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Note

On the $\{k\}$-domination number of Cartesian products of graphs☆

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ABSTRACT

Let $G\Box H$ denote the Cartesian product of graphs G and H. In this paper, we study the $\{k\}$-domination number of Cartesian product of graphs and give a new lower bound of $\gamma^{\{k\}}(G\Box H)$ in terms of packing and $\{k\}$-domination numbers of G and H. As applications of this lower bound, we prove that: (i) For $k = 1$, the new lower bound improves the bound given by Chen, et al. [G. Chen, W. Piotrowski, W. Shreve, A partition approach to Vizing's conjecture, J. Graph Theory 21 (1996) 103–111]. (ii) The product of the $\{k\}$-domination numbers of two any graphs G and H, at least one of which is a (ρ, γ)-graph, is no more than $k\gamma^{\{k\}}(G\Box H)$. (iii) The product of the $\{2\}$-domination numbers of any graphs G and H, at least one of which is a $(\rho, \gamma - 1)$-graph, is no more than $2\gamma^{\{2\}}(G\Box H)$.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The open neighborhood of a vertex $v \in V$, denoted by $N_G(v)$, is the set of adjacent vertices of v, and the closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \cup_{v\in S} N_G(v)$ and $N_G[S] = N_G(S) \cup S$. The subgraph induced by S is denoted by $G[S]$. A set $S \subseteq V(G)$ is an independent set if no two vertices in S are adjacent.

For $S \subseteq V(G)$, S is a dominating set if for any vertex $v \in V(G)$, $|N_G[v] \cap S| \geq 1$, and a packing of G if for any two distinct vertices u and v in S, $N_G[u] \cap N_G[v] = \emptyset$. The minimum cardinality of a dominating set of G is the domination number $\gamma(G)$. The packing number $\rho(G)$ is the maximum cardinality of a packing of G. For any graph G, $\rho(G) \leq \gamma(G)$. For $0 \leq k < \gamma(G)$, we define graph G to be a $(\rho, \gamma - k)$-graph if $\rho(G) = \gamma(G) - k$. It is known that a tree T is a (ρ, γ)-graph.

For a subset Y of the reals $\mathbb{R}$, $Y \subseteq \mathbb{R}$, the weight of a function $f : V(G) \to Y$ is $\omega(f) = \sum_{v\in V(G)} f(v)$, and for $S \subseteq V(G)$ we define $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V(G))$. For any fixed positive integer k, the function $f : V(G) \to \mathbb{N}$ is called a $\{k\}$-dominating function of G if for every $v \in V(G)$, $f(N_G[v]) \geq k$, where $\mathbb{N} = \{0, 1, 2 \ldots\}$ is the set of nonnegative integers. The $\{k\}$-domination number $\gamma^{\{k\}}(G)$ of G is the minimum weight of a $\{k\}$-dominating function. One can clearly restrict to functions with range $\{0, 1, 2, \ldots, k\}$.

For the parameters $\rho(G)$, $\gamma(G)$ and $\gamma^{\{k\}}(G)$, Domke, et al. [4] showed that

Proposition 1 ([4,7]). *For any graph G and positive integer k, $k\rho(G) \leq \gamma^{\{k\}}(G) \leq k\gamma(G)$.*

As an immediate consequence of Proposition 1, we have the following result.

Corollary 2. *If G is a (ρ, γ)-graph, then $\gamma^{\{k\}}(G) = k\rho(G) = k\gamma(G)$.*

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For graphs G and H, the Cartesian product $G\Box H$ is the graph with vertex set $V(G)\times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$. The most famous open problem involving domination in graphs is Vizing's conjecture [8] which states that: For any graphs G and H, $\gamma(G)\gamma(H) \le \gamma(G\Box H)$. The best general upper bound to date on $\gamma(G)\gamma(H)$ in terms of $\gamma(G\Box H)$ is due to Clark and Suen [3]. They proved that: For any graphs G and H, $\gamma(G)\gamma(H) \le 2\gamma(G\Box H)$. In [2], Chen, et al. gave a lower bound of $\gamma(G\Box H)$ in terms of the packing and the domination numbers of G and H. They proved that

$$\gamma(G\Box H) \ge \rho(G)\gamma(H) + \rho(H)(\gamma(G) - \rho(G)). \tag{1}$$

In [7,6], Hartnell and Rall proved that Vizing's conjecture is true for (ρ, γ)-graphs and $(\rho, \gamma - 1)$-graphs.

The inability of proving or disproving Vizing's conjecture lead authors to pose different variations of the original problem. The $\{k\}$-domination version of Vizing's conjecture has been studied by Brešar, Henning and Klarzar [1]. They proved that

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H) \le k(k+1)\gamma^{\{k\}}(G\Box H) \tag{2}$$

and

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H) \le 2k\gamma^{\{k\}}(G\Box H) + k\psi(G, H), \tag{3}$$

where $\psi(G, H) = \min\{|V(H)|(k\gamma(G) - \gamma^{\{k\}}(G)), |V(G)|(k\gamma(H) - \gamma^{\{k\}}(H))\}$.

In this paper, we give a lower bound of $\gamma^{\{k\}}(G\Box H)$ in terms of packing and $\{k\}$-domination numbers of G and H. The main result is

Theorem 3. *For any graphs G and H and positive integer $k \ge 1$, the following statements are true.*

(i) $\gamma^{\{k\}}(G\Box H) \ge \max\{\rho(G)\gamma^{\{k\}}(H), \rho(H)\gamma^{\{k\}}(G)\}$.

(ii) *Let $m = \gamma^{\{k\}}(G) - k\rho(G)$. If $m > 0$, then $\gamma^{\{k\}}(G\Box H) \ge \rho(G)\gamma^{\{k\}}(H) + \gamma^{\{m\}}(H)$.*

Some applications of the theorem are given in Section 3.

2. The proof of Theorem 3

In this section, we give a proof of Theorem 3.

Let $A = \{v_1, \ldots, v_{\rho(G)}\}$ be a maximum packing of G. Then the sets $N_G[v_i]$ are pairwise disjoint for $i = 1, \ldots, \rho(G)$. Let $\Pi_i = N_G[v_i]$ and $\Pi_0 = V(G) - \cup_{i=1}^{\rho(G)} \Pi_i$. Then $\{\Pi_0, \Pi_1, \ldots, \Pi_{\rho(G)}\}$ is a partition of V(G) (note that Π_0 is possibly empty). For $i = 1, \ldots, \rho(G)$ and $w \in V(H)$, let $H_{v_i} = \{v_i\} \times V(H)$, $G_w = V(G) \times \{w\}$, and $G_{i,w} = \Pi_i \times \{w\}$. Clearly, the subgraphs induced by H_{v_i} (resp. G_w) is isomorphic to H (resp. G). In the following, instead of $(G\Box H)[H_{v_i}]$ and $(G\Box H)[G_w]$, we simply use H_{v_i} and G_w.

Let f be a minimum $\{k\}$-dominating function of $G\Box H$, then $\omega(f) = \gamma^{\{k\}}(G\Box H)$. For any (u, v) in $G\Box H$, we simply use $f(u, v)$ instead of $f((u, v))$.

Lemma 4. $\gamma^{\{k\}}(G\Box H) \ge \max\{\rho(G)\gamma^{\{k\}}(H), \rho(H)\gamma^{\{k\}}(G)\}$.

Proof. For $i = 1, \ldots, \rho(G)$, define $h_i(w) = f(G_{i,w})$ for $w \in V(H)$. Then, for any $w \in V(H)$,

$$\begin{aligned} h_i(N_H[w]) &= \sum_{u\in N_H[w]} h_i(u) = \sum_{u\in N_H[w]} f(G_{i,u}) = f(G_{i,w}) + \sum_{u\in N_H(w)} f(G_{i,u}) \\ &\ge f(G_{i,w}) + \sum_{u\in N_H(w)} f(v_i, u) = f(N_{G\Box H}[(v_i, w)]) \ge k. \end{aligned}$$

Hence, h_i is a $\{k\}$-dominating function of H, so $w(h_i) = \sum_{u\in V(H)} f(G_{i,u}) \ge \gamma^{\{k\}}(H)$.

Thus,

$$\gamma^{\{k\}}(G\Box H) \ge \sum_{i=1}^{\rho(G)} \sum_{w\in V(H)} f(G_{i,w}) \ge \sum_{i=1}^{\rho(G)} \gamma^{\{k\}}(H) = \rho(G)\gamma^{\{k\}}(H).$$

By the symmetry of G and H, the result follows. $\Box$

Lemma 5. *Let $m = \gamma^{\{k\}}(G) - k\rho(G)$. If $m > 0$, then*

$$\gamma^{\{k\}}(G\Box H) \ge \rho(G)\gamma^{\{k\}}(H) + \gamma^{\{m\}}(H).$$

Proof. If $m = \gamma^{\{k\}}(G) - k\rho(G) > 0$, then $\Pi_0 \neq \emptyset$. (If not, define $\phi(v) = k$ if $v \in A$; 0 otherwise. Then ϕ is a $\{k\}$-dominating function of G with weight $k|A| = k\rho(G)$, a contradiction.) Let $G_{0,w} = \Pi_0 \times \{w\}$, $w \in V(H)$.

For $i = 1, \ldots, \rho(G)$ and $w \in V(H)$, define

$$r(G_{i,w}) = \min\left\{\sum_{u\in N_H[w]} f(G_{i,u}) - k, f(G_{i,w}) - f(v_i, w)\right\}.$$

Claim 1. *Let $\sum_{u\in N_H[w]} f(G_{0,u}) = l_w$. If $l_w < m$, then*

$$\sum_{i=1}^{\rho(G)} r(G_{i,w}) \geq m - l_w.$$

Proof of Claim 1. Let $t = \max\{r(G_{i,w}) \mid 1 \leq i \leq \rho(G)\}$. For $0 \leq j \leq t$, let $A_j = \{v_i \mid r(G_{i,w}) = j\}$, let $B_j = \{v_i \mid r(G_{i,w}) = \sum_{u\in N_H[w]} f(G_{i,u}) - k = j\}$ and $C_j = A_j - B_j$, that is $C_j = \{v_i \mid r(G_{i,w}) = f(G_{i,w}) - f(v_i, w) = j < \sum_{u\in N_H[w]} f(G_{i,u}) - k\}$. Clearly, $A_0, A_1, \ldots, A_t$ is a partition of A, and B_j, C_j is a partition of A_j. Let $B = \cup_{j=0}^{t} B_j$ and $C = \cup_{j=0}^{t} C_j$. Then $B \cup C = A$ and $|B| + |C| = |A| = \rho(G)$. Further,

$$\sum_{i=1}^{\rho(G)} r(G_{i,w}) = \sum_{j=0}^{t} j|A_j| = \sum_{j=0}^{t} j(|B_j| + |C_j|). \tag{4}$$

For $v \in V(G)$, define

$$f_w(v) = \begin{cases} f(v_i, w) + \sum\limits_{u\in N_H(w)} f(G_{i,u}) & : v = v_i \in B \\ k & : v = v_i \in C \\ \sum\limits_{u\in N_H[w]} f(v, u) & : v \in \Pi_0 \\ f(v, w) & : v \in V(G) - \Pi_0 - A. \end{cases}$$

We claim that f_w is a $\{k\}$-dominating function of G, that is for any $v \in V(G)$, $f_w(N_G[v]) \geq k$. Clearly, $f_w(v) \geq f(v, w)$ for any $v \in V(G)$.

If $v \in \Pi_0$, then

$$\begin{aligned} f_w(N_G[v]) &= f_w(v) + \sum_{v'\in N_G(v)} f_w(v') \\ &\geq \sum_{u\in N_H[w]} f(v, u) + \sum_{v'\in N_G(v)} f(v', w) \\ &= f(N_{G\square H}[(v, w)]) \geq k. \end{aligned}$$

If $v \in V(G) - \Pi_0$, then there exists some $v_i \in A$ ($1 \leq i \leq \rho(G)$) such that $v \in N_G[v_i]$, and so $v_i \in N_G[v]$. If $v_i \in B$, then

$$\begin{aligned} f_w(N_G[v]) &= \sum_{\substack{v'\in N_G[v] \\ v'\neq v_i}} f_w(v') + f_w(v_i) \\ &\geq \sum_{\substack{v'\in N_G[v] \\ v'\neq v_i}} f(v', w) + f(v_i, w) + \sum_{u\in N_H(w)} f(G_{i,u}) \\ &\geq \sum_{v'\in N_G[v]} f(v', w) + \sum_{u\in N_H(w)} f(v, u) \\ &= f(N_{G\square H}[(v, w)]) \geq k. \end{aligned}$$

If $v_i \in C$, then $f_w(N_G[v]) = \sum_{v'\in N_G[v]} f_w(v') \geq f_w(v_i) = k$. The claim follows.

Since G_w is isomorphic to G, $\omega(f_w) \geq \gamma^{\{k\}}(G)$. So,

$$\begin{aligned} k\rho(G) + m &= \gamma^{\{k\}}(G) \leq \omega(f_w) \\ &= \sum_{v_i\in B} f_w(v_i) + \sum_{v_i\in C} k + \sum_{v\in \Pi_0} \sum_{u\in N_H[w]} f(v, u) + \sum_{v\notin(\Pi_0\cup A)} f(v, w) \\ &= \sum_{v_i\in B} \sum_{u\in N_H(w)} f(G_{i,u}) + \sum_{v_i\in C} (k - f(v_i, w)) + l_w + \sum_{i=1}^{\rho(G)} f(G_{i,w}) \end{aligned}$$

$$= \sum_{v_i\in B}\sum_{u\in N_H[w]} f(G_{i,u}) + \sum_{v_i\in C}(f(G_{i,w}) + k - f(v_i, w)) + l_w$$
$$= \sum_{j=0}^{t}\left(\sum_{v_i\in B_j}(k+j) + \sum_{v_i\in C_j}(k+j)\right) + l_w$$
$$= k\sum_{j=0}^{t}(|B_j| + |C_j|) + \sum_{j=0}^{t} j(|B_j| + |C_j|) + l_w$$
$$= k\rho(G) + \sum_{i=1}^{\rho(G)} r(G_{i,w}) + l_w.$$

The third equality follows from

$$\sum_{v\in\Pi_0}\sum_{u\in N_H[w]} f(v,u) = \sum_{u\in N_H[w]}\sum_{v\in\Pi_0} f(v,u) = \sum_{u\in N_H[w]} f(G_{0,u}) = l_w$$

and

$$\sum_{v\notin(\Pi_0\cup A)} f(v,w) = \sum_{i=1}^{\rho(G)}(f(G_{i,w}) - f(v_i,w)) = \sum_{i=1}^{\rho(G)} f(G_{i,w}) - \sum_{v_i\in B} f(v_i,w) - \sum_{v_i\in C} f(v_i,w).$$

The fifth equality follows from the definition of B_j and C_j. The last equality follows from $\sum_{j=0}^{t}(|B_j| + |C_j|) = \sum_{j=0}^{t}|A_j| = |A| = \rho(G)$ and Eq. (4).

Therefore, we have $\sum_{i=1}^{\rho(G)} r(G_{i,w}) \geq m - l_w$. The proof of the claim is completed.

In the rest of the proof, we will construct a $\{k\}$-dominating function, say f_i, of $H \cong H_{v_i}$ for each $1 \leq i \leq \rho(G)$ and an $\{m\}$-dominating function, f_0 say, of H such that $\omega(f) = \sum_{i=1}^{\rho(G)}\omega(f_i) + \omega(f_0)$.

We proceed as follows. Recall that l_w is defined as $\sum_{u\in N_H[w]} f(G_{0,u})$. For $0 \leq t \leq m-1$, define $S'_t = \{w \mid l_w = t, w \in V(H)\}$. Let S_0 be a maximum independent set of $H[S'_0]$. Let S_t be a maximum independent set of $H\left[S'_t - N_H[\cup_{j=0}^{t-1} S_j]\right]$ for $1 \leq t \leq m-1$ and $m \geq 2$. By the definition of S_t, the vertices in S_t are not adjacent with the vertices in S_j for $0 \leq j \leq t-1$. Hence $\cup_{t=0}^{m-1} S_t$ is an independent set of H. Let $S = \cup_{t=0}^{m-1} S_t$ and $\bar{S} = V(H) - S$.

For given $1 \leq i \leq \rho(G)$, define

$$f_i(w) = \begin{cases} f(G_{i,w}) - r(G_{i,w}) & : w \in S\\ f(G_{i,w}) & : w \in \bar{S}\end{cases}$$

for $w \in V(H)$. We show that f_i is a $\{k\}$-dominating function of H. Let w be any vertex of $V(H)$. Recall that $r(G_{i,w}) = \min\{\sum_{u\in N_H[w]} f(G_{i,u}) - k, f(G_{i,w}) - f(v_i, w)\}$.

If $w \in \bar{S}$, then

$$\begin{aligned} f_i(N_H[w]) &= \sum_{u\in N_H[w]} f_i(u) = f(G_{i,w}) + \sum_{u\in N_H(w)} f_i(u)\\ &\geq f(G_{i,w}) + \sum_{u\in N_H(w)}(f(G_{i,u}) - r(G_{i,u}))\\ &\geq f(G_{i,w}) + \sum_{u\in N_H(w)} f(v_i, u)\\ &= f(N_{G\Box H}[(v_i, w)]) \geq k.\end{aligned}$$

If $w \in S$, then, note that S is an independent set,

$$\begin{aligned} f_i(N_H[w]) &= \sum_{u\in N_H[w]} f_i(u) = f_i(w) + \sum_{u\in N_H(w)} f_i(u)\\ &= f(G_{i,w}) - r(G_{i,w}) + \sum_{u\in N_H(w)} f(G_{i,u})\\ &= \sum_{u\in N_H[w]} f(G_{i,u}) - r(G_{i,w})\\ &\geq k.\end{aligned}$$

Hence, f_i is a $\{k\}$-dominating function of H. So,

$$\begin{aligned}\omega(f_i) &= \sum_{w\in V(H)} f_i(w)\\ &= \sum_{w\in S}(f(G_{i,w}) - r(G_{i,w})) + \sum_{w\in \bar{S}} f(G_{i,w})\\ &= \sum_{w\in V(H)} f(G_{i,w}) - \sum_{w\in S} r(G_{i,w})\\ &\geq \gamma^{\{k\}}(H),\end{aligned}$$

for $i = 1, \ldots, \rho(G)$.

Now define

$$f_0(w) = \begin{cases} f(G_{0,w}) + \sum_{i=1}^{\rho(G)} r(G_{i,w}) & : w \in S\\ f(G_{0,w}) & : w \in \bar{S}\end{cases}$$

for $w \in V(H)$. We prove that f_0 is an $\{m\}$-dominating function of H.

Recall that $l_w = \sum_{u\in N_H[w]} f(G_{0,u})$ for any $w \in V(H)$.

Case 1. $w \in S$. Note that S is an independent set in H.

$$\begin{aligned}f_0(N_H[w]) &= f_0(w) + \sum_{u\in N_H(w)} f_0(u)\\ &= f(G_{0,w}) + \sum_{i=1}^{\rho(G)} r(G_{i,w}) + \sum_{u\in N_H(w)} f(G_{0,u})\\ &= l_w + \sum_{i=1}^{\rho(G)} r(G_{i,w}).\end{aligned}$$

If $l_w \geq m$, then $f_0(N_H[w]) \geq l_w \geq m$.

If $l_w < m$, then, by Claim 1,

$$f_0(N_H[w]) = l_w + \sum_{i=1}^{\rho(G)} r(G_{i,w}) \geq l_w + m - l_w = m.$$

Case 2. $w \in \bar{S}$.

If $l_w \geq m$, then

$$f_0(N_H[w]) = \sum_{u\in N_H[w]} f_0(u) \geq \sum_{u\in N_H[w]} f(G_{0,u}) = l_w \geq m.$$

If $1 \leq l_w < m$, then $w \in S'_{l_w}$ and $w \notin S_{l_w}$. Since S_{l_w} is a maximum independent set of $H\left[S'_{l_w} - N_H[\cup_{j=0}^{l_w-1} S_j]\right]$, $w \in N_H[S_{l_w}]$ or $w \in N_H[\cup_{j=0}^{l_w-1} S_j]$. Hence, there exists $0 \leq p \leq l_w$ and a vertex w' such that $w' \in S_p$ and $w' \in N_H[w]$. So, by Claim 1,

$$\begin{aligned}f_0(N_H[w]) &= \sum_{u\in N_H[w]-\{w'\}} f_0(u) + f_0(w')\\ &\geq \sum_{u\in N_H[w]-\{w'\}} f(G_{0,u}) + f(G_{0,w'}) + \sum_{i=1}^{\rho(G)} r(G_{i,w'})\\ &\geq \sum_{u\in N_H[w]} f(G_{0,u}) + (m-p)\\ &\geq l_w + m - l_w = m.\end{aligned}$$

If $l_w = 0$, then $w \in S'_0$ and $w \notin S_0$. Since S_0 is a maximum independent set of $H[S'_0]$, there exists a vertex $w' \in S_0$ such that $w' \in N_H[w]$. So, by Claim 1,

$$f_0(N_H[w]) = \sum_{u\in N_H[w]-\{w'\}} f_0(u) + f_0(w') \geq f_0(w') = f(G_{0,w'}) + \sum_{i=1}^{\rho(G)} r(G_{i,w}) \geq m.$$

Therefore, f_0 is an $\{m\}$-dominating function of H. So,

$$\begin{aligned}\omega(f_0) &= \sum_{w\in V(H)} f_0(w)\\ &= \sum_{w\in S}\Big(f(G_{0,w}) + \sum_{i=1}^{\rho(G)} r(G_{i,w})\Big) + \sum_{w\in\bar{S}} f(G_{0,w})\\ &= \sum_{w\in V(H)} f(G_{0,w}) + \sum_{w\in S}\sum_{i=1}^{\rho(G)} r(G_{i,w})\\ &\geq \gamma^{\{m\}}(H).\end{aligned}$$

Thus,

$$\begin{aligned}\gamma^{\{k\}}(G\Box H) = w(f) &= \sum_{i=1}^{\rho(G)}\sum_{w\in V(H)} f(G_{i,w}) + \sum_{w\in V(H)} f(G_{0,w})\\ &= \sum_{i=1}^{\rho(G)}\left[\sum_{w\in V(H)} f(G_{i,w}) - \sum_{w\in S} r(G_{i,w})\right] + \sum_{w\in V(H)} f(G_{0,w}) + \sum_{i=1}^{\rho(G)}\sum_{w\in S} r(G_{i,w})\\ &= \sum_{i=1}^{\rho(G)} w(f_i) + w(f_0) \geq \sum_{i=1}^{\rho(G)}\gamma^{\{k\}}(H) + \gamma^{\{m\}}(H)\\ &= \rho(G)\gamma^{\{k\}}(H) + \gamma^{\{m\}}(H).\end{aligned}$$

The proof of the lemma is completed. □

Theorem 3 follows directly from Lemmas 4 and 5.

3. Some applications of Theorem 3

Define $\gamma^{\{0\}}(H) = 0$. When $k = 1$, Theorem 3 implies that

$$\gamma(G\Box H) \geq \rho(G)\gamma(H) + \gamma^{\{m\}}(H), \tag{5}$$

where $m = \gamma(G) - \rho(G)$. By Proposition 1,

$$\gamma(G\Box H) \geq \rho(G)\gamma(H) + \gamma^{\{m\}}(H) \geq \rho(G)\gamma(H) + m\rho(H).$$

This improves the lower bound (1) given by Chen, Piotrowski and Shreve [2].

Note that if $\rho(G) = \gamma(G)$ or $\rho(G) = \gamma(G) - 1$, then (5) implies that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$. That is

Corollary 6. *Vizing's conjecture is true for (ρ,γ)-graphs and $(\rho,\gamma-1)$-graphs.*

This is a result originally given by Hartnell and Rall [6,7].

In [5], the authors proved that

Lemma 7 *([5]). If both G and H are connected graphs of order at least four and have domination number one-half their order, then $\gamma(G)\gamma(H) = \gamma(G\Box H)$.*

The following result gives a tight upper bound for $\gamma^{\{k\}}(G)\gamma^{\{k\}}(H)$ in term of $\gamma^{\{k\}}(G\Box H)$ for (ρ,γ)-graphs.

Corollary 8. *For integer $k \geq 1$ and any graphs G and H, at least one of which is a (ρ,γ)-graph,*

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H) \leq k\gamma^{\{k\}}(G\Box H)$$

and this bound is sharp.

Proof. We may assume the graph G is a (ρ,γ)-graph, and so $\rho(G) = \gamma(G)$. Hence, by Corollary 2, $\gamma^{\{k\}}(G) = k\rho(G)$. By (i) of Theorem 3,

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H) = k\rho(G)\gamma^{\{k\}}(H) \leq k\gamma^{\{k\}}(G\Box H).$$

That the bound is sharp may be seen as follows. Let G' and H' are any connected graph of order at least two. Let G (resp. H) be obtained from G' (resp. H') by adding exactly one vertex of degree one adjacent to each vertex $V(G')$ (resp. $V(H')$).

Then $\rho(G)=\gamma(G)=|V(G')|=\frac{1}{2}|V(G)|$ and $\rho(H)=\gamma(H)=|V(H')|=\frac{1}{2}|V(H)|$. Thus, by Proposition 1, Corollary 2 and Lemma 7,

$$k^2\gamma(G)\gamma(H)=\gamma^{\{k\}}(G)\gamma^{\{k\}}(H)\le k\gamma^{\{k\}}(G\Box H)\le k^2\gamma(G\Box H)=k^2\gamma(G)\gamma(H).$$

So,

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H)=k\gamma^{\{k\}}(G\Box H).\quad\Box$$

The next result shows that the bound of Corollary 8 for $k=2$ is valid not only for (ρ,γ)-graphs but also for $(\rho,\gamma-1)$-graphs.

Corollary 9. *For any graphs G and H, at least one of which is a $(\rho,\gamma-1)$-graph,*

$$\gamma^{\{2\}}(G)\gamma^{\{2\}}(H)\le 2\gamma^{\{2\}}(G\Box H).$$

Proof. We may assume the graph G is a $(\rho,\gamma-1)$-graph, and so $\rho(G)=\gamma(G)-1$. By Proposition 1, $2\rho(G)\le\gamma^{\{2\}}(G)\le 2\rho(G)+2$.

If $\gamma^{\{2\}}(G)=2\rho(G)$, then, by (i) of Theorem 3,

$$\gamma^{\{2\}}(G)\gamma^{\{2\}}(H)=2\rho(G)\gamma^{\{2\}}(H)\le 2\gamma^{\{2\}}(G\Box H).$$

If $\gamma^{\{2\}}(G)=2\rho(G)+1$, then $m=1$. By Proposition 1 and (ii) of Theorem 3,

$$\begin{aligned}\gamma^{\{2\}}(G)\gamma^{\{2\}}(H)&=2\rho(G)\gamma^{\{2\}}(H)+\gamma^{\{2\}}(H)\\&\le 2\rho(G)\gamma^{\{2\}}(H)+2\gamma(H)\\&=2\rho(G)\gamma^{\{2\}}(H)+2\gamma^{\{1\}}(H)\\&\le 2\gamma^{\{2\}}(G\Box H).\end{aligned}$$

If $\gamma^{\{2\}}(G)=2\rho(G)+2$, then $m=2$. By (ii) of Theorem 3,

$$\gamma^{\{2\}}(G)\gamma^{\{2\}}(H)=2\rho(G)\gamma^{\{2\}}(H)+2\gamma^{\{2\}}(H)\le 2\gamma^{\{2\}}(G\Box H).\quad\Box$$

Note that a tree T is a (ρ,γ)-graph. By Corollary 8, for any tree T and any graph H,

$$\gamma^{\{k\}}(T)\gamma^{\{k\}}(H)\le k\gamma^{\{k\}}(T\Box H).$$

Let C_n be a cycle on n vertices. It is easy to check that C_n is a (ρ,γ)-graph if $n\equiv 0(\bmod 3)$; a $(\rho,\gamma-1)$-graph otherwise. Hence, by Corollaries 8 and 9,

$$\gamma^{\{2\}}(C_n)\gamma^{\{2\}}(H)\le 2\gamma^{\{2\}}(C_n\Box H)$$

for any graph H.

Clearly, the bounds given in Corollaries 8 and 9 are smaller than the bounds (2) and (3) given by Brešar, Henning, and Klavzar [1], hence improve the bounds (2) and (3) for (ρ,γ)-graphs and $(\rho,\gamma-1)$-graphs. We conclude with an open problem.

Question 1. *For any graphs G and H and any positive integer k, is it true that*

$$\gamma^{\{k\}}(G)\gamma^{\{k\}}(H)\le k\gamma^{\{k\}}(G\Box H)?$$

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