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# Path and cycle decompositions of complete equipartite graphs: Four parts

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ABSTRACT

We show that a complete equipartite graph with four partite sets has an edge-disjoint decomposition into cycles of length $k$ if and only if $k \geq 3$, the partite set size is even, $k$ divides the number of edges in the equipartite graph and the total number of vertices in the graph is at least $k$. We also show that a complete equipartite graph with four *even* partite sets has an edge-disjoint decomposition into paths with $k$ edges if and only if $k$ divides the number of edges in the equipartite graph and the total number of vertices in the graph is at least $k + 1$.

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## 1. Introduction and preliminaries

In this paper we give necessary and sufficient conditions for the existence of an edge-disjoint decomposition of a complete equipartite graph, having four parts of equal even size, into cycles and paths of length $k$. Before we venture further, we remind the reader of some definitions. A *complete equipartite graph* $K_{n(m)}$ has its $nm$ vertices partitioned into $n$ parts (often referred to as partite sets), of size $m$, and there is an edge between any two vertices in different partite sets, but no edge between any two vertices in the same partite set.

The lexicographic product $G * H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$, and with an edge joining $(g_1, h_1)$ to $(g_2, h_2)$ if and only if: $g_1$ is adjacent to $g_2$ in $G$; or $g_1 = g_2$ and $h_1, h_2$ are adjacent in $H$. Here we shall be concerned with graphs such as $K_n * \overline{K}_m$, which is the same as the complete equipartite graph $K_{n(m)}$ having $n$ parts of size $m$. (Here as usual $\overline{K}_m$ denotes the complement of $K_m$.) We shall also use the notation $K(a_1, a_2, \ldots, a_n)$ for the complete multipartite graph with $n$ parts of sizes $a_1, a_2, \ldots, a_n$. We point out that if $G$ has an edge-disjoint decomposition into subgraphs $G_1, G_2, \ldots, G_t$, then $G * \overline{K}_m$ has an edge-disjoint decomposition into subgraphs $G_1 * \overline{K}_m,\ G_2 * \overline{K}_m, \ldots, G_t * \overline{K}_m$.

The standard notation for a path on $n$ vertices is $P_n$. In this paper, in the context of edge-disjoint decompositions, we are more interested in a path's length, so we let $L_k$ denote a path of *length* $k$ (on $k + 1$ vertices), and $T_k$ a trail of length $k$ (which may be on fewer than $k + 1$ vertices).

The problem of determining necessary and sufficient conditions for the existence of an edge-disjoint decomposition of a complete graph $K_n$ ($n$ odd) into $k$-cycles was finally completed in [1,12]. This graph can be regarded as a complete equipartite graph in which all the parts have size 1. The same cycle decomposition problem for the graph $K_n - F$ where $F$ is a 1-factor, when $n$ is even, was also solved in these papers. This latter graph can be regarded as a complete multipartite graph with $n/2$ parts of size 2.

For paths, Tarsi [15] gave necessary and sufficient conditions for an edge-disjoint decomposition of $K_n$ into paths $L_k$.

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Some work has also been done on cycle decompositions of complete equipartite graphs with arbitrary part size $m$, $K_{n(m)}$ or $K_n * \overline{K}_m$, chiefly when the cycle length is small and specified (see [5] for small even length and arbitrary part sizes), or else of prime length [10] or of length twice a prime [13]. Less work seems to have been done for paths in equipartite or multipartite graphs, although for short paths on at most five vertices the paper [3] deals with arbitrary complete multipartite graphs. Liu [8,9] deals with *resolvable* cycle decompositions of complete equipartite graphs with any number of parts; the resolvability of course means a greater restriction on possible cycle lengths.

In this paper and its partner paper [2], instead of restricting the cycle (or path) length, or restricting the part sizes, we restrict the total *number* of parts. The paper [4] dealt with cycles in tripartite graphs, and [2] deals with paths in tripartite graphs, and both cycles and paths in equipartite graphs having 5 parts. When the number of parts is even, as in this paper where we concentrate on four parts, unless the part size is also even, the graph has odd degree, rendering cycle decompositions impossible and path decompositions more restricted. So here we deal with complete equipartite graphs having four parts of *even* size, and give necessary and sufficient conditions for an edge-disjoint decomposition into cycles and paths of length $k$.

In what follows we can assume that all paths have length at least 3. Of course *any* graph can be decomposed into paths of length 1! Also it has long been known that *any* connected graph with an even number of edges has an edge-disjoint decomposition into paths of length 2. The following is due to Hoffman [6], although it may first have been shown by Kotzig. Let $G$ be a connected graph with an even number of edges, and replace each (undirected) edge by a single directed edge, randomly directed. If all the outdegrees of vertices are even, we stop; otherwise take two vertices with odd outdegree (there must be an even number!), and find a path (not directed) between them. Reversing the orientations along this path will ensure that the end vertices of this path now have even outdegree; the inner vertices on this path have their outdegrees unchanged. Since $G$ is finite, continue until all outdegrees are even. Now each path of length 2 can be taken as a pair of directed edges in this graph, both directed away from the same vertex which is the centre vertex of the path.

Subsequently we denote a $k$-cycle on the vertex set $\{x_i \mid 1 \leqslant i \leqslant k\}$, with edges $\{x_1, x_k\}$ and $\{x_i, x_{i+1}\}$ for $1 \leqslant i \leqslant k-1$, by $(x_1, x_2, \ldots, x_k)$ or $(x_k, x_{k-1}, \ldots, x_2, x_1)$ or by any cyclic shift of these. A path on the vertex set $\{x_i \mid 1 \leqslant i \leqslant k+1\}$, with edges $\{x_i, x_{i+1}\}$ for $1 \leqslant i \leqslant k$, will be denoted by $[x_1, x_2, \ldots, x_{k+1}]$ or by $[x_{k+1}, x_k, \ldots, x_2, x_1]$.

The notation $\ell$ MOLS$(n)$ will refer to a set of $\ell$ mutually orthogonal latin squares of order $n$. (Here, if $\ell = 1$, we really mean any one latin square of order $n$.)

In Section 2 we state some "blowing up" type lemmas which we use. Then Section 3 starts with cycles and paths of length 0 (mod 4), goes on to deal with cycles and paths of length 2 (mod 4), and concludes with odd length cycles and paths, first those which have length a multiple of 3, and then those of length coprime to 3. Perhaps surprisingly, most of the methods used are so similar for paths and cycles that we deal with them simultaneously in Section 3. Section 4 gives a concluding theorem and includes some remarks on decomposition into paths when the four part sizes are odd.

## 2. Some useful lemmas

We need the following results which are proved in our partner paper [2] or in Cavenagh [4].

**Theorem 2.1.** *If the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ has a decomposition into cycles of length $k$, then the graph $K(a_1\ell, a_2\ell, \ldots, a_n\ell)$*

(i) *has a decomposition into cycles of length $k$; and*
(ii) *has a decomposition into cycles of length $k\ell$.*

There is a corresponding theorem for paths, which follows from Lemmas 2.1 and 2.2 in [2]:

**Theorem 2.2.** *If the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ has a decomposition into paths of length $k$, then the graph $K(a_1\ell, a_2\ell, \ldots, a_n\ell)$*

(i) *has a decomposition into paths of length $k$; and*
(ii) *has a decomposition into paths of length $k\ell$.*

Applying Theorem 2.5 and Corollary 2.6 from our paper [2], we have the following two results, which we use frequently throughout this paper.

**Theorem 2.3.** *If the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ has a decomposition into closed trails of length $k$, each having maximum degree $\Delta(T_k) = \Delta$ and (vertex) chromatic number $\chi(T_k) = \chi$, then for all $\ell \geqslant \Delta/2$, provided there exist at least $\chi - 2$ MOLS $(\ell)$, the graph $K(a_1\ell, a_2\ell, \ldots, a_n\ell)$ has a decomposition into cycles of length $k$.*

**Corollary 2.4.** *Suppose the complete multipartite graph $K(a_1, a_2, \ldots, a_n)$ has a decomposition into closed trails of length $k$, each having vertex chromatic number $\chi(T_k) = \chi$ with maximum degree $\Delta$. Let $\ell \geqslant \Delta/2$ (if the minimum degree of $T_k$ is strictly less than $\Delta$) or $\ell > \Delta/2$ if $T_k$ has regular degree $\Delta$. Then, provided there exist at least $\chi - 2$ MOLS $(\ell)$, the graph $K(a_1\ell, a_2\ell, \ldots, a_n\ell)$ has a decomposition into paths of length $k$.*

The following lemma will be used to deal with two special cases of Theorem 3.12 in Section 3. The graph $\lambda K$ denotes the multigraph obtained from the graph $K$ by replacing each of its edges with $\lambda$ edges.

**Lemma 2.5.** *Let $\lambda$ be an odd prime. Suppose that $\lambda K_{n(m)}$ decomposes into cycles of length k. Suppose, in turn, that for each cycle $C_1 = (v_0, v_1, \ldots, v_{k-1})$ in the decomposition, there exists a cycle $C_2 = (v'_0, v'_1, v'_2, \ldots, v'_{k-1})$ in the decomposition which shares at least two edges with $C_1$ which have the same "direction" within the cycle (for some orientation of the cycle). That is, there exist $i, j, g, h$ such that $i \neq j$, $g \neq h$ and $v_i = v'_g$, $v_{i+1} = v'_{g+1}$, $v_j = v'_h$ and $v_{j+1} = v'_{h+1}$, where subscripts are calculated modulo k. Then there exists a decomposition of $K_{n(m\lambda)}$ into cycles of length $k\lambda$.*

**Proof.** First, assign to each edge of $\lambda K_{n(m)}$ an integer between 0 and $\lambda - 1$ (inclusive) in such a way that if two distinct edges have the same end vertices, they are labelled differently. Label the partite sets of $\lambda K_{n(m)}$ with $A_1, A_2, \ldots, A_n$. Let $C = (v_0, v_1, \ldots, v_{k-1})$ be a cycle in the decomposition. Suppose, for a particular $i$, that $v_i \in A_j$ and $v_{i+1} \in A_k$ and that edge $e = \{v_i, v_{i+1}\}$ (in this cycle) has been assigned integer $x$. If $j < k$, we assign the edge $e$ a *weight* $x$; otherwise $j > k$ and we assign this edge $e$ *weight* $-x$. We then define the weight of the cycle $C$ to be the sum of the weights of its edges. We say that a cycle is *good* if its weight is *not* divisible by $\lambda$; otherwise it is *bad*.

Our aim is to, if necessary, swap labellings of the edges so that every cycle is good. If every cycle is already good we are done. Otherwise assume $C_1$ is some bad cycle and a "partner" cycle is $C_2$. Let $e_1 = \{v_i, v_{i+1}\}$ and $f_1 = \{v_j, v_{j+1}\}$ be edges in $C_1$, and suppose (by the conditions of this lemma) that these pairs of vertices appear in the same order in $C_2$, as edges $e_2$ and $f_2$. Suppose furthermore that the edges $e_1$ and $e_2$ are labelled with $w$ and $x$, respectively. Then clearly $w \neq x$. Similarly, suppose that the edges $f_1$ and $f_2$ are labelled with $y$ and $z$, respectively. Again, we have $y \neq z$.

If both $C_1$ and $C_2$ are bad, then swapping the labels $w$ and $x$ between edges $e_1$ and $e_2$ will make both of them good. Assume then that $C_2$ is good. If using label $x$ on edge $e_2$ with label $y$ on edge $f_2$ makes $C_2$ good, then we swap $y$ and $z$, making both $C_1$ and $C_2$ good. Similarly, if using label $w$ on edge $e_2$ with label $z$ on edge $f_2$ makes $C_2$ good, then we swap $w$ and $x$, making both $C_1$ and $C_2$ good.

Otherwise, the pairs $(w, z)$ and $(x, y)$ both give a weight for $C_2$ which is divisible by $\lambda$. Thus the total weight induced by edges $e_2$ and $f_2$ must be the same for $(w, z)$ and $(x, y)$. Let $v_i \in A_p$, $v_{i+1} \in A_q$, $v_j \in A_r$ and $v_{j+1} \in A_s$. If ($p < q$ and $r < s$) or ($p > q$ and $r > s$), then, since $\lambda$ is prime,

$$w + z \equiv x + y \pmod{\lambda}. \tag{1}$$

Otherwise either ($p < q$ and $r > s$) or ($p > q$ and $r < s$) and

$$w - z \equiv x - y \pmod{\lambda}. \tag{2}$$

Now, if we swap *both* $w$ with $x$ *and* $y$ with $z$, then $C_2$ must be good. If such a swap also makes $C_1$ good we are done. Otherwise, the pairs $(w, y)$ and $(x, z)$ both give a weight for $C_1$ which is divisible by $\lambda$. This implies that either $w + y \equiv x + z \pmod{\lambda}$ (if ($p < q$ and $r < s$) or ($p > q$ and $r > s$)) or $w - y \equiv x - z \pmod{\lambda}$ (otherwise). In both cases, using Eq. (1) or (2), we have $w = x$, a contradiction.

Repeating this process for each bad cycle, we have a labelling for which all cycles are good. Moreover, we still have the property that each edge between the same pair of vertices uses a different label. Next, we replace each multi-edge of $\lambda K_{n(m)}$ with a copy of the complete bipartite graph $K(\lambda, \lambda)$, to obtain the graph $K_{n(m\lambda)}$. Specifically, we replace each vertex $v$ with $v^1, v^2, \ldots, v^\lambda$ and we replace each edge $\{v, w\}$ in $\lambda K_{n(m)}$ with the edges of the form $\{v^i, w^j\}$, for all $i, j$, $1 \le i, j \le \lambda$.

We associate an edge $\{v, w\}$ in $\lambda K_{n(m)}$ labelled with integer $x$ with the edges of difference $x$ in the copy of $K(\lambda, \lambda)$, where if $v \in A_q$ and $w \in A_r$, then we take edges of the form $\{v^i, w^{i+x}\}$ or $\{v^i, w^{i-x}\}$, depending on whether $q < r$ or $q > r$, respectively.

We then simply replace each cycle of length $k$ in $\lambda K_{n(m)}$ with the $k$ sets of $\lambda$ edges in $K_{n(m\lambda)}$ that correspond to the edges of the cycle. Since $\lambda$ is prime, these edges either form $k$ disjoint $\lambda$-cycles, or precisely one $k\lambda$-cycle. In fact, the condition that each cycle is good guarantees the latter. □

## 3. The 4-partite case

As remarked in Section 1, we assume that our paths always have at least 3 edges.

We first list some known results that will be exploited. The following well-known theorem is an easy part of a more general result due to Sotteau [14].

**Theorem 3.1** (*[14]*). *The complete bipartite graph $K(2m, 2m)$ decomposes into $2k$-cycles if and only if $k$ divides $4m^2$ and $k \leqslant 2m$.*

A corresponding theorem for paths is due to Parker [11]; we give the equipartite case here, since that suffices for our purposes.

**Theorem 3.2** (*[11]*). *The complete bipartite graph $K(2m, 2m)$ decomposes into k-paths if and only if $k$ divides $4m^2$ and $k < 4m$. If $m$ is odd, the complete bipartite graph $K(m, m)$ decomposes into k-paths if and only if $k$ is odd, $k$ divides $m^2$ and $k \leqslant m$.*

The tripartite equipartite case has been solved for cycles [4].

**Theorem 3.3.** *The complete tripartite graph $K(m, m, m)$ decomposes into cycles of length k if and only if $k \le 3m$ and $k | 3m^2$.*

We need the following result, completed by Liu [8,9]. (Note that this result includes hamiltonian decompositions, when $k = mn$.)

**Table 1**
Possible values of $k$

| Case | Cycle/path length $k$ |
|---|---|
| I | $24s^2t$ |
| II | $12s^2t$ with $s^2t$ odd |
| III | $8s^2t$ with $s^2t$ not divisible by 3 |
| IV | $4s^2t$ with $s^2t$ odd and not divisible by 3 |

**Theorem 3.4** ([8,9]). *When $k \geqslant 3$ and $n \geqslant 2$, there is a resolvable $k$-cycle decomposition of $K_{n(m)}$ (i.e., a $C_k$-factorization of $K_{n(m)}$) if and only if*

$$k|mn, \ m(n-1) \text{ is even}, \quad \text{and} \quad k \text{ is even if } n = 2,$$

*except there is no resolvable 3-cycle decomposition of $K(2, 2, 2)$, $K(6, 6, 6)$ or $K(2, 2, 2, 2, 2, 2)$, nor any resolvable 6-cycle decomposition of $K(6, 6)$.*

The next theorem follows from a more general result in [5].

**Theorem 3.5** ([5]). *When $k \in \{4, 6, 8\}$ and $n \geqslant 2$, there is a $k$-cycle decomposition of $K_{n(m)}$ if and only if $k$ divides $mn$ and $m(n-1)$ is even.*

The following theorem, due to Leach and Rodger [7], will also be useful in constructing closed even trails from unions of bipartite 2-factors.

**Theorem 3.6** ([7]). *Let $U$ be any 2-factor of the complete bipartite graph $K(2s, 2s)$. Let $H$ be the graph formed by removing the edges of $U$ from $K(2s, 2s)$. Then there exists a hamilton decomposition of $H$ except when $s = 2$ and $U$ consists of two 4-cycles.*

We next give an outline of the approach taken in the following proofs. Our aim is to decompose $K(2m, 2m, 2m, 2m)$ into cycles or paths of length $k = as^2t$, where $a$, $s$ and $t$ are positive integers such that $a$ is the largest factor of $k$ which divides 24, and $t$ is square-free. Since $k$ divides $24m^2$, it follows that we can write $m = m'st$, for some integer $m'$. The necessary condition $k \leqslant 8m$ (or $k < 8m$ for paths) is then equivalent to $as \leqslant 8m'$ (respectively, $as < 8m'$). (In fact, when $as = 8m'$, we can work with a hamiltonian decomposition into cycles, a case usually dealt with by Theorem 3.4. So we assume for now that $as < 8m'$.)

For *most* cases, we first decompose $K(2s, 2s, 2s, 2s)$ into (at worst) tripartite closed trails of length $as^2$ (using all $8s$ vertices) in such a way that the difference between minimum and maximum degree in each trail is at most 2. (Note that closed trails are by definition *connected* graphs; care is taken in the following proofs to ensure this!)

Since all vertices in a closed trail have even degree, the maximum degree in each trail of length $as^2$ in $K(2s, 2s, 2s, 2s)$ must be at most

$$2\left\lfloor \frac{as^2}{8s} \right\rfloor + 2 = 2\left(\left\lfloor \frac{as}{8} \right\rfloor + 1\right) < 2(m'+1),$$

and so $2(\lfloor as/8 \rfloor + 1) \leqslant 2m'$. Thus either Theorem 2.3 (for cycles) or Corollary 2.4 (for paths) may be applied to obtain a decomposition of $K(2sm', 2sm', 2sm', 2sm')$ into cycles or paths of length $as^2$. Finally, we apply Theorem 2.1 (for cycles) or Theorem 2.2 (for paths) to obtain a decomposition of $K(2sm't, 2sm't, 2sm't, 2sm't)$ (which is $K(2m, 2m, 2m, 2m)$) into cycles or paths of length $as^2t = k$. There are some exceptions and variations to the above strategy, particularly for small values of $s$ and $t$, which will be dealt with as they arise.

In the following two theorems we deal with cycles and paths of even length. Since both of these are bipartite graphs, these cases are easier to deal with.

**Theorem 3.7.** *Let $k$ be a positive integer divisible by 4. Then the graph $K(2m, 2m, 2m, 2m)$ has an edge-disjoint decomposition into cycles (paths) of length $k$ if and only if $k|24m^2$ and $k \leqslant 8m$ ($k < 8m$).*

**Proof.** The necessity of the conditions is clear. Table 1 is a summary of the division of cases.

Henceforth the partite sets of $K(2s, 2s, 2s, 2s)$ are labelled $A_1, A_2, A_3$ and $A_4$.

*Case* I: Suppose first that 24 divides $k$. Let $k = 24s^2t$, where $s$ and $t$ are positive integers and $t$ is square-free. Then since $k|24m^2$, we have $st|m$, so let $m = stm'$ for some positive integer $m'$. Moreover, $k \leqslant 8m$ means that $3s \leqslant m'$, with strict inequality for paths.

*Case* Ia: We deal with the case $m' = 6$ separately since there do not exist two MOLS of order 6. So (since $3s \leqslant m'$ for cycles and $3s < m'$ for paths) we need only consider $s = 1$ or 2 (for cycles) and $s = 1$ (for paths). If $s = 1$, we first take a hamilton decomposition of $K(6, 6, 6, 6)$ into 24-cycles, which exists by Theorem 3.4. From Theorem 2.1(i), there is thus a decomposition of $K(12, 12, 12, 12)$ into 24-cycles. Moreover, since 24-cycles are bipartite closed trails, we can apply Corollary 2.4 to obtain a decomposition of $K(12, 12, 12, 12)$ into paths of length 24. We then apply Theorems 2.1(ii) and 2.2(ii) to obtain decompositions of $K(12t, 12t, 12t, 12t)$ into cycles (respectively, paths) of length $24t$. If $s = 2$, then $2m = 24t$ and $2k = 96t$, so the required decomposition is a hamilton decomposition into cycles, which exists from Theorem 3.4.

**Table 2**
Possible values of $k$

| Case | Cycle/path length $k$ |
|---|---|
| I | $6s^2t$ with $s^2t$ odd |
| II | $2s^2t$ with $s^2t$ odd and not divisible by 3 |

*Case* Ib: Now assume that $m' \neq 6$. Since $3s \leqslant m'$ and $s \geqslant 1$, we have $m' > 2$, so there exists a pair of MOLS of order $m'$. We begin with the graph $K(2s, 2s, 2s, 2s)$ which may be thought of as a closed trail of length $24s^2$ with vertex chromatic number 4 and maximum degree $6s$. Thus we may apply Theorem 2.3 or Corollary 2.4 (since $m' \geqslant 3s$) to obtain a decomposition of $K(2sm', 2sm', 2sm', 2sm')$ into cycles or paths of length $24s^2m'$, respectively. Finally we apply Theorems 2.1(ii) or 2.2(ii) to obtain the required decomposition.

*Case* II: Here $k/12$ is an odd integer. We have $k = 12s^2t$, where $s$ and $t$ are positive integers and $t$ is square-free. Then since $k|24m^2$, it follows that $st|m$, and so let $m = stm'$ for some positive integer $m'$. Moreover, $k \leqslant 8m$ implies that $3s \leqslant 2m'$. In fact, since $s$ is odd, we have $3s < 2m'$.

Our aim is to decompose $K(2s, 2s, 2s, 2s)$ into two closed trails, each of which has length $12s^2$, is bipartite and has maximum degree $3s + 1$. The result will then follow from either Theorems 2.1 and 2.3, or Theorem 2.2 and Corollary 2.4. From Theorem 3.1, the edges between the partite sets $A_1$ and $A_2$ may be decomposed into cycles of length $4s$. Similarly the edges between $A_3$ and $A_4$ may be decomposed into cycles of length $4s$. We construct one closed trail of length $12s^2$ by taking *all* the $4s^2$ edges between $A_1$ and $A_3$, all the $4s^2$ edges between $A_2$ and $A_4$, $(s + 1)/2$ of the $4s$-cycles between $A_1$ and $A_2$ and $(s - 1)/2$ of the $4s$-cycles between $A_3$ and $A_4$. Note that this subgraph is connected even when $s = 1$. The unused edges from $K(2s, 2s, 2s, 2s)$ form the second, isomorphic closed trail of length $12s^2$.

*Case* III: Here $k$ is divisible by 8 but *not* divisible by 3. Let $k = 8s^2t$, where $s$ and $t$ are positive integers and $t$ is square-free. Then since $k|24m^2$, we have $m = stm'$ for some positive integer $m'$. Moreover, $s \leqslant m'$, with strict inequality for paths.

*Case* IIIa: We deal with the case $s = 1$ separately. A decomposition of both $K(2, 2, 2, 2)$ and $K(2m', 2m', 2m', 2m')$ into 8-cycles exists by Theorem 3.5. For paths, since $m' > 1$ and 8-cycles are bipartite, we can apply Corollary 2.4 to obtain a decomposition of $K(2m', 2m', 2m', 2m')$ into paths of length 8. The result then follows as in previous cases.

*Case* IIIb: Now suppose $s > 1$. Our aim is to decompose $K(2s, 2s, 2s, 2s)$ into three closed trails, each of which has length $8s^2$, is bipartite and has maximum degree $2s$. The result will then follow, using results in Section 2, as described in Case II.

From Theorem 3.1, the edges between each pair of partite sets may be decomposed into cycles of length $4s$. The first closed trail of length $8s^2$ is constructed from $(s - 1)/2$ of the $4s$-cycles between partite sets $A_1$ and $A_2$, $(s + 1)/2$ of the $4s$-cycles between partite sets $A_2$ and $A_3$, $(s - 1)/2$ of the $4s$-cycles between partite sets $A_3$ and $A_4$ and $(s + 1)/2$ of the $4s$-cycles between partite sets $A_4$ and $A_1$. The second closed trail of length $8s^2$ is constructed from $(s + 1)/2$ of the $4s$-cycles between partite sets $A_1$ and $A_2$, $(s - 1)/2$ of the $4s$-cycles between partite sets $A_2$ and $A_4$, $(s + 1)/2$ of the $4s$-cycles between partite sets $A_4$ and $A_3$ and $(s - 1)/2$ of the $4s$-cycles between partite sets $A_3$ and $A_1$. The third closed trail is made up of the remaining edges.

*Case* IV: Finally $k/4$ is odd and not divisible by 3. Let $k = 4s^2t$, where $s$ and $t$ are positive integers and $t$ is square-free. It follows that $m = stm'$ for some positive integer $m'$. Moreover, $s \leqslant 2m'$. In fact since $s$ is odd we must have $s < 2m'$.

Our aim is to decompose $K(2s, 2s, 2s, 2s)$ into six closed trails, each of which has length $4s^2$, is bipartite and near-regular (specifically, the difference between maximum and minimum degree must be 2). The result will then follow as in previous cases.

We first decompose the edges between each pair of partite sets into $4s$-cycles. Next, take a decomposition of $3K_4$ into six paths of length 3, such that the *middle* edges of each path form a copy of $K_4$. (Such a decomposition is straightforward to verify). We label the vertices of $K_4$ with 1, 2, 3 and 4. Then for each path $[i, j, k, l]$ in the decomposition of $3K_4$, we construct a closed trail of length $4s^2$ from $(s - 1)/2$ $4s$-cycles between partite sets $A_i$ and $A_j$, one $4s$-cycle between partite sets $A_j$ and $A_k$ and $(s - 1)/2$ $4s$-cycles between partite sets $A_k$ and $A_l$. □

When the cycle or path length is congruent to 2 (mod 4) we must construct our closed trails using some cycles of length $2s$.

**Theorem 3.8.** *Let $k$ be a positive even integer such that $k \equiv 2 \pmod 4$. Then the graph $K(2m, 2m, 2m, 2m)$ has an edge-disjoint decomposition into cycles (paths) of length $k$ if and only if $k|24m^2$ and $k \leqslant 8m$ ($k < 8m$).*

**Proof.** The necessity of the conditions is clear. Table 2 is a summary of the division of cases. Henceforth the partite sets of $K(2s, 2s, 2s, 2s)$ are labelled $A_1, A_2, A_3$ and $A_4$.

*Case* I: Here $k$ is divisible by 6 and $k/6$ is odd. Let $k = 6s^2t$, where $s$ and $t$ are odd positive integers and $t$ is square-free. Then since $k|24m^2$, we have $m = stm'$ for some positive integer $m'$. Moreover, $3s \leqslant 4m'$. In fact since $s$ is odd we must have $3s < 4m'$.

Our aim is to decompose $K(2s, 2s, 2s, 2s)$ into four closed trails, each of which has length $6s^2$, is bipartite and has maximum degree $2\lceil 3s/4\rceil \leqslant 2m'$. The result will then follow as in previous cases (in the theorem above).

**Table 3**
The numbers of $4s$-cycles between partite sets, $3s \equiv 1 \pmod 4$

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $T_1$ | $(s-1)/2$ | $(s-1)/2$ | 0 | 0 | $(s+1)/4$ | $(s+1)/4$ |
| $T_2$ | $(s-1)/2$ | $(s-1)/2$ | $(s+1)/4$ | $(s+1)/4$ | 0 | 0 |
| $T_3$ | 0 | 0 | $(3s-1)/4$ | $(3s-1)/4$ | 0 | 0 |
| $T_4$ | 0 | 0 | 0 | 0 | $(3s-1)/4$ | $(3s-1)/4$ |

**Table 4**
The numbers of $4s$-cycles between partite sets, $3s \equiv 3 \pmod 4$

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $T_1$ | $(s-3)/2$ | $(s-3)/2$ | 0 | 0 | $(s+3)/4$ | $(s+3)/4$ |
| $T_2$ | $(s-3)/2$ | $(s-3)/2$ | $(s+3)/4$ | $(s+3)/4$ | 0 | 0 |
| $T_3$ | 0 | 0 | $(3s-3)/4$ | $(3s-3)/4$ | 0 | 0 |
| $T_4$ | 0 | 0 | 0 | 0 | $(3s-3)/4$ | $(3s-3)/4$ |

From Theorem 3.6, the edges between partite sets $A_1$ and $A_2$ can be decomposed into one 2-factor (made up of two $2s$-cycles) and $s-1$ $4s$-cycles. We perform a similar decomposition of the edges between $A_3$ and $A_4$. The edges between all other pairs of partite sets are decomposed into $4s$-cycles.

*Case* Ia: Suppose firstly that $3s \equiv 1 \pmod 4$. Then $6s^2 = ((3s-1)/4) \times 8s + 2s$.

We first construct four disjoint *bipartite* $(3s-1)/2$-regular spanning subgraphs ($T_1$, $T_2$, $T_3$ and $T_4$) of $K(2s, 2s, 2s, 2s)$, omitting the four $2s$-cycles which are a 2-factor of $K(2s, 2s, 2s, 2s)$. The required decomposition will then follow by attaching the four $2s$-cycles to these subgraphs in any order to form bipartite, connected graphs.

Table 3 shows the number of $4s$-cycles used between each pair of partite set by each $T_i$, $1 \leqslant i \leqslant 4$.

*Case* Ib: Suppose next that $3s \equiv 3 \pmod 4$. We first deal with the case $s = 1$. Then $K(2, 2, 2, 2)$ decomposes into 6-cycles by Theorem 3.5. A decomposition of $K(2, 2, 2, 2)$ into paths of length 6 is also easily obtained: take a decomposition of $K_4$ into two paths of length 3, and then blow up points two-fold, increasing the path length by 2 at the same time; see Theorem 2.2.

Otherwise $s > 1$. Then we write $6s^2 = ((3s-3)/4) \times 8s + 6s$. We first construct four disjoint *bipartite* $(3s-3)/2$-regular spanning subgraphs ($T_1$, $T_2$, $T_3$ and $T_4$) of $K(2s, 2s, 2s, 2s)$, omitting a set of four $2s$-cycles which are a 2-factor of $K(2s, 2s, 2s, 2s)$ and also four $4s$-cycles. Each of these omitted cycles occurs either between $A_1$ and $A_2$ or between $A_3$ and $A_4$. We match these $2s$-cycles and $4s$-cycles in one-to-one correspondence to form four graphs, each on $6s$ edges with degree at most 2. The required decomposition will then follow by attaching these subgraphs in one-to-one correspondence with the $(3s-3)/2$-regular graphs to form bipartite, connected graphs.

Table 4 shows the number of $4s$-cycles used between each pair of partite set by each $T_i$, $1 \leqslant i \leqslant 4$.

*Case* II: Finally, we consider when $k/2$ is odd and not divisible by 3. Let $k = 2s^2t$, where $s$ and $t$ are odd positive integers and $t$ is square-free. Then since $k|24m^2$, we let $m = stm'$ for some positive integer $m'$. Moreover, $s \leqslant 4m'$. In fact since $s$ is odd we must have $s < 4m'$.

If $s = 1$, then $k > 2$ implies that $t > 1$. By Theorems 3.1 and 3.2, there exists a decomposition of $K(2m't, 2m't)$ into $2t$-cycles or $2t$-paths. Thus there exists a decomposition of $K(2m't, 2m't, 2m't, 2m't)$ into $2t$-cycles or $2t$-paths.

Otherwise $s \geqslant 5$. Our aim is to decompose $K(2s, 2s, 2s, 2s)$ into twelve closed trails, each of which has length $2s^2$, is bipartite and near-regular of maximum degree $(s+1)/2$ (specifically, the difference between maximum and minimum degree must be 2). The result will then follow as in previous cases. From Theorem 3.6, the edges between every pair of partite sets can be decomposed into one 2-factor (made up of two $2s$-cycles) and $s-1$ $4s$-cycles. Consider a decomposition of $6K_4$ into twelve paths of length 3, such that the *middle* edges of each path form the graph $2K_4$. (Such a decomposition exists by taking every possible 3-path within $K_4$.)

*Case* IIa: Suppose firstly that $s \equiv 1 \pmod 4$. Then we write $2s^2 = ((s-1)/4) \times 8s + 2s$.

Then for each path $[i, j, k, l]$ in the decomposition of $6K_4$, we construct a bipartite closed trail of length $2s^2$ from $(s-1)/4$ of the $4s$-cycles between partite sets $A_i$ and $A_j$, one $2s$-cycle between partite sets $A_j$ and $A_k$ and $(s-1)/4$ of the $4s$-cycles between partite sets $A_k$ and $A_l$.

*Case* IIb: Otherwise $s \equiv 3 \pmod 4$. We write $2s^2 = ((s-3)/4) \times 8s + 6s$.

Then for each path $[i, j, k, l]$ in $6K_4$, we construct a bipartite closed trail of length $2s^2$ from $(s-3)/4$ of the $4s$-cycles between partite sets $A_i$ and $A_j$, one $2s$-cycle between partite sets $A_j$ and $A_k$, $(s-3)/4$ of the $4s$-cycles between partite sets $A_k$ and $A_l$ and finally one $4s$-cycle between partite sets $A_l$ and $A_i$. □

**Theorem 3.9.** *Let $k$ be an odd integer divisible by 3. Let $s$ and $t$ be positive integers such that $k = 3s^2t$, where $t$ is square-free. Then the graph $K(2m, 2m, 2m, 2m)$ has an edge-disjoint decomposition into cycles or paths of length $k$ if and only if $st|m$ and $k \leqslant 8m$ (with strict inequality for paths).*

**Proof.** Let $m = m'st$, for some positive integer $m'$. Then $3s \leq 8m'$, with strict inequality for paths. We split our proof into four cases, depending on the congruency of $s$. In each of the following cases, the partite sets of $K(2s, 2s, 2s, 2s)$ are labelled

**Table 5**
The differences used, between even/odd vertices

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $H_1$ | 0, $(e, e)$ | | 1, $(e, o)$ | | | 1, $(e, o)$ |
| $H_2$ | 0, $(o, o)$ | | 1, $(o, e)$ | | | 1, $(o, e)$ |
| $H_3$ | | 1, $(e, o)$ | | 1, $(e, o)$ | | 0, $(e, e)$ |
| $H_4$ | | 1, $(o, e)$ | | 1, $(o, e)$ | | 0, $(o, o)$ |
| $H_5$ | | 0, $(e, e)$ | 0, $(e, e)$ | | 0, $(e, e)$ | |
| $H_6$ | | 0, $(o, o)$ | 0, $(o, o)$ | | 0, $(o, o)$ | |
| $H_7$ | 1, $(o, e)$ | | | 0, $(e, e)$ | 1, $(o, e)$ | |
| $H_8$ | 1, $(e, o)$ | | | 0, $(o, o)$ | 1, $(e, o)$ | |

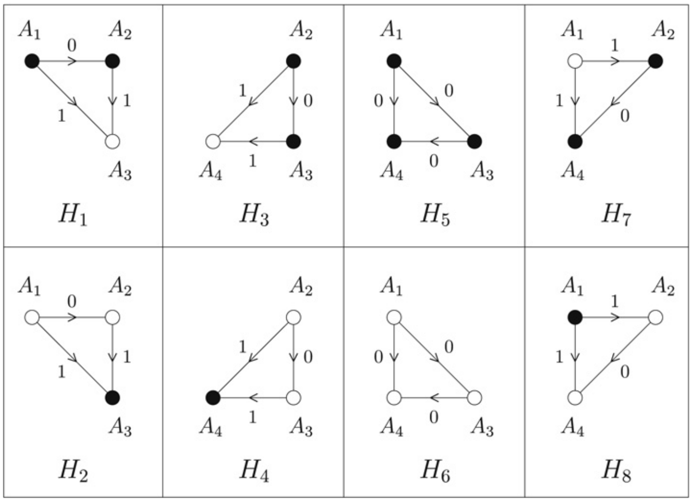


**Fig. 1.**

$A_1, A_2, A_3$ and $A_4$. We also label the vertices of $K(2s, 2s, 2s, 2s)$ with $\{1_i, 2_i, \ldots, (2s)_i \mid 1 \le i \le 4\}$, where subscript $i$ denotes a vertex belonging to $A_i$. We then define the *difference* of an edge $\{x_i, y_j\}$ to be $x - y \pmod{2s}$, where $i < j$ (that is,

$$(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}).$$

Observe that each difference between a pair of partite sets corresponds to a matching between those partite sets.

Our aim is always to decompose $K(2s, 2s, 2s, 2s)$ into eight bipartite or tripartite closed trails of length $3s^2$ with as regular degree as possible. Each trail of length $3s^2$ is essentially made up unions of matchings (or 1-factors) between pairs of partite sets. The result then follows as in previous theorems.

*Case* 1: $s \equiv 1 \pmod 8$. Let $s = 8s' + 1$. Then $s' \ge 0$. We write $3s^2 = 3s' \times 8s + 3s$. If $s = 1$, we can decompose $K(2, 2, 2, 2)$ into cycles of length 6 by Theorem 3.5. We then split each 6-cycle into two 3-paths. It is also easy to obtain a decomposition of $K(2, 2, 2, 2)$ into 3-cycles. Otherwise $s \geqslant 9$.

We first use all the edges of difference 0 and 1 between pairs of partite sets of $K(2s, 2s, 2s, 2s)$ to construct $8s$ triangles. These triangles, in turn, partition into eight sets of $s$ pairwise disjoint triangles: $H_1, H_2, \ldots H_8$. Table 5 shows which differences are used for each $H_i$, $1 \le i \le 8$. The brackets indicate whether odd or even vertices are used in each partite set. For example, 1, $(o, e)$ in column $A_i, A_j$ indicates that difference 1 is used between the odd vertices of $A_i$ and the even vertices of $A_j$. This is also illustrated in Fig. 1, where even vertices are coloured black and odd vertices are coloured white.

Note that two matchings corresponding to consecutive differences between partite sets give a cycle of length $4s$. In such a fashion we can decompose the remaining edges between each pair of partite sets into cycles of length $4s$. We now construct eight regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_8$, each of length $3s^2 - 3s$. Table 6 shows the number of $4s$-cycles used between each pair of partite sets for each $T_i$, $1 \leqslant i \leqslant 8$. Observe that the graphs $T_i \cup H_i$ are each connected and tripartite for $s' \ge 1$. Thus $T_i \cup H_i$, $1 \le i \le 8$, are the required closed trails of length $3s^2$.

*Case* 2: $s \equiv 3 \pmod 8$. Let $s = 8s' + 3$. Then $s' \ge 0$. We write $3s^2 = 3s' \times 8s + 9s$. We use the edges of difference 0 and 1 as in Case 1. Next, we use all the edges of differences 2, 3, 4 and 5 to obtain eight graphs, $J_1, J_2, \ldots, J_8$, each 2-regular on $6s$ vertices. The differences each $J_i$ uses are shown in Table 7. This is also illustrated in Fig. 2.

In the case $s = 3$ each $J_i$ is in fact a $6s$-cycle.

**Table 6**
The numbers of $4s$-cycles used

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $T_1, T_2$ | $2s'$ | $2s'$ | 0 | 0 | $s'$ | $s'$ |
| $T_3, T_4$ | $2s'$ | $2s'$ | $s'$ | $s'$ | 0 | 0 |
| $T_5, T_6$ | 0 | 0 | $3s'$ | $3s'$ | 0 | 0 |
| $T_7, T_8$ | 0 | 0 | 0 | 0 | $3s'$ | $3s'$ |

**Table 7**
The differences used

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $J_1$ | 4 | | 5 | | | 2 |
| $J_2$ | 5 | | 2 | | | 4 |
| $J_3$ | | 3 | | 5 | | 3 |
| $J_4$ | | 4 | | 4 | | 5 |
| $J_5$ | | 2 | 3 | | 4 | |
| $J_6$ | | 5 | 4 | | 2 | |
| $J_7$ | 2 | | | 2 | 3 | |
| $J_8$ | 3 | | | 3 | 5 | |

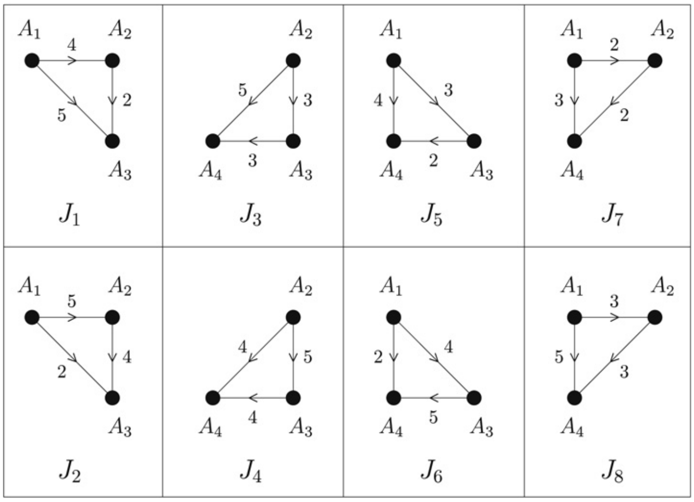


**Fig. 2.**

**Table 8**
The differences used

| | $A_1, A_2$ | $A_3, A_4$ |
|---|---|---|
| $J_1$ | 2, 4, $(o, o)$ | |
| $J_2$ | 2, 4, $(e, e)$ | |
| $J_3$ | | 3, 5, $(e, o)$ |
| $J_4$ | | 3, 5, $(o, e)$ |
| $J_5$ | | 2, 4, $(o, o)$ |
| $J_6$ | | 2, 4, $(e, e)$ |
| $J_7$ | 3, 5, $(e, o)$ | |
| $J_8$ | 3, 5, $(o, e)$ | |

Next, use any remaining differences to construct $4s$-cycles between pairs of partite sets, as in Case 1. We also construct eight regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_8$, each of length $3s^2 - 9s$. This is done also as in Case 1 (using $s' = (s-3)/8$ rather than $s' = (s-1)/8$). Finally, $T_i \cup H_i \cup J_i$ for each $i$, $1 \le i \le 8$ gives the required tripartite closed trails of length $3s^2$. (Note that in the case $s' = 0$, $T_i$ and $H_i$ are empty, but $J_i$ is still connected, for each $i$.)

*Case* 3: $s \equiv 7 \pmod 8$. Let $s = 8s' + 7$. Then $s' \ge 0$. We write $3s^2 = (3s' + 2) \times 8s + 5s$. We use the edges of difference 0 and 1 as in Case 1. Next, we use all the edges of differences 2, 3, 4 and 5 between partite sets $A_1$ and $A_2$ and also between partite sets $A_3$ and $A_4$, to obtain eight $2s$-cycles: $J_1, J_2, \ldots, J_8$, as follows. Table 8 shows which differences are used in each case. Brackets indicate whether odd or even vertices are used in each partite set. For example, 3, 5, $(0, e)$ in column $A_i, A_j$ indicates that the differences 3 and 5 are used between the odd vertices of $A_i$ and the even vertices of $A_j$.

**Table 9**
The numbers of 4*s*-cycles used

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $T_1, T_2$ | $2s'+1$ | $2s'+1$ | 0 | 0 | $s'+1$ | $s'+1$ |
| $T_3, T_4$ | $2s'+1$ | $2s'+1$ | $s'+1$ | $s'+1$ | 0 | 0 |
| $T_5, T_6$ | 0 | 0 | $3s'+2$ | $3s'+2$ | 0 | 0 |
| $T_7, T_8$ | 0 | 0 | 0 | 0 | $3s'+2$ | $3s'+2$ |

**Table 10**
The differences used

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|
| $J_1$ | 2, 4, $(o, o)$ | 3, 5, $(e, o)$ | | |
| $J_2$ | 2, 4, $(e, e)$ | 3, 5, $(o, e)$ | | |
| $J_3$ | | | 3, 5, $(o, e)$ | 2, 4, $(o, o)$ |
| $J_4$ | | | 3, 5, $(e, o)$ | 2, 4, $(e, e)$ |
| $J_5$ | | | 2, 4, $(o, o)$ | 3, 5, $(e, o)$ |
| $J_6$ | | | 2, 4, $(e, e)$ | 3, 5, $(o, e)$ |
| $J_7$ | 3, 5, $(e, o)$ | 2, 4, $(o, o)$ | | |
| $J_8$ | 3, 5, $(o, e)$ | 2, 4, $(e, e)$ | | |

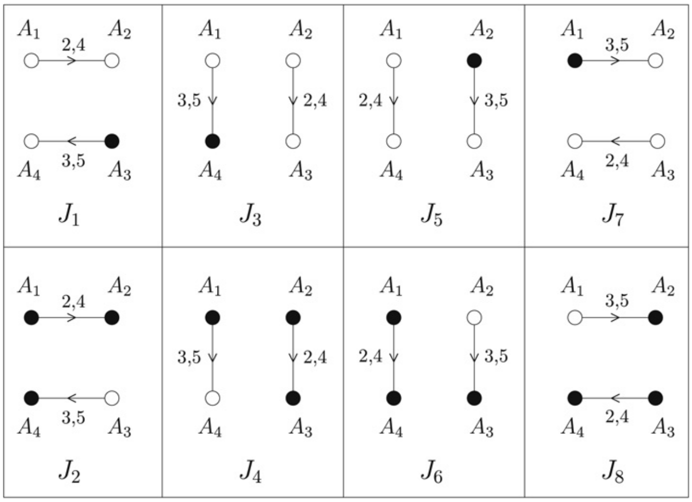


**Fig. 3.**

Next, use any remaining differences to construct 4*s*-cycles between pairs of partite sets, as in Case 1. We then construct eight regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_8$, each of length $3s^2 - 5s$. Table 9 shows the number of 4*s*-cycles used between each pair of partite sets for each $T_i$, $1 \leqslant i \leqslant 8$. Finally, $T_i \cup H_i \cup J_i$ for each $i$, $1 \le i \le 8$ gives the required tripartite closed trails of length $3s^2$.

*Case* 4: $s \equiv 5 \pmod 8$. Let $s = 8s' + 5$. Then $s' \ge 0$. Also $3s^2 = (3s'+1) \times 8s + 7s$. We use the edges of difference 0 and 1 as in Case 1. Next, we use all the edges of differences 2, 3, 4 and 5 between remaining pairs of partite sets (*except* between partite sets $A_1$ and $A_4$ and between partite sets $A_2$ and $A_3$) to obtain eight pairs of 2*s*-cycles: $J_1, J_2, \ldots, J_8$, as follows. Table 10 shows which differences are used in each case. The bracket notation is as in previous cases. Fig. 3 illustrates this, where even vertices are black and odd ones are white.

In the case $s = 3$ each $J_i$ is in fact a 6*s*-cycle.

Next, use any remaining differences to construct 4*s*-cycles between pairs of partite sets, as in Case 1. We then construct eight regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_8$, each of length $3s^2 - 7s$. Table 11 shows the number of 4*s*-cycles used between each pair of partite sets for each $T_i$, $1 \leqslant i \leqslant 8$. Finally, $T_i \cup H_i \cup J_i$ for each $i$, $1 \le i \le 8$ gives the required tripartite closed trails of length $3s^2$. □

The final theorem in this paper requires four special cases, which are dealt with in the following two lemmas.

**Table 11**
The numbers of 4s-cycles used

| | $A_1, A_2$ | $A_3, A_4$ | $A_1, A_3$ | $A_2, A_4$ | $A_1, A_4$ | $A_2, A_3$ |
|---|---|---|---|---|---|---|
| $T_1, T_2$ | $2s'$ | $2s'$ | 0 | 0 | $s'+1$ | $s'+1$ |
| $T_3, T_4$ | $2s'+1$ | $2s'+1$ | $s'$ | $s'$ | 0 | 0 |
| $T_5, T_6$ | 0 | 0 | $3s'+1$ | $3s'+1$ | 0 | 0 |
| $T_7, T_8$ | 0 | 0 | 0 | 0 | $3s'+1$ | $3s'+1$ |

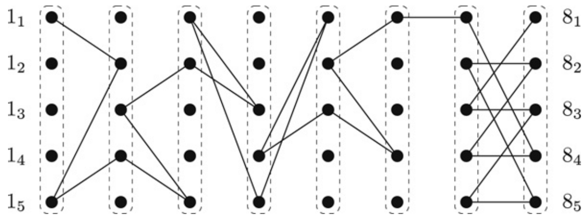


**Fig. 4.** A "starter" path of length 25 in $C_8 * \overline{K}_5$.

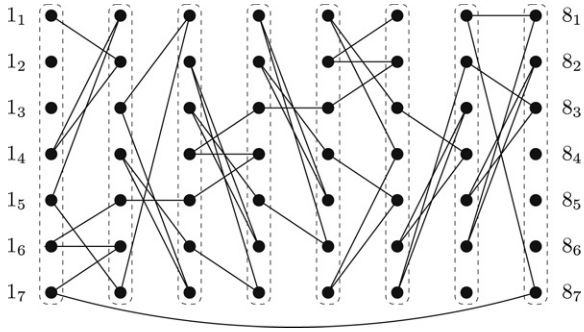


**Fig. 5.** A "starter" path of length 49 in $C_8 * \overline{K}_7$.

**Lemma 3.10.** *The graph* $K(10, 10, 10, 10)$ *decomposes into twenty-four paths, each of length* 25. *The graph* $K(14, 14, 14, 14)$ *decomposes into twenty-four paths, each of length* 49.

**Proof.** A careful examination of the path in Fig. 4 shows that for each $j, k$ with $1 \leqslant j, k \leqslant 5$, there exists $i$ with $1 \leqslant i \leqslant 8$ such that there is an edge joining $i_j$ to $(i+1)_k$. Thus if we take 8 copies of this path by incrementing the partite set labels modulo 8 (using residues $1, 2, \ldots, 8$), every edge between adjacent partite sets is covered, and we obtain a decomposition of $C_8 * \overline{K}_5$ into eight paths of length 25.

Similarly, Fig. 5 illustrates the starter path for a cyclic decomposition of $C_8 * \overline{K}_7$ into paths of length 49. Then, noting that $K(2, 2, 2, 2)$ decomposes into three 8-cycles (see Theorem 3.5), the results follow. □

**Lemma 3.11.** *The graph* $K(10, 10, 10, 10)$ *decomposes into twenty-four cycles, each of length* 25. *The graph* $K(14, 14, 14, 14)$ *decomposes into twenty-four cycles, each of length* 49.

**Proof.** For the first part, it suffices to give a decomposition of $5K(2, 2, 2, 2)$ into 5-cycles which satisfies the conditions of Lemma 2.5. We label the vertices of $5K(2, 2, 2, 2)$ with $\infty_1, \infty_2, 1_1, 1_2, 2_1, 2_2, 3_1$ and $3_2$. Below are eight 5-cycles, listed in rows so that the cycles in the same row share two consecutive edges.

$$\begin{array}{ll} (1_1, 2_1, 1_2, \infty_1, 2_2) & (1_1, 2_1, 1_2, 2_2, \infty_2) \\ (\infty_1, 2_1, \infty_2, 2_2, 1_1) & (\infty_1, 2_1, \infty_2, 1_2, 2_2) \\ (2_2, 3_2, 2_1, 1_2, \infty_2) & (2_2, 3_2, 2_1, 1_1, \infty_1) \\ (2_2, 3_1, 2_1, \infty_1, 1_2) & (2_2, 3_1, 2_1, \infty_2, 1_1). \end{array}$$

The required twenty-four 5-cycles are formed by replacing each vertex $i_j$ with vertex $(i+1)_j$, where $i+1$ is calculated modulo 3 (using residues 1, 2 and 3) with $\infty$ fixed.

For the second part, it suffices to give a decomposition of $7K(2, 2, 2, 2)$ into 7-cycles which satisfies the conditions of Lemma 2.5. Using the same vertex-labelling as above, we list eight 7-cycles below, listed in rows so that the cycles in the

**Table 12**
The differences used

| | $A_2, A_3$ | $A_1, A_4$ | $A_1, A_2$ | $A_3, A_4$ | $A_2, A_4$ | $A_1, A_3$ |
|---|---|---|---|---|---|---|
| $H_1$ | $0, (e, e)$ | | $-3, (e, o)$ | | | $-3, (e, o)$ |
| $H_2$ | $-2, (e, e)$ | | $+1, (e, o)$ | | | $-1, (e, o)$ |
| $H_3$ | $+2, (e, e)$ | | $-1, (e, o)$ | | | $+1, (e, o)$ |
| $H_4$ | | $-3, (e, o)$ | | $-3, (e, o)$ | | $0, (e, e)$ |
| $H_5$ | | $-1, (e, o)$ | | $+1, (e, o)$ | | $-2, (e, e)$ |
| $H_6$ | | $+1, (e, o)$ | | $-1, (e, o)$ | | $+2, (e, e)$ |
| $H_7$ | | $0, (e, e)$ | $-2, (e, e)$ | | $+2, (e, e)$ | |
| $H_8$ | | $-2, (e, e)$ | $0, (e, e)$ | | $-2, (e, e)$ | |
| $H_9$ | | $+2, (e, e)$ | $+2, (e, e)$ | | $0, (e, e)$ | |
| $H_{10}$ | $+1, (e, o)$ | | | $0, (o, o)$ | $+1, (e, o)$ | |
| $H_{11}$ | $-1, (e, o)$ | | | $-2, (o, o)$ | $-3, (e, o)$ | |
| $H_{12}$ | $-3, (e, o)$ | | | $+2, (o, o)$ | $-1, (e, o)$ | |

**Table 13**
The numbers of 4$s$-cycles used

| | $A_2, A_3$ | $A_1, A_4$ | $A_1, A_2$ | $A_3, A_4$ | $A_2, A_4$ | $A_1, A_3$ |
|---|---|---|---|---|---|---|
| $T_1, T_2, \ldots, T_8$ | $s'$ | $s'$ | 0 | 0 | 0 | 0 |
| $T_9, T_{10}, \ldots, T_{16}$ | 0 | 0 | $s'$ | $s'$ | 0 | 0 |
| $T_{17}, T_{18}, \ldots, T_{24}$ | 0 | 0 | 0 | 0 | $s'$ | $s'$ |

same row share two common edges (shown as underlined pairs of vertices).

$$\begin{array}{ll}
(\underline{2_1, 1_2}, 3_2, \underline{1_1, 2_2}, \infty_2, 3_1) & (\underline{2_1, 1_2}, \infty_2, \underline{1_1, 2_2}, 3_1, \infty_1) \\
(\underline{2_1, 1_1}, \infty_1, \underline{1_2, 2_2}, 3_2, \infty_2) & (\underline{2_1, 1_1}, 3_1, \underline{1_2, 2_2}, \infty_1, 3_2) \\
(\underline{\infty_1, 1_2}, 3_2, \underline{1_1, \infty_2}, 2_2, 3_1) & (\underline{\infty_1, 1_2}, 2_2, \underline{1_1, \infty_2}, 3_1, 2_1) \\
(\underline{\infty_1, 1_1}, 2_1, \underline{1_2, \infty_2}, 3_2, 2_2) & (\underline{\infty_1, 1_1}, 3_1, \underline{1_2, \infty_2}, 2_1, 3_2).
\end{array}$$

The required twenty-four 7-cycles are formed by replacing each vertex $i_j$ with vertex $(i + 1)_j$, where $i + 1$ is calculated modulo 3 (using residues 1, 2 and 3) with $\infty$ fixed. □

**Theorem 3.12.** *Let $k$ be an odd integer not divisible by* 3. *Let $s$ and $t$ be positive integers such that $k = s^2 t$, where $t$ is square-free. Then the graph $K(2m, 2m, 2m, 2m)$ has an edge-disjoint decomposition into cycles (paths) of length $k$ if and only if $st|m$ and $k \le 8m$ (with strict inequality for paths).*

**Proof.** We first deal with the case $s = 1$. Then $t \ge 3$. In fact, since $k$ is not divisible by 3, $t \ge 5$. Since $t$ divides $m$, we can decompose the complete bipartite graph $K(2m, 2m)$ (and thus $K(2m, 2m, 2m, 2m)$) into paths of length $t$ by Theorem 3.2. For cycles, we begin with a decomposition of $K(2, 2, 2, 2)$ into triangles. We replace each triangle with the complete tripartite graph $K(t, t, t)$ to obtain the graph $K(2t, 2t, 2t, 2t)$. However, each copy of $K(t, t, t)$ decomposes into $t$-cycles by Theorem 3.3. Finally we can decompose $K(2m, 2m, 2m, 2m)$ into $t$-cycles by Theorem 2.1(i).

The cases $s = 3$ and $s = 9$ do not arise because $k$ is not divisible by 3. The cases $s = 5$ and $s = 7$ are done in Lemmas 3.10 and 3.11. Thus we henceforth assume that $s \ge 11$.

Let $m = m'st$, for some positive integer $m'$. Then $s \le 8m'$, with strict inequality for paths. We split our proof into four cases, depending on the congruency of $s$ modulo 8. In each of the following cases, the partite sets of $K(2s, 2s, 2s, 2s)$ are labelled $A_1, A_2, A_3$ and $A_4$. We also label the vertices of $K(2s, 2s, 2s, 2s)$ with $\{1_i, 2_i, \ldots, (2s)_i \mid 1 \le i \le 4\}$, where subscript $i$ denotes a vertex belonging to $A_i$. We then define the *difference* of an edge $\{x_i, y_j\}$ to be $x - y \pmod{2s}$, where $i < j$. Observe that each difference between a pair of partite sets corresponds to a matching between those partite sets.

Our aim is always to decompose $K(2s, 2s, 2s, 2s)$ into twenty-four bipartite or tripartite closed trails of length $s^2$ with as regular degree as possible. Each trail of length $s^2$ is essentially made up unions of matchings (or 1-factors) between pairs of partite sets. The result then follows as in previous lemmas.

*Case* 1: $s \equiv 3 \pmod 8$. Let $s = 8s' + 3$. Then $s' \ge 1$. We write $s^2 = s' \times 8s + 3s$.

We first use all the edges of differences $-3, -2, -1, 0, 1$ and 2 between pairs of partite sets of $K(2s, 2s, 2s, 2s)$ to construct $24s$ triangles. These triangles, in turn, partition into twenty-four sets of $s$ pairwise disjoint triangles: $H_1, H_2, \ldots H_{24}$. Table 12 shows which differences are used for each $H_i$, $1 \le i \le 12$. The brackets indicate whether odd or even vertices are used in each partite set. For example, 1, $(o, e)$ in column $A_i, A_j$ indicates that difference 1 is used between the odd vertices of $A_i$ and the even vertices of $A_j$. For each $i$, $1 \le i \le 12$, $H_{12+i}$ is obtained from $H_i$ by switching the parities of all vertices.

Note that two matchings corresponding to consecutive differences between partite sets give a cycle of length $4s$. In such a fashion we can decompose the remaining edges between each pair of partite sets into cycles of length $4s$. We now construct twenty-four regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_{24}$, each of length $s^2 - 3s$. Table 13 shows the number of $4s$-cycles used between each pair of partite sets for each $T_i$, $1 \leqslant i \leqslant 8$. Finally, combining $T_i$ with $H_i$ for each $i$, $1 \le i \le 8$, gives the required tripartite closed trails of length $s^2$.

**Table 14**
The differences used

| | $A_2, A_3$ | $A_1, A_4$ | $A_1, A_2$ | $A_3, A_4$ | $A_2, A_4$ | $A_1, A_3$ |
|---|---|---|---|---|---|---|
| $J_1$ | 4, 6, $(o, o)$ | | | | | |
| $J_2$ | | | | 4, 6, $(o, o)$ | | |
| $J_3$ | | 4, 6, $(o, o)$ | | | | |
| $J_4$ | | | | | | 4, 6, $(o, o)$ |
| $J_5$ | | | 4, 6, $(o, o)$ | | | |
| $J_6$ | | | | | 4, 6, $(e, e)$ | |
| $J_7$ | 3, 5, $(o, e)$ | | | | | |
| $J_8$ | | | | 3, 5, $(o, e)$ | | |
| $J_9$ | | | | | | 3, 5, $(o, e)$ |
| $J_{10}$ | | 3, 5, $(o, e)$ | | | | |
| $J_{11}$ | | | | | 3, 5, $(o, e)$ | |
| $J_{12}$ | | | 3, 5, $(o, e)$ | | | |

**Table 15**
The differences used

| | $A_2, A_3$ | $A_1, A_4$ | $A_1, A_2$ | $A_3, A_4$ | $A_2, A_4$ | $A_1, A_3$ |
|---|---|---|---|---|---|---|
| $J_1$ | 4, 6, $(o, o)$ | 7, 9, $(e, o)$ | | | | |
| $J_2$ | | | 7, 9, $(o, e)$ | 4, 6, $(o, o)$ | | |
| $J_3$ | 8, 10, $(o, o)$ | 4, 6, $(o, o)$ | | | | |
| $J_4$ | | | | | 7, 9, $(o, e)$ | 4, 6, $(o, o)$ |
| $J_5$ | | | 4, 6, $(o, o)$ | 7, 9, $(o, e)$ | | |
| $J_6$ | | | | | 4, 6, $(e, e)$ | 8, 10, $(o, o)$ |
| $J_7$ | 3, 5, $(o, e)$ | 8, 10, $(o, o)$ | | | | |
| $J_8$ | | | 8, 10, $(o, o)$ | 3, 5, $(o, e)$ | | |
| $J_9$ | | | | | 8, 10, $(o, o)$ | 3, 5, $(o, e)$ |
| $J_{10}$ | 7, 9, $(o, e)$ | 3, 5, $(o, e)$ | | | | |
| $J_{11}$ | | | | | 3, 5, $(o, e)$ | 7, 9, $(e, o)$ |
| $J_{12}$ | | | 3, 5, $(o, e)$ | 8, 10, $(e, e)$ | | |

*Case* 2: $s \equiv 1 \pmod 8$. Let $s = 8s' + 1$. Then $s' \geq 2$. We write $s^2 = (s' - 1) \times 8s + 9s$. We first use all the edges of differences $-3, -2, -1, 0, 1$ and 2 between pairs of partite sets as in Case 1. Next, we use all the edges of differences 3, 4, 5, 6, 7 and 8 as follows. We construct twenty-four graphs, $J_i$ $(1 \leq i \leq 24)$, each on $6s$ edges and 2-regular. We do this in such a way that each $J_i$ uses three pairs of partite sets, and between each pair of partite sets the edges correspond to precisely one difference. Moreover, we ensure that each $J_i$ uses the same three partite sets as the corresponding graph $H_i$.

We next construct twenty-four regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_{24}$, each of length $s^2 - 9s$, in the same manner as in the previous case. The required tripartite closed trails are then precisely $T_i \cup H_i \cup J_i$, $1 \leq i \leq 24$.

*Case* 3: $s \equiv 5 \pmod 8$. Let $s = 8s' + 5$. Then $s' \geq 1$. We write $s^2 = s' \times 8s + 5s$. We first use all the edges of differences $-3, -2, -1, 0, 1$ and 2 between pairs of partite sets of $K(2s, 2s, 2s, 2s)$, as in Case 1. Next, we use all the edges of differences 3, 4, 5 and 6 as follows. We construct twenty-four $2s$-cycles: $J_i$, $1 \leq i \leq 24$. Table 14 shows which differences are used for each $2s$-cycle, and also the parity of vertices used. (The bracket notation is defined in Case 1). For each $i$, $1 \leq i \leq 24$, $J_{12+i}$ is obtained from $J_i$ by changing the parities of vertices. We next construct twenty-four regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_{24}$, each of length $s^2 - 5s$, in the same manner as in previous cases. The required tripartite closed trails are then precisely $T_i \cup H_i \cup J_i$, $1 \leq i \leq 24$.

*Case* 4: $s \equiv 7 \pmod 8$. Let $s = 8s' + 7$. Then $s' \geq 2$. We write $s^2 = s' \times 8s + 7s$. We first use all the edges of differences $-3, -2, -1, 0, 1$ and 2 between pairs of partite sets of $K(2s, 2s, 2s, 2s)$, as in Case 1. Next, we use all the edges of differences 3, 4, 5, 6, 7, 8, 9 and 10 as follows. We construct twenty-four pairs of $2s$-cycles: $J_i$, $1 \leq i \leq 24$. (The $2s$-cycles in each pair will be vertex-disjoint.) Table 15 shows which differences are used for each $J_i$, and also the parity of vertices used. For each $i$, $1 \leq i \leq 24$, $J_{12+i}$ is obtained from $J_i$ by changing the parities of vertices. We next construct twenty-four regular, bipartite spanning subgraphs: $T_1, T_2, \ldots, T_{24}$, each of length $s^2 - 7s$, in the same manner as in previous cases. The required tripartite closed trails are then precisely $T_i \cup H_i \cup J_i$, $1 \leq i \leq 24$. □

## 4. Concluding comments

Combining the results in the previous section, we have shown the following.

**Theorem 4.1.** *The complete equipartite graph $K(m, m, m, m)$ has an edge-disjoint decomposition into cycles of length k if and only if m is even, the number of edges in the graph,* $6m^2$*, is divisible by k, and* $k \leqslant 4m$*.*

*The complete equipartite graph $K(2m, 2m, 2m, 2m)$ having four even sized parts has an edge-disjoint decomposition into paths of even length k if and only if the number of edges in the graph,* $6m^2$*, is divisible by k, and* $k < 8m$*.*

If we turn our attention to the four-partite case where the part size, say $m$, is odd, a necessary condition for an edge-disjoint decomposition into paths of length $k$ is that twice the number of paths must be at least as large as the number of vertices (since every vertex must be the end of an odd number of paths, and each path has two ends!). So for the graph $K(m, m, m, m)$ when $m$ is odd, besides $k|6m^2$, a further necessary condition is that $k \leqslant 3m$. (There are $6m^2/k$ paths, and $4m$ vertices, so there are $12m^2/k$ path ends, and this must be at least $4m$.)

We conjecture that these necessary conditions are sufficient in this case with four odd partite sets, and have some results to support this. However, the techniques in this paper and its partner paper [2], if applied to a larger number of parts, lead to an unmanageable number of cases; more general techniques will be required.

## References


[1] B.R. Alspach, H.J. Gavlas, Cycle decompositions of $K_n$ and $K_n - I$, J. Combin. Theory Ser. B 81 (2001) 77–99.

[2] E.J. Billington, N.J. Cavenagh, B.R. Smith, Path and cycle decompositions of complete equipartite graphs: 3 and 5 parts, Discrete Math., in press (doi:10.1016/j.disc.2008.08.009).

[3] E.J. Billington, D.G. Hoffman, Short path decompositions of arbitrary complete multipartite graphs, Congr. Numer. 187 (2007) 161–173.

[4] N.J. Cavenagh, Decompositions of complete tripartite graphs into $k$-cycles, Australas. J. Combin. 18 (1998) 193–200.

[5] N.J. Cavenagh, E.J. Billington, Decompositions of complete multipartite graphs into cycles of even length, Graphs Combin. 16 (2000) 49–65.

[6] D.G. Hoffman, Private communication.

[7] C.D. Leach, C.A. Rodger, Non-disconnecting disentanglements of amalgamated 2-factorizations of complete multipartite graphs, J. Combin. Des. 9 (2001) 460–467.

[8] J. Liu, A generalization of the Oberwolfach problem and $C_t$-factorizations of complete equipartite graphs, J. Combin. Des. 8 (2000) 42–49.

[9] J. Liu, The equipartite Oberwolfach problem with uniform tables, J. Combin. Theory Series A 101 (2003) 20–34.

[10] R.S. Manikandan, P. Paulraja, $C_p$-decompositions of some regular graphs, Discrete Math. 306 (2006) 429–451.

[11] C.A. Parker, Complete bipartite graph path decompositions, Ph.D. Thesis, Auburn University, 1998.

[12] M. Šajna, Cycle decompositions III: Complete graphs and fixed length cycles, J. Combin. Des. 10 (2002) 27–78.

[13] B.R. Smith, Decomposing complete equipartite graphs into cycles of length $2p$, J. Combin. Des. 16 (2008) 244–252.

[14] D. Sotteau, Decomposition of $K_{m,n}(K^*_{m,n})$ into cycles (circuits) of length $2k$, J. Combin. Theory Ser. B 30 (1981) 75–81.

[15] M. Tarsi, Decomposition of a complete multigraph into simple paths: nonbalanced handcuffed designs, J. Combin. Theory Ser. A 34 (1983) 60–70.