible. □

We have checked that Conjecture 2 holds for all required values of $d > 5$ up to 50. So, for any of them we can apply Theorem 2 and conclude the nonexistence of the corresponding graphs of defect two and diameter two.

Corollary 1. *No $(d, 2, 2)$-graph exists for odd degree d, $5 < d < 50$.*

3.2. Case of even degree d

Theorem 3. *Let $d > 6$ be an even integer and let us assume that one of the following conditions holds:*

(i) *If $d = l^2 + l + 2$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for every $i = 3, \ldots, d^2 - 1$ unless $i = 3$.*
(ii) *If $d = l^2 + l$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for every $i = 3, \ldots, d^2 - 1$ unless $i = 6$.*
(iii) *If $d \neq l^2 + l$ and $d \neq l^2 + l + 2$ then $F_{i,d}(x)$ is irreducible in $\mathbb{Q}[x]$, for every $i = 3, \ldots, d^2 - 1$.*

Then, no $(d, 2, 2)$-graph exists.

Proof. Let G be a $(d, 2, 2)$-graph, with adjacency matrix A, and let $(m_1, \ldots, m_n)$ be its repeat cycle structure, where $n = d^2 - 1$.

First, let us consider $d = l^2 + l + 2$ $(l > 1)$ and let us assume that condition (i) holds. From Theorem 1 and Lemma 1, and taking into account that $F_{3,d}(x) = x^2 + x + 2 - d = (x - l)(x + l + 1)$,

$$\phi(G, x) = (x - d)(x^2 + x - 1 - d)^{(m(1)-1)/2}(x^2 + x + 3 - d)^{m(2)/2}$$
$$(x - l)^{\alpha}(x + l + 1)^{2m(3)-\alpha} \prod_{i=4}^{n} F_{i,d}(x)^{m(i)},$$

where $0 \le \alpha \le 2m(3)$. Then,

$$\operatorname{tr} A = -\frac{1}{2}(l^4 + 2l^3 + 3l^2 + 2l - 2) + (2l + 1)(\alpha - m(3)).$$

So, the condition $\operatorname{tr} A = 0$ implies that

$$\alpha = m(3) + \frac{1}{32}\left(8l^3 + 12l^2 + 18l + 7 - \frac{39}{2l + 1}\right). \tag{5}$$

Since α must be an integer, $l = 6, 19$. In each of these two cases, a feasible value for α is obtained. Analogously to the proof of Theorem 2, we proceed with the computation of $\operatorname{tr} A^3$,

$$\operatorname{tr} A^3 = -\frac{1}{2}(l^6 + 3l^5 + 7l^4 + 9l^3 + 2l^2 - 2l - 14) + (2l + 1)(l^2 + l + 1)(\alpha - m(3)).$$

By substituting α in the previous expression for (5),

$$\operatorname{tr} A^3 = -\frac{1}{2}(l^4 + 2l^3 - l^2 - 2l - 12).$$

Hence, $\operatorname{tr} A^3 < 0$, which is impossible.

Next, let us assume that $d = l^2 + l$ $(l > 2)$ and that condition (ii) holds. Then,

$$\phi(G, x) = (x - d)(x^2 + x - 1 - d)^{(m(1)-1)/2}(x^2 + x + 3 - d)^{m(2)/2}$$
$$(x - l)^{\alpha}(x + l + 1)^{2m(6)-\alpha} \prod_{\substack{i=3 \\ i \neq 6}}^{n} F_{i,d}(x)^{m(i)},$$

where $0 \le \alpha \le 2m(6)$ is a nonnegative integer. Therefore,

$$\operatorname{tr} A = -\frac{1}{2}(l^4 + 2l^3 - l^2 - 2l - 2) + (2l + 1)(\alpha - m(6)).$$

So, the condition $\operatorname{tr} A = 0$ implies that

$$\alpha = m(6) + \frac{1}{32}\left(8l^3 + 12l^2 - 14l - 9 - \frac{23}{2l + 1}\right). \tag{6}$$

Since α must be an integer, $l = 11$. Besides,

$$\operatorname{tr} A^3 = -\frac{1}{2}(l^6 + 3l^5 + l^4 - 3l^3 - 8l^2 - 6l - 2) + (2l + 1)(l^2 + l + 1)(\alpha - m(6)),$$

and using the expression of α, given in (6), we obtain

$$\operatorname{tr} A^3 = -\frac{1}{2} l(l+1)(l^2+l+2).$$

As a consequence, $\operatorname{tr} A^3 < 0$, which is impossible.

Finally, let us assume that neither $d = l^2 + l$ nor $d = l^2 + l + 2$ and that condition (iii) holds. Then,

$$\phi(G, x) = (x-d)(x^2+x-1-d)^{(m(1)-1)/2}(x^2+x+3-d)^{m(2)/2} \prod_{i=3}^{n} F_{i,d}(x)^{m(i)}.$$

Therefore,

$$\operatorname{tr} A = d + \frac{1}{2} - \frac{1}{2} n.$$

Hence, the condition $\operatorname{tr} A = 0$ implies that $d^2 - 2d - 2 = 0$, which is impossible. □

We have checked that Conjecture 1 holds for all even values of $d > 6$ up to 50. So, for any of them we can apply Theorem 2 and conclude the nonexistence of the corresponding graphs of defect two and diameter two.

Corollary 2. *No $(d, 2, 2)$-graph exists for even degree d, $4 < d \le 50$.*

We end up by noticing that the proof of Conjectures 1 and 2 would imply the nonexistence of $(d, 2, 2)$-graphs for any degree $d > 5$.

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