Discrete Mathematics 309 (2009) 3310–3317

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On t-intersecting families of signed sets and permutations

Peter Borg

Department of Mathematics, Junior College, University of Malta, Pjazza Guzè Debono, Msida MSD 1252, Malta

ARTICLE INFO

Article history:
Received 12 September 2007
Received in revised form 29 August 2008
Accepted 22 September 2008
Available online 29 October 2008

Keywords:
Erdős–Ko–Rado
Intersecting family
Permutation
Signed set

ABSTRACT

A family $\mathcal{A}$ of sets is said to be *t-intersecting* if any two sets in $\mathcal{A}$ contain at least t common elements. A t-intersecting family is said to be *trivial* if there are at least t elements common to all its sets.

Let X be an r-set $\{x_1, \dots, x_r\}$. For $k \geq 2$, we define $\mathcal{S}_{X,k}$ and $\mathcal{S}^*_{X,k}$ to be the families of *k-signed r-sets* given by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), \dots, (x_r, a_r)\}: a_1, \dots, a_r \text{ are elements of } \{1, \dots, k\}\},$$
$$\mathcal{S}^*_{X,k} := \{\{(x_1, a_1), \dots, (x_r, a_r)\}: a_1, \dots, a_r \text{ are distinct elements of } \{1, \dots, k\}\}.$$

$\mathcal{S}^*_{X,k}$ can be interpreted as the family of *permutations* of r-subsets of $\{1, \dots, k\}$. For a family $\mathcal{F}$, we define $\mathcal{S}_{\mathcal{F},k} := \bigcup_{F\in\mathcal{F}} \mathcal{S}_{F,k}$ and $\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F\in\mathcal{F}} \mathcal{S}^*_{F,k}$.

This paper features two theorems. The first one is as follows: For any two integers s and t with $t \leq s$, there exists an integer $k_0(s, t)$ such that, for any $k \geq k_0(s, t)$ and any family $\mathcal{F}$ with $t \leq \max\{|F|: F \in \mathcal{F}\} \leq s$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial. The second theorem is an analogue of the first one for $\mathcal{S}^*_{\mathcal{F},k}$.

1. Introduction

1.1. Notation and definitions

We start with some standard notation for sets. $\mathbb{N}$ is the set $\{1, 2, \dots\}$ of positive integers. For $m, n \in \mathbb{N}$ with $m \leq n$, the set $\{i \in \mathbb{N}: m \leq i \leq n\}$ is denoted by $[m, n]$, and if $m = 1$ then we also write $[n]$. For a set X, the *power set* $\{A: A \subseteq X\}$ of X is denoted by 2^X, and the *uniform* sub-family $\{Y \subseteq X: |Y| = r\}$ of 2^X is denoted by $\binom{X}{r}$.

For a family $\mathcal{F}$ of sets, we denote the union of all sets in $\mathcal{F}$ by $U(\mathcal{F})$. For a set V, we set

$$\mathcal{F}[V] := \{F \in \mathcal{F}: V \subseteq F\}, \qquad \mathcal{F}(V) := \{F \in \mathcal{F}: F \cap V \neq \emptyset\}.$$

For $u \in U(\mathcal{F})$, we abbreviate $\mathcal{F}(\{u\})$ to $\mathcal{F}(u)$. We call $\mathcal{F}(u)$ a *star of* $\mathcal{F}$. More generally, if T is a t-subset of a set in $\mathcal{F}$, then we call $\mathcal{F}[T]$ a *t-star of* $\mathcal{F}$.

A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{A}$. More generally, $\mathcal{A}$ is said to be *t-intersecting* if $|A \cap B| \geq t$ for any $A, B \in \mathcal{A}$. A t-intersecting family $\mathcal{A}$ is said to be *trivial* if $|\bigcap_{A\in\mathcal{A}} A| \geq t$ (i.e. there are at least t elements common to all the sets in $\mathcal{A}$); otherwise, $\mathcal{A}$ is said to be *non-trivial*. Note that a t-star of a family $\mathcal{F}$ is a maximal trivial t-intersecting sub-family of $\mathcal{F}$.

In the following, unless otherwise stated, sets and families are to be assumed non-empty and finite.

E-mail address: p.borg.02@cantab.net.

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.09.039

1.2. Intersecting sub-families of $2^{[n]}$ and $\binom{[n]}{r}$

The study of intersecting families took off with the publication of [13], which features the classical result, known as the Erdős–Ko–Rado (EKR) Theorem, that says that, if $r \leq n/2$ and $\mathcal{A}$ is an intersecting sub-family of $\binom{[n]}{r}$, then $\mathcal{A}$ has size at most $\binom{n-1}{r-1}$, which is the size of a star of $\binom{[n]}{r}$. There are various proofs of this theorem, two of which are particularly short and beautiful: Katona's [21] using the *cycle method* and Daykin's [7] using another fundamental result known as the Kruskal–Katona Theorem [22,25]. Hilton and Milner [19] determined the size of a largest non-trivial intersecting sub-family of $\binom{[n]}{r}$, and consequently they established that, if $r < n/2$, then no non-trivial intersecting sub-family of $\binom{[n]}{r}$ is as large as the stars of $\binom{[n]}{r}$.

The facts we have just mentioned inspire us to make the following definition. We say that a family $\mathcal{F}$ is *EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains a star, and *strictly EKR* if the set of largest intersecting sub-families of $\mathcal{F}$ contains only stars.

Also in [13], Erdős, Ko and Rado initiated the study of t-intersecting families for $t \geq 2$. They pointed out the simple fact that $2^{[n]}$ is EKR, and they posed the following question: What is the size of an extremal (i.e. largest) t-intersecting sub-family of $2^{[n]}$ for $t \geq 2$? The answer in a complete form was given by Katona [23]. It is interesting that, for $n > t \geq 2$, no extremal t-intersecting sub-family of $2^{[n]}$ is a t-star.

For the uniform case, Erdős, Ko and Rado [13] proved that, for $t < r$, there exists an integer $n_0(r,t)$ such that, for all $n \geq n_0(r,t)$, the largest t-intersecting sub-families of $\binom{[n]}{r}$ are the t-stars. For $t \geq 15$, Frankl [14] showed that the smallest such $n_0(r,t)$ is $(r-t+1)(t+1)+1$ and that, if $n = (r-t+1)(t+1)$, then t-stars are extremal but not uniquely so. Subsequently, Wilson [33] proved the sharp upper bound $\binom{n-t}{r-t}$ for the size of a t-intersecting sub-family of $\binom{[n]}{r}$ for all t and $n \geq (r-t+1)(t+1)$. Frankl [14] conjectured that an extremal t-intersecting sub-family of $\binom{[n]}{r}$ has size $\max\{|\{A \in \binom{[n]}{r} : |A \cap [t+2i]| \geq t+i\}| : i \in \{0\} \cup [r-t]\}$. A remarkable proof of this long-standing conjecture together with a complete characterisation of the extremal structures was finally obtained by Ahlswede and Khachatrian [1].

Theorem 1.1 (*Ahlswede and Khachatrian [1]*). *Let $1 \leq t \leq r \leq n$, and let $\mathcal{A}$ be an extremal t-intersecting sub-family of $\binom{[n]}{r}$.*

(i) *If $(r-t+1)(2+\frac{t-1}{i+1}) < n < (r-t+1)(2+\frac{t-1}{i})$ for some $i \in \{0\} \cup \mathbb{N}$ - where, by convention, $(t-1)/i = \infty$ if $i = 0$- then $\mathcal{A} = \{A \in \binom{[n]}{r} : |A \cap X| \geq t+i\}$ for some $X \in \binom{[n]}{t+2i}$.*

(ii) *If $t \geq 2$ and $(r-t+1)(2+\frac{t-1}{i+1}) = n$ for some $i \in \{0\} \cup \mathbb{N}$, then $\mathcal{A} = \{A \in \binom{[n]}{r} : |A \cap X| \geq t+j\}$ for some $j \in \{i, i+1\}$ and $X \in \binom{[n]}{t+2j}$.*

Many other beautiful results were inspired by the seminal Erdős–Ko–Rado paper [13]. The survey papers [10] and [15] are recommended.

We now proceed to the first of the two main themes of the paper.

1.3. Intersecting families of signed sets

Let X be an r-set $\{x_1, \ldots, x_r\}$. Let $y_1, \ldots, y_r \in \mathbb{N}$. We call the set $\{(x_1, y_1), \ldots, (x_r, y_r)\}$ a *k-signed r-set* if $|\{y_1, \ldots, y_r\}| \leq k$. For an integer $k \geq 2$, we define $\mathcal{S}_{X,k}$ to be the family of k-signed r-sets given by

$$\mathcal{S}_{X,k} := \{\{(x_1, a_1), \ldots, (x_r, a_r)\} : a_1, \ldots, a_r \in [k]\}.$$

We shall set $\mathcal{S}_{\emptyset,k} := \emptyset$.

The *Cartesian product* $X \times Y$ of sets X and Y is the set $\{(x,y) : x \in X, y \in Y\}$. So $\mathcal{S}_{X,k} = \{A \subset X \times [k] : |A \cap (\{x\} \times [k])| = 1 \text{ for all } x \in X\}$.

For a family $\mathcal{F}$ of sets, we define

$$\mathcal{S}_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}_{F,k}.$$

We remark that the 'signed sets' terminology was introduced in [4] for a setting that can be re-formulated as $\mathcal{S}_{\binom{[n]}{r},k}$, and the general formulation $\mathcal{S}_{\mathcal{F},k}$ was introduced by the author in [5], the theme of which is the following conjecture.

Conjecture 1.2 (*Borg [5]*). *Let $\mathcal{F}$ be any family, and let $k \geq 2$. Then:*

(i) *$\mathcal{S}_{\mathcal{F},k}$ is EKR;*
(ii) *$\mathcal{S}_{\mathcal{F},k}$ is not strictly EKR iff $k = 2$ and there exist at least three elements u_1, u_2, u_3 of $U(\mathcal{F})$ such that $\mathcal{F}(u_1) = \mathcal{F}(u_2) = \mathcal{F}(u_3)$ and $\mathcal{S}_{\mathcal{F},2}((u_1, 1))$ is a largest star of $\mathcal{S}_{\mathcal{F},2}$.*

The main result in the same paper is that this conjecture is true if $\mathcal{F}$ is compressed with respect to an element u^* of $U(\mathcal{F})$ (i.e. $u \in F \in \mathcal{F} \setminus \mathcal{F}(u^*)$ implies $(F \setminus \{u\}) \cup \{u^*\} \in \mathcal{F}$). This generalises a well-known result that was first stated by Meyer [31] and proved in different ways by Deza and Frankl [10], Bollobás and Leader [4], Engel [11] and Erdős et al. [12], and that can be described by saying that the conjecture is true for $\mathcal{F} = \binom{[n]}{r}$. Berge [3] and Livingston [30] had proved (i) and (ii) respectively for the special case $\mathcal{F} = \{[n]\}$ (other proofs are found in [18,32]). In [5] the conjecture is also verified for $\mathcal{F}$ uniform and EKR; Holroyd and Talbot [20] had essentially proved (i) for such a family $\mathcal{F}$ in a graph-theoretical context.

The t-intersection problem for sub-families of $\mathcal{S}_{[n],k}$ has also been solved. Frankl and Füredi [16] were the first to investigate it, and the following result had been a conjecture that they made and that they verified for $k \geq t + 1 \geq 16$ in [16].

Theorem 1.3 (*Ahlswede, Khachatrian [2]; Frankl, Tokushige [17]*). *If $\mathcal{A}$ is an extremal t-intersecting sub-family of $\mathcal{S}_{[n],k}$, then $|\mathcal{A}| = \max\{|\{A \in \mathcal{S}_{[n],k} : |A \cap ([t + 2i] \times [1])| \geq t + i\}| : i \in \{0\} \cup \mathbb{N}\}$.*

It follows from this result that the set of extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$ contains t-stars iff $k \geq t + 1$. What led to this result was the accomplishment of Theorem 1.1. As in Theorem 1.1, Ahlswede and Khachatrian [2] also determined the extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$, and it turns out that the structure of a t-star of $\mathcal{S}_{[n],k}$ is the unique extremal structure iff $k \geq t + 2$. Kleitman [24] had long established Theorem 1.3 for $k = 2$.

To the best of the author's knowledge, apart from a general result we present later, no results for t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ with $|\mathcal{F}| \geq 2$ have been established. However, some very important results have been obtained for a modification of the problem, which we describe next.

1.4. *Intersecting families of permutations and partial permutations*

For an r-set $X := \{x_1, \ldots, x_r\}$, we define $\mathcal{S}^*_{X,k}$ to be the special sub-family of $\mathcal{S}_{X,k}$ given by

$$\mathcal{S}^*_{X,k} := \left\{ \{(x_1, a_1), \ldots, (x_r, a_r)\} : \{a_1, \ldots, a_r\} \in \binom{[k]}{r} \right\}.$$

Note that $\mathcal{S}^*_{X,k} \neq \emptyset$ iff $r \leq k$.

For a family $\mathcal{F}$, we define $\mathcal{S}^*_{\mathcal{F},k}$ to be the special sub-family of $\mathcal{S}_{\mathcal{F},k}$ given by

$$\mathcal{S}^*_{\mathcal{F},k} := \bigcup_{F \in \mathcal{F}} \mathcal{S}^*_{F,k}.$$

An *r-partial permutation of a set N* is a pair (A, f) where $A \in \binom{N}{r}$ and $f : A \to N$ is an injection. An $|N|$-partial permutation of N is simply called a *permutation of N*. Clearly, the family of permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{[n],n}$, and the family of r-partial permutations of $[n]$ can be re-formulated as $\mathcal{S}^*_{\binom{[n]}{r},n}$.

Let X be as above. $\mathcal{S}^*_{X,k}$ can be interpreted as the family of permutations of sets in $\binom{[k]}{r}$: consider the bijection $\beta : \mathcal{S}^*_{X,k} \to \{(A, f) : A \in \binom{[k]}{r}, f : A \to A \text{ is a bijection}\}$ defined by $\beta(\{(x_1, a_1), \ldots, (x_r, a_r)\}) := (\{a_1, \ldots, a_r\}, f)$ where, for $b_1 < \cdots < b_r$ such that $\{b_1, \ldots, b_r\} = \{a_1, \ldots, a_r\}$, $f(b_i) := a_i$ for $i = 1, \ldots, r$. $\mathcal{S}^*_{X,k}$ can also be interpreted as the sub-family $\mathcal{X} := \{(A, f) : A \in \binom{[k]}{r}, f : A \to [r] \text{ is a bijection}\}$ of the family of r-partial permutations of $[k]$: consider an obvious bijection from $\mathcal{S}^*_{X,k}$ to $\mathcal{S}^*_{\binom{[k]}{r},r}$ and another one from $\mathcal{S}^*_{\binom{[k]}{r},r}$ to $\mathcal{X}$.

In [8,9] the study of intersecting permutations was initiated. Deza and Frankl [9] showed that $\mathcal{S}^*_{[n],n}$ is EKR. So an intersecting sub-family of $\mathcal{S}^*_{[n],n}$ has size at most $(n-1)!$. Only a few years ago, Cameron and Ku [6] and Larose and Malvenuto [28] independently proved that furthermore $\mathcal{S}^*_{[n],n}$ is strictly EKR.

Ku and Leader [27] proved that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is EKR for all $r \in [n]$, and they also showed that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is strictly EKR for all $r \in [8, n-3]$. Naturally, they conjectured that $\mathcal{S}^*_{\binom{[n]}{r},n}$ is also strictly EKR for the few remaining values of r. This was settled by Li and Wang [29] using tools forged by Ku and Leader.

When it comes to t-intersecting families of permutations, things are of course much harder, and the most interesting challenge comes from the following conjecture.

Conjecture 1.4 (*Deza and Frankl [9]*). *For any $t \in \mathbb{N}$, there exists $n_0(t) \in \mathbb{N}$ such that, for any $n \geq n_0(t)$, the size of a t-intersecting sub-family of $\mathcal{S}^*_{[n],n}$ is at most that of a t-star of $\mathcal{S}^*_{[n],n}$, i.e. $(n-t)!$.*

This conjecture suggests an obvious extension for the extremal case. It is worth pointing out that the condition $n \geq n_0(t)$ is necessary; [26, Example 3.1.1] illustrates this fact. An analogue of the statement of the conjecture for partial permutations has been proved by Ku.

Theorem 1.5 (*Ku [26, Theorem 6.6.6]*). *For any $r, t \in \mathbb{N}$ with $r \geq t$, there exists $n_0(r,t) \in \mathbb{N}$ such that, for any $n \geq n_0(r,t)$, the size of a t-intersecting sub-family of $\mathcal{S}^*_{\binom{[n]}{r},n}$ is at most that of a t-star of $\mathcal{S}^*_{\binom{[n]}{r},n}$, i.e. $\binom{n-t}{r-t}\frac{(n-t)!}{(n-r)!}$.*

This result emerges as an immediate consequence of one of the two main theorems in this paper; see next section.

2. Results and conjectures

For a family $\mathcal{F}$, let $\alpha(\mathcal{F})$ denote the size of a largest set in $\mathcal{F}$. Any t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ or $\mathcal{S}^*_{\mathcal{F},k}$ trivially consists of at most one set if $\alpha(\mathcal{F}) \leq t$. We now consider $\alpha(\mathcal{F}) > t$.

In view of Conjecture 1.2, we suggest the following general conjecture for t-intersecting families of signed sets.

Conjecture 2.1. *For any $t \in \mathbb{N}$, there exists $k_0(t) \in \mathbb{N}$ such that, for any $k \geq k_0(t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) > t$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial.*

As we mentioned in Section 1.3, the t-stars of $\mathcal{S}_{[n],k}$ are extremal t-intersecting sub-families of $\mathcal{S}_{[n],k}$ iff $k \geq t+1$, and they are uniquely extremal iff $k \geq t+2$. This suggests that, if Conjecture 2.1 is true, then, as is claimed by Conjecture 1.2 for $t = 1$, the smallest value of $k_0(t)$ is $t+2$ (and the largest t-stars of $\mathcal{S}_{\mathcal{F},t+1}$ are among the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},t+1}$). We are able to prove a relaxation of the statement of Conjecture 2.1.

Theorem 2.2. *For any $r, t \in \mathbb{N}$ with $t < r$, let $k_0(r,t) := \binom{r}{t}\binom{r}{t+1}$. For any $k \geq k_0(r,t)$ and any family $\mathcal{F}$ with $t < \alpha(\mathcal{F}) \leq r$, the largest t-intersecting sub-families of $\mathcal{S}_{\mathcal{F},k}$ are trivial.*

Corollary 2.3. *Conjecture 1.2 is true if $k \geq \alpha(\mathcal{F})\binom{\alpha(\mathcal{F})}{2}$.*

We next pose a similar problem for t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$.

Conjecture 2.4. *For any $t \in \mathbb{N}$, there exists $k^*_0(t) \in \mathbb{N}$ such that, for any $k \geq k^*_0(t)$ and any family $\mathcal{F}$ with $\alpha(\mathcal{F}) > t$, the largest t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ are trivial.*

By taking $k \geq k^*_0(t)$ and $\mathcal{F} = \{[k]\}$, we get Conjecture 1.4. We are able to prove the following analogue of Theorem 2.2.

Theorem 2.5. *For any $r, t \in \mathbb{N}$ with $t < r$, let $k^*_0(r,t) := \binom{r}{t}\binom{3r-2t-1}{\lfloor\frac{3r-2t-1}{2}\rfloor}\frac{r!}{(r-t-1)!} + r + 1$. For any $k \geq k^*_0(r,t)$ and any family $\mathcal{F}$ with $t < \alpha(\mathcal{F}) \leq r$, the largest t-intersecting sub-families of $\mathcal{S}^*_{\mathcal{F},k}$ are trivial.*

By taking $k \geq k^*_0(r,t)$ and $\mathcal{F} = \binom{[k]}{r}$, we get Theorem 1.5.

We now proceed to the proofs of the two theorems above.

3. Proof of Theorem 2.2

We shall base the proof of Theorem 2.2 on the compression technique used in [10] and in [16]. We point out that this can be avoided by applying an argument similar to the one for Theorem 2.5; however, the compression technique enables us to obtain a neater proof and a value of $k_0(r,t)$ that is better than what we would obtain without using it.

For $(a,b) \in [n] \times [2,k]$, let $\Delta_{a,b}\colon 2^{\mathcal{S}_{2^{[n]},k}} \to 2^{\mathcal{S}_{2^{[n]},k}}$ be defined by

$$\Delta_{a,b}(\mathcal{A}) := \{\delta_{a,b}(A)\colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{a,b}(A) \in \mathcal{A}\},$$

where $\delta_{a,b}\colon \mathcal{S}_{2^{[n]},k} \to \mathcal{S}_{2^{[n]},k}$ is defined by

$$\delta_{a,b}(A) := \begin{cases} A \setminus \{(a,b)\} \cup \{(a,1)\} & \text{if } (a,b) \in A; \\ A & \text{otherwise.} \end{cases}$$

Note that $|\Delta_{a,b}(\mathcal{A})| = |\mathcal{A}|$. It is known and easy to check that, if $\mathcal{A}$ is t-intersecting, then $\Delta_{a,b}(\mathcal{A})$ is t-intersecting. We prove a bit more than this.

Lemma 3.1. *Let $\mathcal{A} \subset \mathcal{S}_{2^{[n]},k}$ and $V \subseteq [n] \times [2,k]$ such that $|(A \cap B) \setminus V| \geq t$ for any $A, B \in \mathcal{A}$. Then $|(C \cap D) \setminus (V \cup \{(a,b)\})| \geq t$ for any $C, D \in \Delta_{a,b}(\mathcal{A})$.*

Proof. Let $C, D \in \Delta_{a,b}(\mathcal{A})$. Let $C' := (C \setminus \{(a,1)\}) \cup \{(a,b)\}$, $D' := (D \setminus \{(a,1)\}) \cup \{(a,b)\}$. Suppose $|(C \cap D) \setminus V| < t$. So C and D cannot both be in $\mathcal{A}$. Suppose $C, D \notin \mathcal{A}$; then $(a,1)$ is in both C and D, C' and D' are in $\mathcal{A}$, and $|(C' \cap D') \setminus V| \leq |(C \cap D) \setminus V| < t$, a contradiction. Thus, without loss of generality, $C \notin \mathcal{A}$ and $D \in \mathcal{A}$. So $(a, 1) \in C$ and $C' \in \mathcal{A}$. If $(a, b) \notin D$ then $|(C' \cap D) \setminus V| \le |(C \cap D) \setminus V| < t$, contradicting $C', D \in \mathcal{A}$. So $(a, b) \in D$ and hence $\delta_{a,b}(D) \in \mathcal{A}$ (because otherwise $D \notin \Delta_{a,b}(\mathcal{A})$). But then $|(C' \cap \delta_{a,b}(D)) \setminus V| = |(C \cap D) \setminus V| < t$, contradicting $C', \delta_{a,b}(D) \in \mathcal{A}$. We therefore conclude that $|(C \cap D) \setminus V| \ge t$.

Now suppose $|(C \cap D) \setminus (V \cup \{(a, b)\})| < t$. Since $|(C \cap D) \setminus V| \ge t$, $(a, b) \in C \cap D$. So $C, \delta_{a,b}(C), D, \delta_{a,b}(D) \in \mathcal{A}$ and $|(C \cap \delta_{a,b}(D)) \setminus V| = |(C \cap D) \setminus (V \cup \{(a, b)\})| < t$, a contradiction. □

Corollary 3.2. *Let $\mathcal{A}^*$ be a t-intersecting sub-family of $\mathcal{S}_{2^{[n]},k}$. Let*

$$\mathcal{A} := \Delta_{n,k} \circ \cdots \circ \Delta_{n,2} \circ \cdots \circ \Delta_{1,k} \circ \cdots \circ \Delta_{1,2}(\mathcal{A}^*).$$

Then $|A \cap B \cap ([n] \times [1])| \ge t$ for any $A, B \in \mathcal{A}$.

Proof. By repeated application of Lemma 3.1, $|(A \cap B) \setminus ([n] \times [2, k])| \ge t$ for any $A, B \in \mathcal{A}$. The result follows since $(A \cap B) \setminus ([n] \times [2, k]) = A \cap B \cap ([n] \times [1])$. □

Lemma 3.3. *Let $\mathcal{F} \subseteq 2^{[n]}$, $k \ge 3$ and $(a, b) \in [n] \times [2, k]$. Suppose $\mathcal{A}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ and $\Delta_{a,b}(\mathcal{A})$ is a sub-family of a t-star $\mathcal{S}_{\mathcal{F},k}[Z]$ ($Z \in \mathcal{S}_{\binom{[n]}{t},k}$) of $\mathcal{S}_{\mathcal{F},k}$. Then $|\mathcal{A}| < |\mathcal{S}_{\mathcal{F},k}[Z]|$.*

Proof. Let $Y := \{z : (z, l) \in Z \text{ for some } l \in [k]\}$. Given that $\Delta_{a,b}(\mathcal{A}) \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$, we have $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and, since $\mathcal{A}$ is non-trivial, there exists $A \in \mathcal{A}$ such that $|A \cap Z| = t - 1$ and $Z \subseteq \delta_{a,b}(A)$. So $(a, 1) \in Z$ and $Z' := Z \setminus \{(a, 1)\} \subset A$ for all $A \in \mathcal{A}$. Let $Y' := Y \setminus \{a\}$. Setting $\mathcal{F}' := \{F \setminus Y' : F \in \mathcal{F}[Y']\}$ and $\mathcal{A}' := \{A \setminus Z' : A \in \mathcal{A}[Z']\}$, we then have $\mathcal{A}' \subset \mathcal{S}_{\mathcal{F}'(a),k}$ (as $\mathcal{A} \subset \mathcal{S}_{\mathcal{F}[Y],k}$ and $Y = Y' \cup \{a\}$) and $|\mathcal{A}'| = |\mathcal{A}|$. Since $\mathcal{A}$ is a non-trivial t-intersecting family and $|Z'| = t - 1$, $\mathcal{A}'$ is a non-trivial intersecting family.

For $F' \in \mathcal{F}'(a)$, let $\mathcal{A}'_{F'} := \mathcal{A}' \cap \mathcal{S}_{F',k}$. Since $\mathcal{A}'$ is intersecting, $\mathcal{A}'_{F'}$ is intersecting. Suppose $\mathcal{A}'_{F'} \neq \emptyset$. If $\mathcal{A}'_{F'}$ is non-trivial, then, by Livingston's theorem [30] (see Section 1.3), $|\mathcal{A}'_{F'}| < k^{|F'|-1}$. Suppose $\mathcal{A}'_{F'}$ is trivial; so $\mathcal{A}'_{F'} \subseteq \mathcal{S}_{F',k}((c, d))$ for some $(c, d) \in F' \times [k]$. Since $\mathcal{A}'$ is non-trivial, there exists $A' \in \mathcal{A}'$ such that $(c, d) \notin A'$. Thus, since $\mathcal{A}'$ is intersecting, we actually have $\mathcal{A}'_{F'} \subseteq \{A \in \mathcal{S}_{F',k}((c, d)) : A \cap A' \neq \emptyset\}$, and hence we again get $|\mathcal{A}'_{F'}| < k^{|F'|-1}$.

We therefore have

$$|\mathcal{A}| = |\mathcal{A}'| = \sum_{F' \in \mathcal{F}'(a)} |\mathcal{A}'_{F'}| < \sum_{F' \in \mathcal{F}'(a)} k^{|F'|-1} = \sum_{F \in \mathcal{F}[Y]} k^{|F|-t},$$

and the result follows since $\sum_{F \in \mathcal{F}[Y]} k^{|F|-t} = |\mathcal{S}_{\mathcal{F},k}[Z]|$. □

Proof of Theorem 2.2. Let $\mathcal{F}$ be a family with $t < \alpha(\mathcal{F}) \le r$. We may assume that $\mathcal{F} \subseteq 2^{[n]}$ for some $n \in \mathbb{N}$. Let $k \ge k_0(r, t)$. We prove the result by showing that, for any non-trivial t-intersecting sub-family $\mathcal{B}$ of $\mathcal{S}_{\mathcal{F},k}$, there exists a trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$ that is larger than $\mathcal{B}$.

Let $\mathcal{A}^*$ be a non-trivial t-intersecting sub-family of $\mathcal{S}_{\mathcal{F},k}$. Let $\mathcal{A} := \Delta_{n,k} \circ \cdots \circ \Delta_{n,2} \circ \cdots \circ \Delta_{1,k} \circ \cdots \circ \Delta_{1,2}(\mathcal{A}^*)$. So $\mathcal{A} \subset \mathcal{S}_{\mathcal{F},k}$ and $|\mathcal{A}| = |\mathcal{A}^*|$. Let $X := [n] \times [1]$. By Corollary 3.2,

$$|A \cap B \cap X| \ge t \quad \text{for any } A, B \in \mathcal{A}. \tag{1}$$

Suppose $\mathcal{A}$ is a trivial t-intersecting family, i.e. $\mathcal{A} \subseteq \mathcal{S}_{\mathcal{F},k}[Z]$ for some $Z \in \binom{S}{t}$, $S \in \mathcal{S}_{\mathcal{F},k}$. By Lemma 3.3, we then have $|\mathcal{A}^*| < |\mathcal{S}_{\mathcal{F},k}[Z]|$, and hence we are done.

We now assume that $\mathcal{A}$ is a non-trivial t-intersecting family. Suppose $|A' \cap X| = t$ for some $A' \in \mathcal{A}$. Then, by (1), $A' \cap X \subseteq A$ for all $A \in \mathcal{A}$; but this contradicts the assumption that $\mathcal{A}$ is non-trivial. So $|A \cap X| \ge t + 1$ for all $A \in \mathcal{A}$, and hence we obtain a crude bound for the size of $\mathcal{A}_F := \mathcal{A} \cap \mathcal{S}_{F,k}$ ($F \in \mathcal{F}$) as follows:

$$|\mathcal{A}_F| \le |\{A \in \mathcal{S}_{F,k} : |A \cap (F \times [1])| \ge t + 1\}| < \binom{|F|}{t+1} k^{|F|-t-1} \le \binom{r}{t+1} k^{|F|-t-1}. \tag{2}$$

Let $B \in \mathcal{A}$. Since $\mathcal{A}$ is t-intersecting (by (1)), each $A \in \mathcal{A}$ must contain at least one of the sets in $\binom{B}{t}$, and hence $\mathcal{A} = \bigcup_{C \in \binom{B}{t}} \mathcal{A}[C]$. Choose $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \le |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$. We then have

$$|\mathcal{A}| = \left| \bigcup_{C \in \binom{B}{t}} \mathcal{A}[C] \right| \le \sum_{C \in \binom{B}{t}} |\mathcal{A}[C]| \le \binom{|B|}{t} |\mathcal{A}[C^*]| \le \binom{r}{t} |\mathcal{A}[C^*]|. \tag{3}$$

Set $\mathcal{G} := \{F \in \mathcal{F} : \mathcal{A}[C^*] \cap \mathcal{S}_{F,k} \neq \emptyset\}$. Let $\mathcal{C}$ be the trivial t-intersecting sub-family $\bigcup_{G \in \mathcal{G}} \mathcal{S}_{G,k}[C^*]$ of $\mathcal{S}_{\mathcal{F},k}$. Bringing all the pieces together, we get

$$\begin{aligned}
|\mathcal{A}| &\le \binom{r}{t} |\mathcal{A}[C^*]| \quad \text{(by (3))} \\
&\le \binom{r}{t} \sum_{G \in \mathcal{G}} |\mathcal{A}_G| = \sum_{G \in \mathcal{G}} \binom{r}{t} |\mathcal{A}_G| \\
&< \sum_{G \in \mathcal{G}} \binom{r}{t} \binom{r}{t+1} k^{|G|-t-1} \quad \text{(by (2))} \\
&= \sum_{G \in \mathcal{G}} k_0(r,t) k^{|G|-t-1} \le \sum_{G \in \mathcal{G}} k^{|G|-t} = |\mathcal{C}|.
\end{aligned}$$

So $|\mathcal{A}^*| < |\mathcal{C}|$ as $|\mathcal{A}^*| = |\mathcal{A}|$. Hence the result. $\Box$

4. Proof of Theorem 2.5

The proof of Theorem 2.5 is based on ideas from the preceding section and ideas used by Erdős, Ko and Rado [13] for their result concerning t-intersecting sub-families of $\binom{[n]}{r}$. Unfortunately, the compression technique fails to work for intersecting sub-families of $\mathcal{S}^*_{[n],k}$.

Let $l(n,k,t)$ be the size of a largest non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$, and let $P_j := \{(i,i) : i \in [j]\}$.

Lemma 4.1. *For any $c, n, t \in \mathbb{N}$ with $t < n$, let $k_0(c,n,t) := c\binom{3n-2t-1}{\lfloor \frac{3n-2t-1}{2} \rfloor}\frac{n!}{(n-t-1)!} + n + 1$. For any $k \ge k_0(c,n,t)$,*

$$|\mathcal{S}^*_{[n],k}[P_t]| > c(\max\{l(n,k,t), |\mathcal{S}^*_{[n],k}[P_{t+1}]|\}).$$

Proof. Let $k \ge k_0(c,n,t)$, and let $\mathcal{A} \subset \mathcal{S}^*_{[n],k}$ be a non-trivial t-intersecting family of size $l(n,k,t)$. Choose $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| \le |A \cap B|$ for all $A, B \in \mathcal{A}$.

Suppose $|A_1 \cap A_2| \ge t+1$. Let $(i^*, j^*) \in [n] \times [k]$ such that $(i^*, j^*) \in A_1 \cap A_2$. Let $j' \in [k]$ such that $(i, j') \notin A_1 \cup A_2$ for all $i \in [n]$ (note that such a j' exists since $k \ge k_0(c,n,t) > |A_1 \cup A_2|$). Let $A'_1 := (A_1 \setminus \{(i^*, j^*)\}) \cup (i^*, j')$. By choice of j', $A'_1 \in \mathcal{S}^*_{[n],k}$. Let $\mathcal{A}' := \mathcal{A} \cup \{A'_1\}$. Since $|A'_1 \cap A_2| < |A_1 \cap A_2|$, it follows by choice of A_1 and A_2 that $A'_1 \notin \mathcal{A}$ and hence $|\mathcal{A}'| = |\mathcal{A}| + 1$. Also by choice of A_1 and A_2, we have $|A \cap B| \ge t+1$ for all $A, B \in \mathcal{A}$, which implies that $\mathcal{A}'$ is t-intersecting. Since $\mathcal{A} \subset \mathcal{A}'$ and $\mathcal{A}$ is non-trivially t-intersecting, $|\bigcap_{A' \in \mathcal{A}'} A'| \le |\bigcap_{A \in \mathcal{A}} A| < t$. So $\mathcal{A}'$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{[n],k}$ of size greater than $|\mathcal{A}|$; but this contradicts $|\mathcal{A}| = l(n,k,t)$. We therefore conclude that $|A_1 \cap A_2| = t$. Thus, since $\mathcal{A}$ is non-trivially t-intersecting, there exists $A_3 \in \mathcal{A}$ such that $A_1 \cap A_2 \not\subseteq A_3$ and hence $|A_1 \cap A_2 \cap A_3| < t$.

Let $I := A_1 \cup A_2 \cup A_3$. Suppose there exists $A^* \in \mathcal{A}$ such that $|A^* \cap I| < t+1$. Since $|A_1 \cap A_2| = t$ and $|A^* \cap A_i| \ge t$ for each $i \in [2]$, we must then have $A^* \cap (A_1 \cup A_2) = A_1 \cap A_2$. Thus, by our supposition, $A^* \cap I = A_1 \cap A_2$. But then $A^* \cap A_3 = A_1 \cap A_2 \cap A_3$, which gives the contradiction that $|A^* \cap A_3| < t$. Therefore

$$|A \cap I| \ge t+1 \quad \text{for all } A \in \mathcal{A}. \tag{4}$$

Now $|I| = |A_1 \cup A_2| + |A_3| - |A_3 \cap (A_1 \cup A_2)|$. Since $|A_1 \cup A_2| = 2n - |A_1 \cap A_2| = 2n - t$ and $|A_3 \cap (A_1 \cup A_2)| = |A_3 \cap A_1| + |(A_3 \cap A_2) \setminus A_1| \ge t + (t - |A_3 \cap A_2 \cap A_1|) \ge 2t - (t-1) = t+1$, it follows that

$$|I| \le (2n-t) + n - (t+1) = 3n - 2t - 1.$$

Taking J to be the smallest set such that $I \subset [n] \times J$, we then have

$$n \le |J| \le 3n - 2t - 1.$$

For each $i \in [t+1, n]$, let $\mathcal{A}_i := \{A \in \mathcal{A} : |A \cap ([n] \times J)| = i\}$. By (4), $\bigcup_{i=t+1}^n \mathcal{A}_i$ is a partition for $\mathcal{A}$. Let $x := \sum_{i=t+1}^n |\{A \in \mathcal{S}^*_{[n],k} : |A \cap ([n] \times J)| = i\}|$. We therefore have

$$\begin{aligned}
l(n,k,t) = |\mathcal{A}| = \sum_{i=t+1}^n |\mathcal{A}_i| < x &= \sum_{i=t+1}^n \binom{|J|}{i}\binom{n}{i} i! \binom{k-|J|}{n-i}(n-i)! \\
&< \sum_{i=t+1}^n \binom{3n-2t-1}{i}\binom{n}{i} i! \binom{k-n}{n-i}(n-i)! \\
&\le \sum_{i=t+1}^n \binom{3n-2t-1}{i} \frac{n!}{(n-i)!}(k-n)^{(n-i)}
\end{aligned}$$

$$\leq \binom{3n-2t-1}{\lfloor \frac{3n-2t-1}{2} \rfloor} \frac{n!}{(n-t-1)!} \sum_{i=t+1}^{n} (k-n)^{(n-i)}$$
$$= \left(\frac{k_0(c,n,t)-n-1}{c}\right)\left(\frac{1-(k-n)^{n-t}}{1-(k-n)}\right) \leq \frac{(k-n)^{n-t}-1}{c}$$
$$< \frac{1}{c}\left(\frac{(k-t)!}{(k-n)!}\right) = \frac{|\mathcal{S}^*_{[n],k}[P_t]|}{c}.$$

The result now follows since we also have $|\mathcal{S}^*_{[n],k}[P_{t+1}]| < x$. □

Proof of Theorem 2.5. Let $\mathcal{F}$ be a family with $t < \alpha(\mathcal{F}) \leq r$. Let $k_0(\binom{r}{t}, n, t)$ be as in the statement of Lemma 4.1 with $c = \binom{r}{t}$. Let $k \geq k_0^*(r,t)$. So we have

$$k \geq k_0\left(\binom{r}{t}, r, t\right) = \max\left\{k_0\left(\binom{r}{t}, n, t\right) : n \in [r]\right\}. \quad (5)$$

Let $\mathcal{A}$ be a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{\mathcal{F},k}$.

For any $F \in \mathcal{F}$ and any family $\mathcal{B} \subseteq \mathcal{S}^*_{\mathcal{F},k}$, set $\mathcal{B}_F := \mathcal{B} \cap \mathcal{S}^*_{F,k}$. For all $F \in \mathcal{F}$, choose $F' \in \mathcal{S}^*_{\binom{F}{t},k}$. We show that, for all $F \in \mathcal{F}$,

$$\binom{r}{t} |\mathcal{A}_F| < |\mathcal{S}^*_{F,k}[F']|. \quad (6)$$

If $\mathcal{A}_F$ is a non-trivial t-intersecting family, then (6) follows immediately from (5) and Lemma 4.1. Now suppose $\mathcal{A}_F$ is a trivial t-intersecting family. Setting $T := \bigcap_{A \in \mathcal{A}_F} A$, we then have $|T| \geq t$. If $|T| \geq t+1$, then (6) again follows immediately from (5) and Lemma 4.1. It remains to consider $|T| = t$. Since $\mathcal{A}$ is a non-trivial t-intersecting family, there exists $A_1 \in \mathcal{A}$ such that $T \not\subseteq A_1$ and hence $|T \cap A_1| < t$. Let $D_1 := A_1 \cap (F \times [k])$. Let F_1 be the subset of F such that $D_1 \in \mathcal{S}^*_{F_1,k}$. Let $F_2 := F \setminus F_1$. Let $Y := \{y \in [k] : (x,y) \notin D_1 \cup T \text{ for all } x \in F\}$, and let $y_1, \ldots, y_{|Y|}$ be the distinct elements of Y. We have $|Y| \geq k - |D_1| - |T| = k - |F_1| - t = k - (|F| - |F_2|) - t \geq k_0^*(r,t) - r - t + |F_2| > |F_2|$. If $F_2 \neq \emptyset$ and $x_1, \ldots, x_{|F_2|}$ are the distinct elements of F_2, then we take D_2 to be the set $\{(x_1,y_1), \ldots, (x_{|F_2|}, y_{|F_2|})\}$ in $\mathcal{S}^*_{F_2,k}$; otherwise we take $D_2 := \emptyset$. Let $A_2 := D_1 \cup D_2$. Clearly $A_2 \in \mathcal{S}^*_{F,k}$. Therefore $\mathcal{A}_F \cup \{A_2\}$ is a non-trivial t-intersecting sub-family of $\mathcal{S}^*_{F,k}$ because $|\bigcap_{A' \in \mathcal{A}_F \cup \{A_2\}} A'| = |T \cap A_2| = |T \cap D_1| = |T \cap A_1| < t$ and, for all $A \in \mathcal{A}_F$, $|A_2 \cap A| \geq |D_1 \cap A| = |A_1 \cap A| \geq t$. By (5) and Lemma 4.1, it follows that $\binom{r}{t} |\mathcal{A}_F \cup \{A_2\}| < |\mathcal{S}^*_{F,k}[F']|$, and hence (6).

Now, as in the proof of Theorem 2.2, by choosing $B \in \mathcal{A}$ and $C^* \in \binom{B}{t}$ such that $|\mathcal{A}[C]| \leq |\mathcal{A}[C^*]|$ for all $C \in \binom{B}{t}$, we get

$$|\mathcal{A}| \leq \binom{r}{t} |\mathcal{A}[C^*]|.$$

Set $\mathcal{G} := \{F \in \mathcal{F} : \mathcal{A}[C^*] \cap \mathcal{S}^*_{F,k} \neq \emptyset\}$. Let $\mathcal{C}$ be the trivial t-intersecting sub-family $\bigcup_{G \in \mathcal{G}} \mathcal{S}^*_{G,k}[C^*]$ of $\mathcal{S}^*_{\mathcal{F},k}$. Bringing all the pieces together, we get

$$|\mathcal{A}| \leq \binom{r}{t} |\mathcal{A}[C^*]| \leq \sum_{G \in \mathcal{G}} \binom{r}{t} |\mathcal{A}_G| < \sum_{G \in \mathcal{G}} |\mathcal{C}_G| = |\mathcal{C}|,$$

where the strict inequality follows by (6). Hence the result. □

References

[1] R. Ahlswede, L.H. Khachatrian, The complete intersection theorem for systems of finite sets, European J. Combin. 18 (1997) 125–136.
[2] R. Ahlswede, L.H. Khachatrian, The diametric theorem in Hamming spaces — Optimal anticodes, Adv. in Appl. Math. 20 (1998) 429–449.
[3] C. Berge, Nombres de coloration de l'hypergraphe h-parti complet, in: Hypergraph Seminar, Columbus, Ohio 1972, in: Lecture Notes in Math., vol. 411, Springer, Berlin, 1974, pp. 13–20.
[4] B. Bollobás, I. Leader, An Erdős–Ko–Rado theorem for signed sets, Comput. Math. Appl. 34 (1997) 9–13.
[5] P. Borg, Intersecting systems of signed sets, Electron. J. Combin. 14 (2007) #R41.
[6] P.J. Cameron, C.Y. Ku, Intersecting families of permutations, European J. Combin. 24 (2003) 881–890.
[7] D.E. Daykin, Erdős–Ko–Rado from Kruskal–Katona, J. Combin. Theory Ser. A 17 (1974) 254–255.
[8] M. Deza, Matrices dont deux lignes quelconques coincident dans un nombre donne' de positions communes, J. Combin. Theory Ser. A 20 (1976) 306–318.
[9] M. Deza, P. Frankl, On the maximum number of permutations with given maximal or minimal distance, J. Combin. Theory Ser. A 22 (1977) 352–360.
[10] M. Deza, P. Frankl, The Erdős–Ko–Rado theorem — 22 years later, SIAM J. Algebraic Discrete Methods 4 (1983) 419–431.
[11] K. Engel, An Erdős–Ko–Rado theorem for the subcubes of a cube, Combinatorica 4 (1984) 133–140.
[12] P.L. Erdős, U. Faigle, W. Kern, A group-theoretic setting for some intersecting Sperner families, Combin. Probab. Comput. 1 (1992) 323–334.
[13] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 12 (1961) 313–320.
[14] P. Frankl, The Erdős–Ko–Rado Theorem is true for $n = ckt$, in: Proc. Fifth Hung. Comb. Coll, North-Holland, Amsterdam, 1978, pp. 365–375.

[15] P. Frankl, The shifting technique in extremal set theory, in: C. Whitehead (Ed.), Combinatorial Surveys, Cambridge Univ. Press, London, New York, 1987, pp. 81–110.
[16] P. Frankl, Z. Füredi, The Erdős–Ko–Rado Theorem for integer sequences, SIAM J. Algebraic Discrete Methods 1 (4) (1980) 376–381.
[17] P. Frankl, N. Tokushige, The Erdős–Ko–Rado theorem for integer sequences, Combinatorica 19 (1999) 55–63.
[18] H.-D.O.F. Gronau, More on the Erdős–Ko–Rado theorem for integer sequences, J. Combin. Theory Ser. A 35 (1983) 279–288.
[19] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets, Quart. J. Math. Oxford (2) 18 (1967) 369–384.
[20] F.C. Holroyd, J. Talbot, Graphs with the Erdős–Ko–Rado property, Discrete Math. 293 (2005) 165–176.
[21] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, J. Combin. Theory Ser. B 13 (1972) 183–184.
[22] G.O.H. Katona, A theorem of finite sets, in: Theory of Graphs, Proc. Colloq. Tihany, Akadémiai Kiadó, 1968, pp. 187–207.
[23] G.O.H. Katona, Intersection theorems for finite sets, Acta Math. Acad. Sci. Hungar. 15 (1964) 329–337.
[24] D.J. Kleitman, On a combinatorial conjecture of Erdős, J. Combin. Theory Ser. A 1 (1966) 209–214.
[25] J.B. Kruskal, The number of simplices in a complex, in: Mathematical Optimization Techniques, University of California Press, Berkeley, California, 1963, pp. 251–278.
[26] C.Y. Ku, Intersecting families of permutations and partial permutations, Ph.D. Dissertation, Queen Mary College, University of London, December, 2004.
[27] C.Y. Ku, I. Leader, An Erdős–Ko–Rado theorem for partial permutations, Discrete Math. 306 (2006) 74–86.
[28] B. Larose, C. Malvenuto, Stable sets of maximal size in Kneser-type graphs, European J. Combin. 25 (2004) 657–673.
[29] Yu-Shuang Li, Jun Wang, Erdős–Ko–Rado-type theorems for colored sets, Electron. J. Combin. 14 (2007) #R1.
[30] M.L. Livingston, An ordered version of the Erdős–Ko–Rado Theorem, J. Combin. Theory Ser. A 26 (1979) 162–165.
[31] J.-C. Meyer, Quelques problémes concernant les cliques des hypergraphes k-complets et q-parti h-complets, in: Hypergraph Seminar, Columbus, Ohio 1972, in: Lecture Notes in Math., vol. 411, Springer, Berlin, 1974, pp. 127–139.
[32] A. Moon, An analogue of the Erdős–Ko–Rado theorem for the Hamming schemes $H(n, q)$, J. Combin. Theory Ser. A 32 (1982) 386–390.
[33] R.M. Wilson, The exact bound in the Erdős–Ko–Rado theorem, Combinatorica 4 (1984) 247–257.