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Partitioning a graph into alliance free sets

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ABSTRACT

A strong defensive alliance in a graph $G = (V, E)$ is a set of vertices $A \subseteq V$, for which every vertex $v \in A$ has at least as many neighbors in A as in $V - A$. We call a partition A, B of vertices to be an alliance-free partition, if neither A nor B contains a strong defensive alliance as a subset. We prove that a connected graph G has an alliance-free partition exactly when G has a block that is other than an odd clique or an odd cycle.

1. Definitions and notation

Consider a graph $G = (V, E)$ without loops or multiple edges with order $n = |V|$ and size $m = |E|$. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u : uv \in E\}$, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is defined as $\deg(v) = |N(v)|$. For a set $S \subseteq V$ and vertex $v \in V$, we denote $N_S(v) = N(v) \cap S$ and $\deg_S(v) = |N(v) \cap S| = |N_S(v)| = \deg(v) - \deg_{V-S}(v)$. Similarly, $N[v] \cap S = N_S[v]$. The open and closed neighborhoods of sets of vertices $S \subseteq V$ are defined as follows: $N(S) = \bigcup_{v \in S} N(v)$, and $N[S] = N(S) \cup S$. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, written $G' \subseteq G$ if $V' \subseteq V$ and $E' \subseteq E \cap V' \times V'$. If $S \subseteq V$, the subgraph induced by S is the graph $G[S] = (S, E \cap S \times S)$. Let V_1 and V_2 partition V. The set of edges, which have one end vertex in V_1 and the other in V_2 is denoted as $\langle V_1, V_2 \rangle$. A cut vertex is a vertex whose removal disconnects the graph. A graph with no cut vertex is called a nonseparable graph. A block is a maximal nonseparable subgraph of a graph. Other definitions and notation will be introduced as needed.

2. Alliance-free sets and alliance covers

Defensive alliances in graphs were first introduced by Hedetniemi, et al. [12]. Other types of alliances have been subsequently proposed, for example, (strong) offensive alliances [8], global alliances [11], and powerful alliances [5]. A nonempty set $A \subseteq V$ is a *strong defensive alliance* [12] (also known as cohesive set [14] or 0-defensive alliance [16]) if for all vertices $v \in A$, $\deg_A(v) \geq \deg_{V-A}(v)$. That is, every vertex in a strong defensive alliance A has at least as many neighbors in A as in $V - A$. Throughout this paper, strong defensive alliances will be simply referred to as alliances. An alliance A is called *minimal* if no proper subset of A is an alliance. Note that if A is a minimal alliance then $G[A]$ is connected. Otherwise, any connected component of $G[A]$ is also an alliance, which contradicts A being a minimal alliance.

A set $X \subseteq V$ is *alliance free* if for all alliances A, $A - X \neq \emptyset$. A set $Y \subseteq V$ is an *alliance cover* if for all alliances A, $A \cap Y \neq \emptyset$. An alliance cover Y is minimal if no proper subset of Y is an alliance cover. A minimum alliance cover is a minimal alliance cover of smallest cardinality. A set $X \subset V$ is an alliance cover if and only if $V - X$ is alliance free [15].

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3. Alliance-free partitions

In this paper, we deal with the problem of partitioning the vertex set of a graph G into alliance-free sets. We refer to such a partition as an *alliance-free partition* and say G is partitionable if it has an alliance-free partition. Problems of partitioning the vertex set of a graph with constraints on the degrees of vertices in the sets can be traced to the problem of *unfriendly partition* of graphs introduced by Borodin and Kostochka [4] in 1977. A partition is said to be unfriendly if each vertex has as many or more neighbors outside the set in which it occurs than inside it. The problem has also been studied in [1,3,6,17]. Note that, in an unfriendly partition, if every vertex has *strictly* more neighbors outside the set in which it occurs than inside it, then the partition is an alliance-free partition. However, the converse is not true, i.e., a vertex in an alliance-free partition may have the same number of neighbors inside the set in which it occurs as outside it.

A similar but complementary problem was studied in [9,14], where a bipartition of the vertex set into alliances was sought. Such a partition is called *Satisfactory Partition*. The problem of bi-partitioning the vertex set with constraints on the minimum degrees is addressed in [7,10,13,18,19].

There exists an unfriendly graph bipartition for every finite graph [17]. (There are infinite graphs with no unfriendly bipartition, however, all graphs have an unfriendly 3-partition [17].) This is not the case for satisfactory partitions and alliance-free partitions. For example, odd cliques and complete bipartite graphs $K_{p,q}$ (when p or q is odd) do not have satisfactory partitions, and odd cliques and odd cycles do not have alliance-free partitions. The satisfactory partition problem is known to be NP-complete [2]. In this paper, we characterize graphs having alliance-free partitions. In particular, we show the following:

Theorem 1. *A connected graph G is partitionable if and only if G has a block that is other than an odd clique or an odd cycle.*

Define a set S to be an *alliance-free cover* if S is both alliance free and an alliance cover. Equivalently, S is an alliance-free cover if for all alliances X, $X \cap S \neq \emptyset$ and $X \cap (V - S) \neq \emptyset$. Thus, we have the following:

Lemma 2. *A set S is an alliance-free cover if and only if $V - S$ is an alliance-free cover.*

From Lemma 2, we conclude the following:

Theorem 3. *A graph G is partitionable if and only if G has an alliance-free cover.*

4. When *G* is not partitionable

We call an alliance cover X to be *special* if X contains an alliance U_X and a vertex $u \in U_X$ such that $X - u$ is alliance free.

Lemma 4. *If G is not partitionable and X is a special alliance cover in G then X contains a unique minimal alliance U_X, such that $G[U_X]$ is a connected component of $G[X]$ and $\forall x \in U_X$:*

(1) $\deg_X(x) = \deg_{V-X}(x)$, *and*
(2) $(V - X) \cup \{x\}$ *is also a special alliance cover.*

Proof. Since, by definition of special alliance, there exists a vertex $u \in X$ such that $X - \{u\}$ is alliance free, the alliance U_X containing u is the only alliance in X. Since X is an alliance cover, $V - X$ is alliance free. Also, since G is not partitionable and $X - \{u\}$ is alliance free, the set $(V - X) \cup \{u\}$ must contain an alliance. Hence $\deg_X(u) = \deg_{V-X}(u)$.

Suppose now that there exists $v \in U_X$, such that $\deg_X(v) > \deg_{V-X}(v)$. Let v be the nearest such vertex to u in $G[U_X]$ and let $P : u = v_1, v_2, \ldots, v_k, v$ be a shortest path from u to v. Since $V - X$ is alliance free and $\deg_{V-X}(v) < \deg_X(v)$, $(V - X) \cup \{v\}$ is alliance free. Also, since $\deg_X(v_k) = \deg_{V-X}(v_k)$ and $v \in N(v_k)$, $U_X - \{v\}$ is not an alliance. This implies that $X - \{v\}$ is also alliance free, which is contrary to G not being partitionable. Hence $\forall x \in U_X$, $\deg_X(x) = \deg_{V-X}(x)$ and the graph $G[U_X]$, induced by U_X, is a connected component of the graph $G[X]$. Since G is not partitionable, for any $x \in U_X$, the set $(V - X) \cup \{x\}$ must contain an alliance and hence, is a special alliance cover. □

The following result is immediate from Lemma 4.

Corollary 5. *If G is not partitionable and X is a special alliance cover in G then for any $x \in U_X \subseteq X$ and $y \in U_{(V-X)\cup\{x\}}$, $X' = (X - \{x\}) \cup \{y\}$ is a special alliance cover, and $y \in U_{X'}$.*

The following result shows the existence of special alliance covers in the graphs that are not partitionable.

Lemma 6. *If G is not partitionable then for every $v \in V(G)$, there exists a special alliance cover X such that the minimal alliance U_X contains v.*

Proof. For any vertex $v \in V(G)$, order the vertices $v_1, v_2, \ldots, v_n = v$, such that v_i is adjacent to at least one v_j with $j > i$, for all $i < n$. Now perform the following procedure:

$X \leftarrow \emptyset, Y \leftarrow \emptyset, i \leftarrow 1$
While $i \leq n$
Begin
If $|N_X(v_i)| \leq |N_Y(v_i)|$, $X \leftarrow X \cup \{v_i\}$ *else* $Y \leftarrow Y \cup \{v_i\}$
$i \leftarrow i + 1$
End.

Since G is not partitionable, assume with out loss of generality that X contains an alliance U_X. Let v_k be the first vertex in the procedure whose addition to X formed an alliance in X. If $k < n$ then by procedure, $|N_X(v_k)| < |N_Y(v_k)| + |N_{V-X-Y}(v_k)|$, a contradiction, hence $k = n$. Thus, both $X - \{v_n\}$ and Y are alliance free, which implies that X is a special alliance cover and $v_n = v$ is in the alliance U_X. $\square$

Corollary 7. *If G is not partitionable, then G is Eulerian.*

The following theorem describes the partitionable graphs in terms of their blocks.

Theorem 8. *A connected graph G is partitionable if and only if some block of G is partitionable.*

Proof. The proof is by induction on the number of blocks in graph G. The statement is true if G is itself a block, and hence, the base case is true. Assume that the statement is true for all graphs with at most r blocks, for a fixed but arbitrary $r \geq 1$. Consider a graph G with $r + 1$ blocks and let x be a cut-vertex in G. Let G_1 be the graph induced by $V_1 \subset V$, where $x \in V_1$ and $V_1 - \{x\}$ induces a connected component in graph $G - \{x\}$. Also, let G_2 be the graph induced by $V_2 = (V - V_1) \cup \{x\}$.

First, assume that G is partitionable and thus has an alliance-free cover, say B'. Further, assume that neither G_1 nor G_2 is partitionable. From Lemma 2, we may assume that $x \in B'$. Note that for $i \in \{1, 2\}$, $B_i = B' \cap V_i$ is an alliance cover in graph G_i. Thus each B_i must contain an alliance T_i in graph G_i. Now we have two cases. Case 1: For some $i \in \{1, 2\}$, $x \notin T_i$. Then, $T_i \subseteq B'$ is also an alliance in graph G, which is contrary to B' being an alliance-free cover in graph G. Case 2: $x \in T_1 \cap T_2$. But then, $T_1 \cup T_2 \subseteq B'$ is an alliance in graph G, again a contradiction.

Since both cases lead to a contradiction, we conclude that at least one of G_1 and G_2 is partitionable. Thus, by induction hypothesis, some block of G_1 or G_2 is partitionable. Hence, some block of G is partitionable.

Next, suppose some block of G is partitionable. We may assume without loss of generality that the block is in G_1 and, hence, by the induction hypothesis, G_1 is partitionable. Let B_1 be an alliance-free cover in G_1. From Lemma 2, we may assume that $x \notin B_1$. There are two cases. Case 1: G_2 is partitionable. Then, there is an alliance-free cover B_2 in G_2. Once again, we may assume that $x \notin B_2$. But then $B_1 \cup B_2$ is an alliance-free cover of graph G, thus G is partitionable. Case 2: If G_2 is not partitionable, every alliance cover in G_2 contains some alliance. By Lemma 6, there exists a special alliance cover B_2 in G_2, such that $x \in U_{B_2}$. If $B' = (B_1 \cup B_2) - \{x\}$ is not an alliance cover of graph G then there must exist an alliance S in G, such that $S \cap B' = \emptyset$ and $x \in S$. Since $x \in U_{B_2}$, $\left|N_{V_2 \cap S}(x)\right| = \left|N_{V_2 - S}(x)\right|$. From Corollary 7, we may assume that G is Eulerian, and $\left|N_{V_1}(x)\right| \geq 2$, hence, $V_1 \cap S \neq \emptyset$ and $\left|N_{V_1 \cap S}(x)\right| \geq \left|N_{V_1 - S}(x)\right|$. But then, $V_1 \cap S$ is also an alliance in graph G_1, which contradicts B_1 being an alliance cover in G_1. Hence, B' is an alliance-free cover of graph G, and G is partitionable. $\square$

5. When a block is not partitionable

From Theorem 8, a graph is not partitionable if and only if every block of G is not partitionable. In this section, we characterize the blocks that are not partitionable.

Let G be an unpartitionable block and let X be a special alliance cover in G containing an alliance U_X. Also let $Y = V - X$.

Lemma 9. *If G is an unpartitionable block then the graph $G[U_X]$ is a block.*

Proof. Assume to the contrary that x is a cut vertex in $G[U_X]$. Let $\{a, b\} \subseteq U_X$, such that every $a - b$ path in $G[U_X]$ contains x. Since G is a block, there must be a path P in G from a to b that does not contain x. Since $N_X(U_X) = U_X$, $P \cap \langle X, Y\rangle \neq \emptyset$. Assume now that the choice of X, x, a and b is such that $|P \cap \langle X, Y\rangle|$ is minimum among all such choices. Further, assume that P is a shortest such path in G. Let $P \cap \langle X, Y\rangle = \{y_1y_2, y_3y_4, \ldots, y_{4k-1}y_{4k}\}$ for some $k \geq 1$, where $\{y_{4i-3}, y_{4i}\} \subseteq X$ and $\{y_{4i-2}, y_{4i-1}\} \subseteq Y$, $1 \leq i \leq k$. In addition, y_{2j} may be the same as y_{2j+1}, $0 < j < 2k$. Since P is a shortest such path, $y_1 = a$ and $y_{4k} = b$. Let $X_0 = X$ and for $1 \leq i \leq k$, define;

$$X_i = (X_{i-1} - \{y_{4i-3}\}) \cup \{y_{4i-1}\}, \quad \text{and}$$
$$Y_i = V - X_i.$$

From Corollary 5, $\forall i$, X_i is a special alliance cover. Also, $\forall i > 0$, $\{y_{4i-1}, y_{4i}, y_{4i+1}\} \subseteq U_{X_i}$ and $y_{4i-1}y_{4i} \in E(G)$.

Let $U' \subseteq U_{X_0}$ such that $G[U']$ is a connected component in $G[U_{X_0} - a]$ and $b \in U'$. Note that, $\forall v \in U' - N(a)$, $\deg_{U'}(v) = \deg_{V-U'}(v)$. In particular, $\deg_{U'}(b) = \deg_{V-U'}(b)$. Since $b \in U_{X_k}$ and $N(U_{X_k}) = U_{X_k}$, $U' \subseteq U_{X_k}$. Since none of the vertices y_i for $i < 4k - 1$ can be a neighbor of b, $\deg_{U'}(b) = \deg_{V-U'}(b)$, and $y_{4k-1}b \in E(G)$, it follows that $\deg_{X_k}(b) > \deg_{Y_k}(b)$, which is contrary to X_k being a special alliance cover. $\square$

Lemma 10. *If G is not partitionable and $\{u, v\} \subseteq U_X$, such that $N_{V-X}(u) \cap N_{V-X}(v) \neq \emptyset$ then $uv \in E(G)$.*

Proof. Let $\{u, v\} \subseteq U_X$, such that $z \in N_{V-X}(u) \cap N_{V-X}(v)$. By Corollary 5, $X' = (X - \{u\}) \cup \{z\}$ is a special alliance cover, and $z \in U_{X'}$. Since $v \in N_{X'}(z)$, $v \in U_{X'}$, i.e., $|N_{V-X'}(v)| = |N_{X'}(v)|$, which is possible only if $uv \in E(G)$. □

Lemma 11. *If G is an unpartitionable block and X is a special alliance cover with $|U_X| > 2$ then for any $\{a, b\} \subset U_X$, $N_Y(a) \cap N_Y(b) \neq \emptyset$, where $Y = V - X$.*

Proof. Let $|U_X| > 2$ and $\{a, b\} \subseteq U_X$. From Lemma 9, $\forall x \in U_X$, $|N_{U_X}(x)| \geq 2$. Let $y_2 \in N_Y(a)$. Since G is a block, there must exist a path P from y_2 to b that does not contain a. Let P be such a path, for which $|P \cap \langle X, Y\rangle|$ is minimum among all such paths. Let $y_1 = a$ and $P \cap \langle X, Y\rangle = \{y_3y_4, y_5y_6, \ldots, y_{4k-1}y_{4k}\}$, $k \geq 1$, where $\{y_{4i-3}, y_{4i}\} \subseteq X$ and $\{y_{4i-2}, y_{4i-1}\} \subseteq Y$, $1 \leq i \leq k$. Further, y_{2j} may be the same as y_{2j+1}, $0 < j < 2k$. Also, let $y_{4k+1} = b$, $X_0 = X$ and for $1 \leq i \leq k$, define;

$$X_i = (X_{i-1} - \{y_{4i-3}\}) \cup \{y_{4i-1}\}, \quad \text{and}$$
$$Y_i = V - X_i.$$

From Corollary 5, $\forall i$, X_i is a special alliance cover. Also, $\forall i > 0$, $\{y_{4i-1}, y_{4i}, y_{4i+1}\} \subseteq U_{X_i}$ and $y_{4i-1}y_{4i} \in E(G)$. Note that, $\forall i, 0 < i < k$, $U' \cap U_{X_i} = \emptyset$, where $U' = U_X - \{y_1\}$, otherwise, there is a $y_2 - b$ path $P' \subseteq P$ such that $\left|P' \cap \langle X, Y\rangle\right| < |P \cap \langle X, Y\rangle|$, a contradiction. Since $b \in U_{X_k}$, $U' \subseteq U_{X_k}$. Hence, $\forall z_i \in N_{U_X}(a)$, $y_{4k-1}z_i \in E(G)$. Since $|N_{U_X}(a)| > 1$, there are at least two vertices z_1, z_2 in U_X such that $y_{4k-1} \in N_Y(z_1) \cap N_Y(z_2)$. From Lemma 10, $z_1z_2 \in E(G)$.

We now claim that $\forall x \in U_X, y_{4k-1} \in N(x)$. Suppose not. Then there must exist $\{u, v, w\} \subseteq U_X$, such that $\{v, w\} \subseteq N(u)$, and $y_{4k-1} \in (N(u) \cap N(v)) - N(w)$. By Corollary 5, $X' = (X - \{u\}) \cup \{y_{4k-1}\}$ is a special alliance cover, and $y_{4k-1} \in U_{X'}$. Also, since $G[U_X]$ is a block and $N_{X'}(U_{X'}) = U_{X'}$, $N_{X'}(y_{4k-1}) = N_X(u)$, a contradiction. Hence, $\forall x \in U_X, y_{4k-1} \in N(x)$, which completes the proof. □

Theorem 12. *If G is a block, then G is partitionable if and only if G is neither an odd clique nor an odd cycle.*

Proof. It is easy to see that odd complete graphs and odd cycles are not partitionable. To prove the sufficiency of the theorem, let G be a block that is not partitionable and consider two exhaustive cases:

Case 1: There exists a special alliance cover X in G, such that $|U_X| > 2$. Let $Y = V - X$. From Lemmas 10 and 11, $G[U_X]$ is a clique, and $\forall x \in U_X$, $G[U_{Y\cup\{x\}}]$ is also a clique. Hence $\forall x \in U_X$, $N[x] = U_X \cup U_{Y\cup\{x\}}$. Also, from Lemma 4, $N_{Y\cup\{x\}}\left(U_{Y\cup\{x\}}\right) = U_{Y\cup\{x\}}$. Thus, from Lemma 11, for every $\{x, y\} \subset U_X$, $N[x] = N[y]$. By the above arguments, $\forall x \in U_X$, $N[x]$ is a clique, and is a connected component of the graph G. Since G is connected, this is only possible if $G = G[N[x]]$. Hence, G is a complete graph. In addition, since even cliques are partitionable, G has odd order.

Case 2: For all special alliance covers X in G, $|U_X| = 2$. From Lemma 6, for all $w \in V$, there exists a special alliance cover B, such that $w \in U_B$. Since, $|U_B| = 2$ and $\deg_{U_B}(w) = \deg_{V-U_B}(w)$, $\deg(w) = 2$, and hence, G is a cycle. Further, since even cycles are partitionable, G is an odd cycle. □

From Theorems 8 and 12, we conclude that a connected graph G is partitionable if and only if G has a block that is other than an odd clique or an odd cycle, which is our main result (Theorem 1 of Section 3).

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