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Planar graphs without 5-cycles or without 6-cycles[☆]

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ABSTRACT

Let G be a planar graph without 5-cycles or without 6-cycles. In this paper, we prove that if G is connected and $\delta(G) \geq 2$, then there exists an edge $xy \in E(G)$ such that $d(x)+d(y) \leq 9$, or there is a 2-alternating cycle. By using the above result, we obtain that (1) its linear 2-arboricity $la_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$, (2) its list total chromatic number is $\Delta(G)+1$ if $\Delta(G) \geq 8$, and (3) its list edge chromatic number is $\Delta(G)$ if $\Delta(G) \geq 8$.

1. Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [3]. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Given a graph $G = (V, E)$, let $N(v) = \{u \mid uv \in E(G)\}$, $d(v) = |N(v)|$ is the *degree* of the vertex v, $N_k(v) = \{u \mid u \in N(v) \text{ and } d(u) = k\}$, and $n_k(v) = |N_k(v)|$. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum (vertex) degree and the minimum (vertex) degree, respectively. A k-, k^+- or k^--vertex is a vertex of degree k, at least k or at most k, respectively. For $s \geq 2$, an even cycle $C = v_1v_2 \cdots v_{2s}v_1$ is called a 2-*alternating cycle* if $d(v_1) = d(v_3) = \cdots d(v_{2s-1}) = 2$.

An *edge-partition* of a graph G is a decomposition of G into subgraphs $G_1, G_2, \ldots, G_m$ such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_m)$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. A *linear k-forest* is a graph whose components are paths of length at most k. The *linear k-arboricity* of G, denoted by $la_k(G)$, is the least integer m such that G can be *edge-partitioned* into m linear k-forests. The case $la_1(G)$ is the edge chromatic number χ' of G.

The linear k-arboricity of a graph was first introduced by Habib and Péroche [9]. They posed the following conjecture.

Conjecture A. *For a graph G of order n and a positive integer i,*

$$la_i(G) \leq \begin{cases} \left\lceil (\Delta n + 1)/2 \left\lfloor \dfrac{in}{i+1} \right\rfloor \right\rceil & \text{if } \Delta \neq n-1, \\ \left\lceil (\Delta n)/2 \left\lfloor \dfrac{in}{i+1} \right\rfloor \right\rceil & \text{if } \Delta = n-1. \end{cases}$$

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The linear k-arboricity of cycles, trees, complete graphs, and complete bipartite graphs has been determined in [7,8]. Thomassen [15] proved that $la_k(G) \leq 2$ for a cubic graph G, where $k \geq 5$, and this result is the best possible. Chang [5] and Chang et al. [6] investigated the algorithmic aspects of the linear k-arboricity. It was further studied by Bermond et al. [2], Jackson and Wormald [11], and Aldred and Wormald [1]. Lih, Tong, and Wang [13] proved that for a planar graph G, we have $la_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 12$; moreover, $la_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$ if G does not contain 3-cycles. Qian and Wang [14] proved that for a planar graph G without 4-cycles, $la_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 3$. In this paper, we will prove that for a planar graph G without 5-cycles or without 6-cycles, $la_2(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil + 6$.

A *proper total coloring* of a graph G is a coloring of $V(G) \cup E(G)$ such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ is the smallest number of colors such that G has a proper total coloring. A graph G is said to be *totally f-choosable* if, whenever we give lists of $f(x)$ colors to each element $x \in V(G) \cup E(G)$, there exists a proper total coloring of G where each element is colored with a color from its own list. If $f(x) = k$ for every element $x \in V(G) \cup E(G)$, we say G is *totally k-choosable*. The *list total chromatic number* $\chi''_{\text{list}}(G)$ is the smallest integer k such that G is totally k-choosable. The *list edge chromatic number* $\chi'_{\text{list}}(G)$ of G is defined similarly in terms of coloring edges alone. Obviously, $\chi'_{\text{list}}(G) \geq \chi'(G) \geq \Delta(G)$ and $\chi''_{\text{list}}(G) \geq \chi''(G) \geq \Delta(G) + 1$.

Conjecture B. *For any graph G, (a)* $\chi'_{\text{list}}(G) = \chi'(G)$ *and (b)* $\chi''_{\text{list}}(G) = \chi''(G)$.

Part (a) of Conjecture B was posed independently by Vizing, by Gupta, by Abertson and Collins, and by Bollobás and Harris (see [4]), and is well-known as the List Coloring Conjecture. Part (b) of the conjecture was posed by Borodin, Kostochka and Woodall [4]. Both parts of this conjecture are still very much open. For a planar graph G, it is proved that $\chi'_{\text{list}}(G) = \chi'(G) = \Delta(G)$ and $\chi''_{\text{list}}(G) = \chi''(G) = \Delta(G) + 1$ if $\Delta(G) \geq 12$ [4], or $\Delta(G) \geq 7$ and G does not contain 3-cycles [4], or $\Delta(G) \geq 7$ and G does not contain 4-cycles [10]. In the paper, we will prove both these results if G is a planar graph with maximum degree at least 8 and without 5-cycles or without 6-cycles.

In the next section, we will prove that if G is a connected planar graph with $\delta(G) \geq 2$ and without 5-cycles or without 6-cycles, then there exists an edge $xy \in E(G)$ such that $d(x) + d(y) \leq 9$, or there exists a 2-alternating cycle. In Section 3, we will use the above result to prove our main results.

2. Planar graphs without 5- or without 6-cycles

In the section, all graphs are planar graphs which have been embedded in the plane. For a planar graph G, the degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut edge is counted twice. A k-, k^+- or k^--face is a face of degree k, at least k or at most k, respectively. For a face f of G, let $n_i(f)$ denote the number of the i-vertices on the boundary of f. For $v \in V(G)$, we use $f_i(v)$ to denote the number of i-faces incident with v. A 2-vertex in G is called *improper* if it is incident with a 3-face. Let $S_2(v)$ be the number of 2-vertices any of which is adjacent to v and is incident with a 3-face and a 4-face.

First, let us prove some structural properties for the graphs without 5-cycles.

Lemma 1. *Let G be a planar graph without* 5*-cycles and* $\delta(G) \geq 2$. *If* $d(x) + d(y) \geq 10$ *for any edge* $xy \in E(G)$, *and there are no* 2*-alternating cycles, then all of the following results hold.*

(a) *Any vertex* v *is incident with at most* $\lfloor \frac{2d(v)}{3} \rfloor$ 3*-faces.*
(b) *A* 3*-face is adjacent to a* 4*-face if and only if the two faces are incident with a common* 2*-vertex.*
(c) *If a face is adjacent to two nonadjacent* 3*-faces then the face must be a* 6^+*-face.*
(d) *For any vertex* v, *if* $d(v) \geq 7$ *and* v *is incident with a* 3*-face, then* v *is incident with at most* $d(v) - 2$ *faces of degree at most* 4.

Proof. Since if there are three 3-faces f_1, f_2, f_3 such that they are incident with a common vertex and f_2 is incident with f_1 and f_3, then vertices incident with them form a 5-cycle, so (a) holds. If a 3-face is incident with a 4-face, then all three vertices incident with the 3-face f must be incident with the 4-face, too. So there is a vertex just incident with these two faces and it follows that the vertex is a 2-vertex. Hence, (b) holds. For (c), suppose that a face f is adjacent to two nonadjacent 3-faces. It is obvious that f is not a 3-face for otherwise a 5-cycle appears. By (b), f is not a 4-face. So f must be a 6^+-face and (c) holds.

For (d), suppose that $d(v) \geq 7$ and v is incident with a 3-face. If v is a cut vertex, then (d) is obvious. So assume that v is not a cut vertex. Let $f_1, f_2, \ldots, f_d$ be faces incident with v in a clockwise order, and $v_1, v_2, \ldots, v_d$ be vertices incident with v, where v_i is incident with f_i, f_{i+1}, $i = 1, 2, \ldots, d-1$, and v_d is incident with f_d and f_1. Assume that f_1 is the 3-face. Then by (a), f_2 or f_d is not a 3-face. Without loss of generality, assume that f_d is not a 3-face.

Suppose that f_d is a 4-face. Then $d(v_d) = 2$ by (b). Thus f_2 must be a 3-face or a 6^+-face. If f_2 is a 3-face, then f_3 must be a 6^+-face. So one of f_2 and f_3 is a 6^+-face. Similarly, by (c), f_{d-1} must be a 4-face or a 6^+-face. If f_{d-1} is a 4-face, then $C = vv_dv_1v_{d-1}v$ is a 2-alternating cycle. Hence, one of f_d and f_{d-1} is a 6^+-face.

Suppose that f_d is a 6^+-face. If f_2 is a 3-face, then f_3 must be a 4-face or 6^+-face. If f_3 is a 4-face, then $d(v_2) = 2$ and $d(v_3) \neq 2$ by (b). So f_4 must be a 6^+-face. If f_2 is a 4-face, then f_3 must be a 4-face or a 6^+-face by (c). If f_3 is a 4-face, then $C = vv_1v_dv_2v$ is a 2-alternating cycle. Thus we have $\max\{d(f_2), d(f_3), d(f_4)\} \geq 6$. The proof of (d) is completed. □

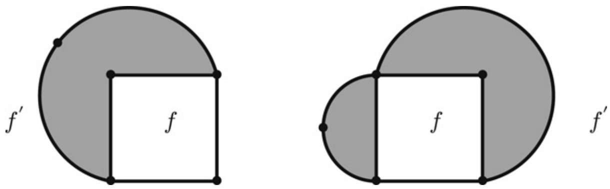

Fig. 1. $d(f) = d(f') = 4$ and the other vertices and edges of G are in the shaded regions.

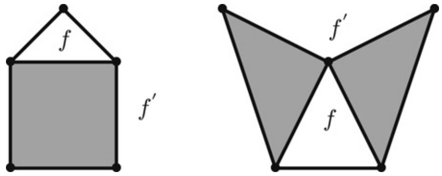

Fig. 2. $d(f) = 3, d(f') = 5$, and the other vertices and edges of G are in the shaded regions.

Lemma 2. *Let G be a 2-connected planar graph without 6-cycles. Then the following two results hold.*

(a) *Two 4-faces, f and f', are adjacent if and only if they are isomorphic to one of the configurations in Fig. 1.*
(b) *A 3-face f is adjacent to a 5-face f' if and only if they are isomorphic to one of the configurations in Fig. 2.*

Lemma 3. *Let G be a 2-connected planar graph without 6-cycles, and $d(x) + d(y) \geq 10$ for any edge $xy \in E(G)$, and there are no 2-alternating cycles in G. Let v be a vertex with $d(v) = d \geq 5$, let $f_1, f_2, \ldots, f_d$ be the faces incident with v in a clockwise order, and $v_1, v_2, \ldots, v_d$ be neighbors of v, where v_i is incident with $f_i, f_{i+1}, i = 1, 2, \ldots, d-1$, and v_d is incident with f_d and f_1. Then all of the following statements hold.*

(a) *If $d(f_1) = d(f_2) = 4$, then $d(f_d) \neq 4$, $d(v_d) > 2$ and there is at most one 3-face in $\{f_3, f_d\}$. Moreover if $d(f_3) = 4$, then $d(v_1) > 2$ and $d(v_2) = 2$.*

(b) *If $d(f_1) = 3$ and $d(f_2) = 5$, then $d(f_d) \neq 4, 5, 6$ and $d(v_d) > 2$. Moreover if $d(f_3) = 4$, then $d(v_2) = 2$.*

(c) *If $d(f_1) = d(f_2) = d(f_3) = 3$, then $\min\{d(f_d), d(f_4)\} \geq 4$, f_4 and f_d are not 5-faces. This implies that v is incident with at most $\lfloor \frac{3d(v)}{4} \rfloor$ 3-faces. Moreover if f_4 is a 4-face, then v_3 must be a 2-vertex and $d(v_4) > 2$. Similarly, if $d(f_d) = 4$, then $d(v_d) = 2$ and $d(v_{d-1}) > 2$.*

(d) *If $d(f_1) = d(f_2) = 3$ and $\min\{d(f_3), d(f_d)\} \geq 4$, then both f_3 and f_d cannot simultaneously be 4-cycles or 5-cycles. Moreover if $\max\{d(f_3), d(f_d)\} \leq 5$, then $\min\{d(v_2), d(v_d)\} = 2$ and $\min\{d(v_3), d(v_{d-1})\} > 2$.*

(e) *Suppose that $d(f_1) = d(f_3) = 3$ and $d(f_2) \geq 4$. Then $d(f_2) = 5$ if and only if $d(v_1) = d(v_2) = 2$ and $v_d v_3 \in E(G)$, and $d(f_2) = 4$ if and only if there is just one 2-vertex in $\{v_1, v_2\}$ and $v_1 v_3 \in E(G)$.*

(f) *Suppose that $d(f_1) = d(f_4) = 3$ and $\min\{d(f_2), d(f_3)\} \geq 4$. Then $\max\{d(f_2), d(f_3)\} \geq 5$; moreover if $\min\{d(v_1), d(v_2)\} \geq 3$, then $\max\{d(f_2), d(f_3)\} \geq 7$.*

(g) *If $d(v) \geq 7$ and v is incident with a 3-face, then v is incident with at most $d(v)-2$ 4^--faces; moreover, if $f_{4^-}(v) = d(v)-2$ and $f_3(v) \geq f_4(v)$, then $f_{7^+}(v) = 2$.*

(h) *If $d(v) \geq 8$ and $f_3(v) + S_2(v) > \lfloor \frac{d(v)}{2} \rfloor$, then $f_{7^+}(v) \geq 2$.*

Proof. (a) and (b) are obvious by Lemma 2. And it is easy to check (c)-(e) by (a) and (b). For (f), if $\max\{d(f_2), d(f_3)\} \leq 4$, then $d(f_2) = d(f_3) = 4$. It is a contradiction to (a). Hence $\max\{d(f_2), d(f_3)\} \geq 5$. Moreover, if $\min\{d(v_1), d(v_2)\} \geq 3$, we suppose $\max\{d(f_2), d(f_3)\} \leq 5$, then $\max\{d(f_2), d(f_3)\} = 5$. From (b), we have $\min\{d(v_1), d(v_2)\} = 2$. This leads to a contradiction. So we have $\max\{d(f_2), d(f_3)\} \geq 7$. Thus we prove (*f*).

Before proving (g), we give some basic notions needed in the following. Let $F(v) = \{f \in F(G)$: the face f is incident with $v\}$, $F_3(v) = \{f \in F(v)$: $d(f) = 3$ and f is incident with $v\}$. A cluster of $F_3(v)$ is a subgraph of G which consists of a nonempty minimal set of 3-faces in $F_3(v)$ such that no other 3-face in $F_3(v)$ is adjacent to a member of this set. A cluster of $F(v) \setminus F_3(v)$ is defined similarly. We use p and q to denote the number of faces in the largest cluster of $F_3(v)$ and $F(v) \setminus F_3(v)$.

Let us begin to prove (g). By (c), we have $p \leq 3$.

Case 1. $p = 1$. We assume that $d(f_1) = 3$ and $\min\{d(f_2), d(f_d)\} \geq 4$. Suppose $d(v) = 7$. Then there is no 2-vertex incident with v. If $d(f_2) = 4$, then f_3 cannot be a 3-face, otherwise, a 6-cycle appears. If $d(f_3) = 4$, then $d(f_4) \geq 5$ by (a). So $\max\{d(f_2), d(f_3), d(f_4)\} \geq 5$. Similarly, $\max\{d(f_d), d(f_{d-1}), d(f_{d-2})\} \geq 5$, so $f_{4^-}(v) \leq d(v) - 2$. Suppose $d(v) \geq 8$. If one of v_1 and v_d is a 2-vertex, without loss of generality, we assume $d(v_1) = 2$, then f_2 can be a 4-face. If $d(f_2) = 4$, then f_3 must be a 3-face or a 7^+-face. If $d(f_3) = 3$, then f_4 must be a 4-face or a 7^+-face. If $d(f_4) = 4$, then $d(v_4) > 2$ by Lemma (a), so $d(f_5) \geq 7$. Hence $\max\{d(f_2), d(f_3), d(f_4), d(f_5)\} \geq 5$. Similarly, we have $\max\{d(f_d), d(f_{d-1}), d(f_{d-2})\} \geq 5$. So $f_{4^-}(v) \leq d(v) - 2$. If $\min\{d(v_1), d(v_d)\} > 2$, it is easy to check that $\max\{d(f_d), d(f_{d-1}), d(f_{d-2})\} \geq 5$ and $\max\{d(f_2), d(f_3), d(f_4)\} \geq 5$. We omit the details here. Hence, $f_{4^-}(v) \leq d(v) - 2$.

Now suppose $f_{4^-}(v) = d(v) - 2$ and $f_3(v) \geq f_4(v)$. If $d(v)$ is odd, then $f_3(v) \geq \frac{d(v)-1}{2}$. Since $p = 1$, $q = 1$ or 2. And there is only one cluster of $F(v) \setminus F_3(v)$ having two faces. If $d(v)$ is even, then $f_3(v) \geq \frac{d(v)-2}{2}$ and $f_{4^+}(v) \leq \frac{d(v)-2}{2}$. Since $p = 1$,

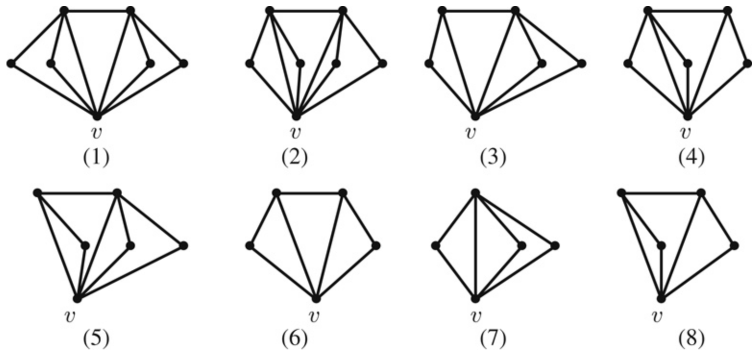

Fig. 3.

there exists a cluster of $F(v) \setminus F_3(v)$ with $q = 1$, otherwise, $f_{4^+}(v) \geq \frac{d(v)-2}{2} \times 2 = d(v) - 2 > \frac{d(v)-2}{2}$, a contradiction. And there are at least two clusters of $F(v) \setminus F_3(v)$ with $q \leq 2$, otherwise, $f_{4^+}(v) \geq (\frac{d(v)-2}{2} - 1) \times 3 + 1 \geq \frac{3d(v)-10}{2} > \frac{d(v)+2}{2}$, a contradiction. In the following, we discuss by the value of q.

Suppose $q = 1$. Without loss of generality, we assume that $d(f_1) = d(f_3) = 3$ and $\min\{d(f_2), d(f_4), d(f_d)\} \geq 4$. By (e) and the proof above, if f_2 is not a 7^+-face, then there are at least two 7^+-faces in $\{f_4, \dots, f_d\}$.

Suppose $q = 2$. Without loss of generality, we assume that $d(f_1) = d(f_4) = 3$ and $\min\{d(f_2), d(f_3), d(f_5), d(f_d)\} \geq 4$. By (f), if $\max\{d(f_2), d(f_3)\} \leq 5$, then it is easy to show that there are at least two 7^+-faces in $\{f_5, \dots, f_d\}$.

Since there are at least two clusters of $F(v) \setminus F_3(v)$ with $q \leq 2$, we have $f_{7^+}(v) \geq 2$.

Case 2. $p = 2$. We assume that $d(f_1) = d(f_2) = 3$ and $\min\{d(f_3), d(f_d)\} \geq 4$. Suppose $d(v) = 7$. Then there is no 2-vertex incident with v, so there is at most one 4-face in $\{f_3, f_d\}$ by (a). Without loss of generality, we assume $d(f_3) = 4$. Then $d(f_4) \geq 7$ and $d(f_d) \geq 7$ by (d). Suppose $d(v) \geq 8$. If $d(f_3) = 5$ and $d(f_d) = 4$ (the case $d(f_3) = 4$ and $d(f_d) = 5$ can be settled similarly), then $d(v_d) = 2$ and $\min\{d(v_3), d(v_{d-1})\} > 2$ by (d). So one of f_4 and f_{d-1} is a 7^+-face. Now we assume $\max\{d(f_3), d(f_d)\} \geq 7$. Without loss of generality, we assume $d(f_d) \geq 7$. Suppose $d(f_3) = 4$. Then f_4 cannot be a 5-face, otherwise, a 6-cycle appears. If $d(f_4) = 3$, then f_5 must be a 4-face or a 7^+-face. If $d(f_5) = 4$, then $d(f_6) \geq 7$. If $d(f_4) = 4$, then $d(v_3) > 2$ by (a). So $d(f_5) \geq 7$. Hence, $\max\{d(f_4), d(f_5), d(f_6)\} \geq 7$. Suppose $d(f_3) = 5$, it is easy to check that there is one 7^+-face in $\{f_4, f_5, f_6\}$. Hence, $f_{7^+}(v) \geq 2$. Certainly, $f_{4^-}(v) \leq d(v) - 2$.

Case 3. $p = 3$. Without loss of generality, we assume that $d(f_1) = d(f_2) = d(f_3) = 3$. If $d(v) = 7$, then there is no 2-vertex in $N(v)$, so $d(f_4) \geq 7$ and $d(f_d) \geq 7$ by (c). Suppose $d(v) \geq 8$. Then f_4 must be a 4-face or a 7^+-face by (c). If f_4 is a 4-face, then $d(v_3) = 2$ and $d(v_4) > 2$, so f_5 must be a 7^+-face. Thus one of f_4 and f_5 is a 7^+-face. Similarly, one of f_d and f_{d-1} is a 7^+-face. Hence, $f_{7^+}(v) \geq 2$. Certainly, $f_{4^-}(v) \leq d(v) - 2$. Hence the proof of (g) is completed.

Before proving (h), we also need to give some basic notions as follows. A 4-face in G is called *improper* if it is incident with an improper 2-vertex. $F_3^*(v) = F_3(v) \cup \{f \in F(v) : f$ is an improper 4-*face*$\}$. A cluster of $F_3^*(v)$ and $F(v) \setminus F_3^*(v)$ is defined similarly to a cluster of $F_3(v)$ above. We use p^* and q^* to denote the number of faces in the largest cluster of $F_3^*(v)$ and $F(v) \setminus F_3^*(v)$, respectively.

For (h), it is obvious that $p^* \leq 5$. Suppose $p^* = 5$. Then there are only two isomorphic configurations in Fig. 3(1) and (2). Suppose $p^* = 4$. Then there are three isomorphic configurations in Fig. 3(3), (4) and (5). Suppose $p^* = 3$. Then there are three isomorphic configurations in Fig. 3(6), (7) and (8). By the proof of (g), it is easy to check that if any case in Fig. 3 appears, then $f_{7^+}(v) \geq 2$.

It remains to show that $f_{7^+}(v) \geq 2$ if $p^* \leq 2$ and $f_3(v) + S_2(v) > \lfloor \frac{d(v)}{2} \rfloor$. It is obvious that $|F_3^*(v)| = f_3(v) + S_2(v)$. If $p^* \leq 2$, then there exists one cluster of $F(v) \setminus F_3^*(v)$ with $q^* = 1$, otherwise, $|F(v) \setminus F_3^*(v)| \geq \lceil \frac{f_3(v)+S_2(v)}{p^*} \rceil \times 2 \geq \lceil \frac{f_3(v)+S_2(v)}{2} \rceil \times 2 \geq f_3(v) + S_2(v) > d(v) - (f_3(v) + S_2(v)) = d(v) - |F_3^*(v)| = |F(v) \setminus F_3^*(v)|$ for $f_3(v) + S_2(v) > \lfloor \frac{d(v)}{2} \rfloor$, a contradiction. And there are at least two clusters of $F(v) \setminus F_3^*(v)$ with $q^* \leq 2$, otherwise, $|F(v) \setminus F_3^*(v)| \geq (\lceil \frac{f_3(v)+S_2(v)}{p^*} \rceil - 1) \times 3 + 1 \geq (\lceil \frac{f_3(v)+S_2(v)}{2} \rceil - 1) \times 3 + 1 > d(v) - (f_3(v) + S_2(v)) = |F(v) \setminus F_3^*(v)|$ for $f_3(v) + S_2(v) > \lfloor \frac{d(v)}{2} \rfloor$, a contradiction. In the following, we discuss by the value of q^*.

Suppose $q^* = 1$. Without loss of generality, we assume that $f_2 \in F(v) \setminus F_3^*(v)$ and $\{f_1, f_3\} \subset Q_3^*(v)$. If $d(f_1) = d(f_3) = 3$, then the proof is similar to the proof of $q = 1$ in (g). If $d(f_1) = 3$ and f_3 is an improper 4-face (the case $d(f_3) = 3$ and f_1 is an improper 4-face can be settled similarly), then $d(v_2) > 2$, so $d(f_2) \neq 5$ by (e) and $d(f_2) \neq 4$ for $f_3 \in F(v) \setminus F_3^*(v)$. Hence, $d(f_2) \geq 7$. If f_1, f_3 are improper 4-faces, then $d(v_1) > 2$ and $d(v_2) > 2$, then $d(f_2) \neq 4, 5$ by (e). Hence, $d(f_2) \geq 7$.

Suppose $q^* = 2$. Without loss of generality, we assume that $\{f_2, f_3\} \subset F(v) \setminus F_3^*(v)$ and $\{f_1, f_4\} \subset F_3^*(v)$. If $d(f_1) = d(f_4) = 3$, then the proof is similar to the proof of $q = 2$ in (g). If $d(f_1) = 3$ and f_4 is an improper 4-face (the case $d(f_4) = 3$ and f_1 is an improper 4-face can be settled similarly), then $d(v_3) > 2$, so $d(f_3) \geq 7$ by (f). If f_1, f_4 are improper 4-faces, then $d(v_1) > 2$ and $d(v_3) > 2$. Since $f_2, f_3 \in F(v) \setminus F_3^*(v)$, then $\max\{d(f_2), d(f_3)\} \geq 7$ by (f).

Hence, if $f_3(v) + S_2(v) > \lfloor \frac{d(v)}{2} \rfloor$ and there are at least two clusters of $F(v) \setminus F_3^*(v)$ with $q^* \leq 2$, then $f_{7^+}(v) \geq 2$. The proof of (h) is completed. □

Theorem 4. *Let G be a connected planar graph with $\delta(G) \geq 2$. If G contains no 5-cycles or contains no 6-cycles, then G contains an edge xy such that $d(x) + d(y) \leq 9$, or G contains a 2-alternating cycle.*

Proof. Suppose, to the contrary, that G is such a connected planar graph not satisfying the theorem. Let G_2 be the subgraph induced by the edges incident with the 2-vertices of G. Since $d(x) + d(y) \geq 10$ for every edge $xy \in E(G)$, every pair of 2-vertices is nonadjacent. Since G does not contain any 2-alternating cycle, G_2 does not contain any cycle at all. So every component of G_2 is a tree and there exists a matching M such that all 2-vertices in G_2 are saturated. Here if $uv \in M$ and $d(u) = 2$, we call v the 2-*master* of u.

From Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, we can derive the following identity.

$$\sum_{v\in V(G)} (2d(v) - 6) + \sum_{f\in F(G)} (d(f) - 6) = -12 < 0.$$

Let ω denote the weight function defined on $V(G) \bigcup F(G)$ by $\omega(v) = 2d(v) - 6$ if $v \in V(G)$ and $\omega(f) = d_G(f) - 6$ if $f \in F(G)$. Next, we will define a set of discharging rules. Once the discharging is finished, a new weight function ω' is produced. We will show that ω' is nowhere negative. This leads to the following obvious contradiction since the total sum of weights is kept fixed during discharging.

$$0 \leq \sum_{x\in V(G)\cup F(G)} \omega'(x) = \sum_{x\in V(G)\cup F(G)} \omega(x) = -12 < 0.$$

Hence, the contradiction proves the theorem.

First, suppose that G contains no 5-cycles. The discharging rules are defined as follows.

R1.1. Each 2-vertex receives 2 from its 2-master.

R1.2. For a 3-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $d(v) = 4$, 1 if $d(v) = 5$, $\frac{5}{4}$ if $d(v) = 6$ and $\frac{3}{2}$ if $d(v) \geq 7$.

R1.3. For a 4-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $4 \leq d(v) \leq 6$, 1 if $d(v) \geq 7$.

Let $f \in F(G)$. Clearly, $\omega'(f) = \omega(f) = d(f) - 6 \geq 0$ if $d(f) \geq 6$. Suppose $d(f) = 3$. Then $\omega(f) = 3 - 6 = -3$. If f is incident with a 3^--vertex, then other incident vertices of f are 7^+-vertices and it follows that $\omega'(f) \geq \omega(f) + 2 \times \frac{3}{2} = 0$. If f is incident with a 4-vertex, then $\omega'(f) \geq \omega(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$. If all vertices incident with f are 5^+-vertices, then $\omega'(f) \geq \omega(f) + 3 \times 1 = 0$. Suppose $d(f) = 4$. If f is incident with a vertex of degree at most 3, then f is incident with at least two 7^+-vertices and it follows that $\omega'(f) \geq \omega(f) + 2 \times 1 = 0$. Otherwise, $\omega'(f) \geq \omega(f) + 4 \times \frac{1}{2} = 0$.

Let $v \in V(G)$. If $d(v) = 2$, then $\omega'(v) = \omega(v) + 2 = 0$ by R1.1. If $d(v) = 3$, then $\omega'(v) = \omega(v) \geq 0$. If $d(v) = 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 0$. If $d(v) = 5$, then $\omega'(v) \geq 10 - 6 - \max\{3\times 1 + 2\times\frac{1}{2}, 2\times 1 + 3\times\frac{1}{2}, 1 + 4\times\frac{1}{2}, 5\times\frac{1}{2}\} = 0$. If $d(v) = 6$, then $\omega'(v) \geq \omega(v) - \max\{4\times\frac{5}{4} + 2\times\frac{1}{2}, 6\times\frac{1}{2}\} = 0$. Suppose $d(v) = 7$. Then all neighbors of v are 3^+-vertices. By Lemma 1, v is incident with at most four 3-faces, and if a 3-face f is incident with v, then v is incident with at most five 4^--faces. So $\omega'(v) \geq \omega(v) - \max\{4\times\frac{3}{2} + \frac{1}{2}, 7\times 1\} \geq 0$. Suppose $d(v) = 8$. If v is not incident with a 3-face, then $\omega'(v) = \omega(v) - 2 - 8 \times 1 = 0$. So assume that v is incident with at least one 3-face. By Lemma 1(d), v is incident with at most six 4^--faces. If v is incident with at least five 3-faces, then v is incident with exactly five 3-faces and by Lemma 1(c) all 4^+-faces incident with v must be 6^+-faces; it follows that $\omega'(v) = \omega(v) - 2 - 5\times\frac{3}{2} = \frac{1}{2} > 0$; otherwise, $\omega'(v) = \omega(v) - 2 - 4\times\frac{3}{2} - 2\times 1 = 0$. Suppose $d(v) \geq 9$. Similarly, we have $\omega'(v) \geq \omega(v) - 2 - \max\{\lfloor\frac{2d(v)}{3}\rfloor \times \frac{3}{2} - (d(v) - 2 - \lfloor\frac{2d(v)}{3}\rfloor) \times 1, d(v) \times 1\} \geq 0$. Hence, the proof of the case when G contains no 5-cycles is completed.

Now for the harder part, suppose that G contains no 6-cycles. If G is not 2-connected, then take an end block B of G, let u be the corresponding cut vertex in B. Let $v \in N(u) \bigcap V(B)$, $w \in N(u) \setminus V(B)$, such that u, v, w lie on a common face. Denote by B^* the graph constructed from four copies B_1, B_2, B_3, B_4 of B and one copy $u'v'$ of uv such that the copy u_i of u in B_i is identified with the copy v_{i+1} of v in B_{i+1} for $i = 1, 2, 3$ and u' is identified with v_1. It is easy to see that B^* has an embedding in the plane such that v' and u_4 are on the boundary of the outer face. Since G is not 2-connected, there is a face f which is incident with w, v that are not contained in the same block of G. Now, we identify the vertices u_4 and v' with w and v, respectively, and embed B^* into f. The resulting graph G' has fewer blocks than G. Clearly, G' has no 6-cycles, no 2-alternating cycles and $d(x) + d(y) \geq 10$ for any edge $xy \in E(G)$. Therefore, G' is also a counterexample to the theorem. By repeating the above construction sufficiently many times, we obtain a 2-connected counterexample.

So we may assume that G is 2-connected and hence all its facial walks are cycles. In particular, G has no faces of length 6. The discharging rules are defined as follows.

R2.1. For a 3-face f and its incident vertex v, if there is a 3^--vertex incident with f, then f receives $\frac{3}{2}$ from each 7^+-vertex. Otherwise, f receives $\frac{1}{2}$ from v if $d(v) = 4$, 1 if $d(v) = 5$, $\frac{5}{4}$ if $d(v) \geq 6$.

R2.2. For a 4-face f and its incident vertex v, if there are two 3^--vertices incident with f, then f receives 1 from each 7^+-vertex. Otherwise, f receives $\frac{1}{2}$ from v if $4 \leq d(v) \leq 6$, $\frac{3}{4}$ if $d(v) \geq 7$.

R2.3. For a 5-face f and its incident 5^+-vertex v, f receives $\frac{1}{3}$ from v.

R2.4. Let $u \in G$ be a 2-vertex, $N(u) = \{v, w\}$ and v be its 2-master. If u is incident with a 3-face and a 4-face, then u receives $\frac{3}{2}$ from v and $\frac{1}{2}$ from w. If u is incident with a 3-face and a 8^+-face, then u receives 1 from the 8^+-face and 1 from v. Suppose u is incident with a 4-face and a 7^+-face f. If $d(f) = 7$ and $n_2(f) = 3$, then u receives $\frac{1}{3}$ from f and $\frac{5}{3}$ from v; Otherwise, u receives $\frac{1}{2}$ from f and $\frac{3}{2}$ from v. In all the other cases, u receives 2 from v.

Let $f \in F(G)$. Suppose $d(f) = 3$. Then $\omega(f) = 3 - 6 = -3$. By R2.1, if f is incident with a 3^--vertex, then other incident vertices of f are 7^+-vertices and it follows that $\omega'(f) \geq \omega(f) + 2 \times \frac{3}{2} = 0$. If f is incident with a 4-vertex, then $\omega'(f) \geq \omega(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$. If all vertices incident with f are 5^+-vertices, then $\omega'(f) \geq \omega(f) + 3 \times 1 = 0$. Suppose $d(f) = 4$. By R2.2, if f is incident with two vertices of degree at most 3, then f is incident with two 7^+-vertices and it follows that $\omega'(f) \geq \omega(f) + 2 \times 1 = 0$. Otherwise, $\omega'(f) \geq \omega(f) + \min\{4 \times \frac{1}{2}, \frac{1}{2} + 2 \times \frac{3}{4}\} = 0$. By R2.3, if $d(f) = 5$, then f is incident with at least three 5^+-vertices and it follows that $\omega'(f) \geq \omega(f) + 3 \times \frac{1}{3} = 0$. Suppose $d(f) = 7$. By R2.4, if $n_2(f) = 3$, then we have $\omega'(f) \geq 7 - 6 - 3 \times \frac{1}{3} = 0$, otherwise, $n_2(f) \leq 2$, we have $\omega'(f) \geq 7 - 6 - 2 \times \frac{1}{2} = 0$. Suppose $d(f) \geq 8$. Then $n_2(f) \leq \lfloor \frac{d(f)-1}{2} \rfloor$ for G containing no 2-alternating cycles. And f is incident with at most $(d(f) - 7)$ improper 2-vertices for otherwise after deleting these 2-vertices, f becomes a 6^--cycle and then a 6-cycle appears, a contradiction. By R2.4, each improper 2-vertex incident with f receives 1 from f and the other 2-vertices in $n_2(f)$ receive at most $\frac{1}{2}$ from f, so we have $\omega'(f) \geq \omega(f) - 1 \times (d(f) - 7) - \frac{1}{2} \times [\lfloor \frac{d(f)-1}{2} \rfloor - (d(f) - 7)] \geq 0$.

Let $v \in V(G)$. If $d(v) = 2$, then $\omega'(v) = \omega(v) + 2 = 0$ by R2.4. If $d(v) = 3$, then $\omega'(v) = \omega(v) = 0$. If $d(v) = 4$, then $\omega'(v) \geq \omega(v) - 4 \times \frac{1}{2} = 0$ by R2.1 and R2.2. If $d(v) = 5$, then $f_3(v) \leq 3$ by Lemma 3(c), so we have $\omega'(v) \geq 10 - 6 - \max\{3 \times 1 + 2 \times \frac{1}{2}, 2 \times 1 + 3 \times \frac{1}{2}, 1 + 4 \times \frac{1}{2}, 5 \times \frac{1}{2}\} = 0$ by R2.1 and R2.2. Similarly, if $d(v) = 6$, then $\omega'(v) \geq \omega(v) - \max\{4 \times \frac{5}{4} + 2 \times \frac{1}{2}, 6 \times \frac{1}{2}\} = 0$. Suppose $d(v) = 7$. Then all neighbors of v are 3^+-vertices. We have $f_3(v) \leq 5$ by Lemma 3(c), and if $f_3(v) \geq 1$, then $f_3(v) + f_4(v) \leq 5$ by Lemma 3(g). By Lemma 3(h), if $f_3(v) \geq 4$, then $f_{7^+}(v) \geq 2$. So $\omega'(v) \geq 14 - 6 - \max\{5 \times \frac{3}{2}, 4 \times \frac{3}{2} + 1, 3 \times \frac{3}{2} + 2 \times 1 + 2 \times \frac{1}{3}, 2 \times \frac{3}{2} + 3 \times 1 + 2 \times \frac{1}{3}, \frac{3}{2} + 4 \times 1 + 2 \times \frac{1}{3}, 7 \times 1\} = \frac{1}{3} > 0$.

Our task is now to prove $\omega'(v) \geq 0$ if $d(v) \geq 8$.

Suppose $d(v) = 8$. Then $f_3(v) \leq 6$ by Lemma 3(c) and $\omega(v) = 16 - 6 = 10$. There can be a 2-vertex in $N(v)$, and we assume that v is the 2-master of some 2-vertex, denoted as u, in $N_2(v)$, otherwise, the problem becomes easier. Note that $S_2(v) \leq 3$ for otherwise it is easy to obtain a 6-cycle or a 2-alternating cycle. If $f_3(v) \leq 1$, then $S_2(v) \leq 1$ and it follows that $\omega'(v) \geq \omega(v) - 2 - \frac{3}{2} \times f_3(v) - 1 \times f_4(v) - \frac{1}{2} \times S_2(v) - \frac{1}{3} \times f_5(v) \geq 10 - 2 - \max\{8 \times 1, \frac{3}{2} + 5 \times 1 + \frac{1}{2} + 2 \times \frac{1}{3}\} = 0$. If $n_2(v) = 0$, then $\omega'(v) \geq 10 - \max\{6 \times \frac{3}{2}, 8 \times 1\} > 0$. So assume that $n_2(v) \geq 1$ and $f_3(v) \geq 2$. By Lemma 3(g), $f_3(v) + f_4(v) \leq 6$. In the following, let us discuss by the number of 3-faces.

Case 1. $f_3(v) = 6$. Then all 4^+-faces incident with v must be 7^+-faces by Lemma 3(h), and any 2-vertex adjacent to v is incident with a 8^+-face, so it follows that $\omega'(v) \geq 10 - 1 - 6 \times \frac{3}{2} = 0$ by R2.4.

Case 2. $f_3(v) = 5$. Then $f_7^+(v) \geq 2$ by Lemma 3(h) and $f_4(v) \leq 1$. Suppose $f_4(v) = 1$. Then for any 2-vertex in $N_2(v)$, it is incident with a 3-face. By R2.4, if $u \in S_2(v)$, then u receives $\frac{3}{2}$ from v. If u is incident with a 8^+-face, then u receives 1 from v. So it follows that $\omega'(v) \geq 10 - \max\{1 + (5 \times \frac{3}{2} + 1 + \frac{1}{2}), \frac{3}{2} + (5 \times \frac{3}{2} + 1)\} = 0$. Otherwise, we have $f_4(v) = 0$ and $f_5(v) \leq 1$. Then $\omega'(v) \geq 10 - 2 - (5 \times \frac{3}{2} + \frac{1}{3}) = \frac{1}{6} > 0$.

Case 3. $f_3(v) = 4$. Then $f_4(v) \leq 2$ by Lemma 3(g). Suppose $f_4(v) \leq 1$, then $S_2(v) \leq 1$. If $S_2(v) = 1$, then $f_{7^+}(v) \geq 2$ by Lemma 3(h), otherwise, $S_2(v) = 0$, then $f_4(v) + f_5(v) \leq 6$. So we have $\omega'(v) \geq 10 - 2 - \max\{4 \times \frac{3}{2} + 1 + 3 \times \frac{1}{3}, 4 \times \frac{3}{2} + 1 + \frac{1}{2} + \frac{1}{3}\} = 0$. Suppose $f_4(v) = 2$. Then $f_{7^+}(v) = 2$ by Lemma 3(g) and $S_2(v) \leq 2$. If $S_2(v) = 0$, then $\omega'(v) \geq 10 - 2 - (4 \times \frac{3}{2} + 1 \times 2) = 0$. So we assume that $S_2(v) \geq 1$. Let us denote the two faces of which vu is the common edge as f_{u1} and f_{u2}. Since $S_2(v) \geq 1$, there exists at least a 4-face adjacent to a 3-face. Then it is impossible that $d(f_{u1}) = d(f_{u2}) = 4$. f_{u1} and f_{u2} cannot be 7^+-faces simultaneously, otherwise a 6-cycle or a 2-alternating cycle appears. Without loss of generality, we assume that $d(f_{u2}) \geq d(f_{u1})$. If $d(f_{u1}) = 3$ and $d(f_{u2}) \geq 8$, then $\omega'(v) \geq 10 - 1 - (4 \times \frac{3}{2} + 1 \times 2 + 2 \times \frac{1}{2}) = 0$ by R2.4. If $d(f_{u1}) = 3$ and $d(f_{u2}) = 4$ or $d(f_{u1}) = 4$ and $d(f_{u2}) \geq 8$, then $S_2(v) = 1$. So we have $\omega'(v) \geq 10 - \frac{3}{2} - (4 \times \frac{3}{2} + 1 \times 2 + 1 \times \frac{1}{2}) = 0$ by R2.4. Suppose $d(f_{u1}) = 4$ and $d(f_{u2}) = 7$. Then $S_2(v) = 1$. Let u' be the vertex which is adjacent to v and is incident with f_{u2} and we denote the other face which is incident with vu' as f_{u3}. If $d(u') = 2$, then $d(f_{u3}) = 4$ for $d(f_{u2}) = 7$. So we have $S_2(v) = 0$, a contradiction to $S_2(v) = 1$. Hence, we assume $d(u') \geq 3$. If $3 \leq d(u') \leq 7$, then $n_2(f_{u2}) < 3$, so it follows that $\omega'(v) \geq 10 - \frac{3}{2} - (4 \times \frac{3}{2} + 1 \times 2 + \frac{1}{2}) = 0$ by R2.4. If $d(u') \geq 8$, there must be a 3-face or a 4-face in $\{f_1, f_2, \ldots f_d\} \setminus \{f_{u1}, f_{u2}\}$ receives $\frac{5}{4}$ or $\frac{3}{4}$ from v by R2.1 and R2.2. So it follows that $\omega'(v) \geq 10 - \frac{5}{3} - \max\{4 \times \frac{3}{2} + 1 + \frac{3}{4} + \frac{1}{2}, 3 \times \frac{3}{2} + \frac{5}{4} + 1 \times 2 + \frac{1}{2}\} = \frac{1}{12} > 0$ by R2.4.

Case 4. $f_3(v) = 3$. Then $f_4(v) \leq 3$. If $f_4(v) \leq 1$, then $S_2(v) \leq 1$, so it follows that $\omega'(v) \geq 10 - 2 - (3 \times \frac{3}{2} + 1 + \frac{1}{2} + 4 \times \frac{1}{3}) = \frac{2}{3} > 0$. Suppose $f_4(v) = 2$. Then $S_2(v) \leq 2$. If $S_2(v) \leq 1$, we have $\omega'(v) \geq 10 - 2 - (3 \times \frac{3}{2} + 1 \times 2 + \frac{1}{2} + 3 \times \frac{1}{3}) = 0$. If $S_2(v) = 2$, then $f_{7^+}(v) \geq 2$ by Lemma 3(h). So we have $\omega'(v) \geq 10 - 2 - (3 \times \frac{3}{2} + 1 \times 2 + 2 \times \frac{1}{2} + \frac{1}{3}) = \frac{1}{6} > 0$. Suppose $f_4(v) = 3$. Then $S_2(v) \leq 3$ and $f_{7^+}(v) \geq 2$ by Lemma 3(g). If $S_2(v) \leq 1$, we have $\omega'(v) \geq 10 - 2 - (3 \times \frac{3}{2} + 1 \times 3 + \frac{1}{2}) = 0$. If $S_2(v) = 2$, the proof is similar to the above case that $f_3(v) = 4$ and $f_4(v) = 2$. If $S_2(v) = 3$, then $f_{7^+}(v) = 2$ by Lemma 3(h). We denote the two 7^+-faces as f and f'. It is obvious that f is not adjacent to f', otherwise a 2-alternating cycle appears. Then all the 2-vertices incident with v are improper 2-vertices. So we have $\omega'(v) \geq 10 - \frac{3}{2} - (3 \times \frac{3}{2} + 1 \times 3 + 2 \times \frac{1}{2}) = 0$.

Case 5. $f_3(v) = 2$. Then $f_4(v) \leq 4$ and $S_2(v) \leq 2$. If $f_4(v) \leq 3$, then $f_4(v) + f_5(v) \leq 6$, we have $\omega'(v) \geq 10 - 2 - (2 \times \frac{3}{2} + 1 \times f_4(v) + \frac{1}{3} \times f_5(v) + 2 \times \frac{1}{2}) \geq 2 - \frac{2}{3} \times f_4(v) \geq 0$. Suppose $f_4(v) = 4$. If $S_2(v) = 0$, we have $\omega'(v) \geq 10 - 2 - (2 \times \frac{3}{2} + 1 \times 4 + 2 \times \frac{1}{3}) = \frac{1}{3} > 0$. If $S_2(v) = 1$, then v is incident with at least one 7^+-face. So it follows that $\omega'(v) \geq 10 - 2 - (2 \times \frac{3}{2} + 1 \times 4 + \frac{1}{2} + \frac{1}{3}) = \frac{1}{6} > 0$. If $S_2(v) = 2$, we can prove this case using an argument similar to the above case that $f_3(v) = 4$ and $f_4(v) = 2$.

Suppose $d(v) = 9$. Then $f_3(v) \leq 6$ and $\omega(v) = 18 - 6 = 12$. If $n_2(v) \leq 1$, then we have $\omega'(v) \geq 12 - 2 - \max\{6 \times \frac{3}{2} + 1, 9 \times 1\} = 0$. In the following, we assume that $n_2(v) \geq 2$. If $f_3(v) \leq 2$, then $S_2(v) \leq f_3(v)$. Since $f_3(v) + f_4(v) \leq 7$, we have $\omega'(v) \geq 12 - 2 - \max\{2 \times \frac{3}{2} + 1 \times 5 + 2 \times \frac{1}{3} + 2 \times \frac{1}{2}, \frac{3}{2} + 1 \times 6 + 2 \times \frac{1}{3} + \frac{1}{2}, 9 \times 1\} = \frac{1}{3} > 0$. Suppose $f_3(v) = 3$. Then $f_4(v) \leq 4$ and $S_2(v) \leq 3$. If $f_4(v) \leq 3$, then $f_4(v) + f_5(v) \leq 6$ and $S_2(v) \leq f_4(v)$, we have $\omega'(v) \geq 12 - 2 - (3 \times \frac{3}{2} + 1 \times f_4(v) + \frac{1}{2} \times S_2(v) + \frac{1}{3} \times f_5(v)) \geq 2 - \frac{2}{3} \times f_4(v) \geq 0$. Suppose $f_4(v) = 4$. If $S_2(v) \leq 1$, then we have $\omega'(v) \geq 12 - 2 - (3 \times \frac{3}{2} + 1 \times 4 + \frac{1}{2} \times 1 + 2 \times \frac{1}{3}) = \frac{1}{3} > 0$. Otherwise, $2 \leq S_2(v) \leq 3$, then $f_{7^+}(v) \geq 2$ by Lemma 3(h). So we have $\omega'(v) \geq 12 - 2 - (3 \times \frac{3}{2} + 1 \times 4 + 3 \times \frac{1}{2}) = 0$. Suppose $f_3(v) \geq 4$. Then $f_4(v) \leq 3$. If $S_2(v) = 3$, then $f_{7^+}(v) \geq 2$ by Lemma 3(h). So $f_3(v) = 4$ and $f_4(v) = 3$. All the 2-vertices adjacent to v are incident with a 3-face. By R2.4, we have $\omega'(v) \geq 12 - \max\{1 + 4 \times \frac{3}{2} + 1 \times 3 + 3 \times \frac{1}{2}, \frac{3}{2} + 4 \times \frac{3}{2} + 1 \times 3 + 2 \times \frac{1}{2}\} = \frac{1}{2} > 0$. If $S_2(v) \leq 2$, it is easy to check that $\omega'(v) \geq 0$ if $f_3(v) = 4$, 5 or 6. We omit the details here.

Suppose $d(v) \geq 10$. Then $f_3(v) \leq \lfloor \frac{3d(v)}{4} \rfloor$ and $S_2(v) \leq \min\{f_3(v), f_4(v)\}$. If $f_3(v) = 0$, then we have $\omega'(v) \geq \omega(v) - 2 - 1 \times f_4(v) \geq d(v) - 8 > 0$. Else, $f_3(v) \geq 1$, then $f_3(v) + f_4(v) \leq d(v) - 2$. Suppose $f_3(v) \geq f_4(v)$. Then $S_2(v) \leq f_4(v)$. By Lemma 3(g), if $f_3(v) + f_4(v) = d(v) - 2$, then $f_{7^+}(v) = 2$. So we have $\omega'(v) \geq 2 \times d(v) - 8 - (\frac{3}{2} \times f_3(v) + 1 \times f_4(v) + \frac{1}{2} \times S_2(v)) \geq 2 \times d(v) - 8 - \frac{3}{2}(f_3(v) + f_4(v)) = \frac{1}{2} \times (d(v) - 10) \geq 0$; otherwise, $f_3(v) + f_4(v) \leq d(v) - 3$, then $f_{5^-}(v) \leq d(v)$, so we have $\omega'(v) \geq 2 \times d(v) - 8 - (\frac{3}{2} \times f_3(v) + 1 \times f_4(v) + \frac{1}{3} \times f_5(v) + \frac{1}{2} \times S_2(v)) \geq 2 \times d(v) - 8 - \frac{3}{2}(f_3(v) + f_4(v)) - \frac{1}{3} \times f_5(v) \geq \frac{1}{2} \times (d(v) - 9) > 0$. Suppose $f_3(v) < f_4(v)$. Then $S_2(v) \leq f_3(v)$ and $f_3(v) \leq \lfloor \frac{d(v)-3}{2} \rfloor$ for $f_3(v) + f_4(v) \leq d(v) - 2$. So we have $\omega'(v) \geq 2 \times d(v) - 8 - (\frac{3}{2} \times f_3(v) + 1 \times f_4(v) + \frac{1}{3} \times f_5(v) + \frac{1}{2} \times S_2(v)) \geq 2 \times d(v) - 8 - (2 \times f_3(v) + f_4(v) + \frac{1}{3} \times f_5(v)) \geq d(v) - f_3(v) - \frac{20}{3}$. If $d(v) = 10$, then $f_3(v) \leq 3$. So we have $\omega'(v) \geq 10 - 3 - \frac{20}{3} = \frac{1}{3} > 0$; otherwise, $\omega'(v) \geq d(v) - f_3(v) - \frac{20}{3} \geq \frac{1}{6} \times (3d(v) - 31) > 0$.

Hence the proof of the theorem is completed. □

3. Main results and their proofs

Lemma 5. *If a graph G can be edge-partitioned into m subgraphs $G_1, G_2, \ldots, G_m$, then $la_2(G) \leq \sum_{i=1}^{m} la_2(G_i)$.*

The above lemma is obvious since we just need to use disjoint color sets on the G_i's.

Lemma 6 (*[7]*). *For a forest T, we have $la_2(T) \leq \lceil \frac{\Delta(T)+1}{2} \rceil$.*

Lemma 7 (*[2]*). *For a graph G, we have $la_2(G) \leq \Delta(G)$.*

Lemma 8. *Every planar graph G without 5-cycles or without 6-cycles has an edge-partition into two forests T_1, T_2 and a subgraph H such that for every $v \in V(G)$, $d_{T_1}(v) \leq \lceil d_G(v)/2 \rceil$, $d_{T_2}(v) \leq \lceil d_G(v)/2 \rceil$ and $d_H(v) \leq 4$.*

Proof. We prove the lemma by induction on the number $|V(G)| + |E(G)|$. If $|V(G)| + |E(G)| \leq 5$, then the result holds trivially. Let G be a planar graph with $|V(G)| + |E(G)| \geq 6$. If $\Delta(G) \leq 4$, it suffices to take $H = G$ and $T_1 = T_2 = \emptyset$.

Suppose now that $\Delta(G) \geq 5$. We may assume that G is connected. If G' is a proper subgraph of G, then G' has an edge-partition as desired by the induction hypothesis; call the graphs of this edge-partition T_1', T_2', H'. We will choose an appropriate subgraph G' so that we can extend $T_1' \bigcup T_2' \bigcup H'$ to an edge-partition $T_1 \bigcup T_2 \bigcup H$ of G satisfying the lemma.

If $\delta(G) = 1$, let $uv \in E(G)$ with $d_G(u) = 1$. Define the graph $G' = G - uv$.

If $d_{H'}(v) \leq 3$, we let $H = H' + uv$ and $T_i = T_i'$ for $i = 1$ and 2. It is easy to see that the lemma holds.

If $d_{H'}(v) = 4$, we suppose that $d_{T_1'}(v) \leq d_{T_2'}(v)$. Since $d_{G'}(v) = d_{T_1'}(v) + d_{T_2'}(v) + d_{H'}(v) = d_{T_1'}(v) + d_{T_2'}(v) + 4$ and $d_{G'}(v) = d_G(v) - 1$, we have $d_{T_1'}(v) \leq (d_G(v) - 5)/2$. Let $T_1 = T_1' + uv$, $T_2 = T_2'$, and $H = H'$. Thus $d_{T_2}(x) = d_{T_2'}(x)$ and $d_H(x) = d_{H'}(x)$ for all $x \in V(G')$. Moreover, $d_{T_1}(u) = 1 = \lceil d_G(u)/2 \rceil$, $d_{T_1}(v) = 1 + d_{T_1'}(v) \leq 1 + (d_G(v) - 5)/2 < \lceil d_G(v)/2 \rceil$, and $d_{T_1}(x) = d_{T_1'}(x)$ for all $x \in V(G) \setminus \{u, v\}$.

Suppose next that $\delta(G) \geq 2$. By Theorem 4, we only need to consider two cases.

Case 1. There is an edge $xy \in E(G)$ such that $d_G(x) + d_G(y) \leq 9$.

Define $G' = G - xy$ and assume that $d_{H'}(x) \leq d_{H'}(y)$. If $d_{H'}(y) \leq 3$, let $H = H' + xy$, $T_1 = T_1'$ and $T_2 = T_2'$.

Assume that $d_{H'}(y) = 4$. In that case $1 \leq d_{G'}(x) \leq 3$ and $d_{T_1'}(y) + d_{T_2'}(y) + d_{G'}(x) \leq 3$. We may assume $d_{T_1'}(x) \leq d_{T_2'}(x)$.

If $d_{G'}(x) = 3$, then y belongs to neither T_1' nor T_2'. Let $T_1 = T_1' + xy$, $T_2 = T_2'$, and $H = H'$. If $d_{G'}(x) = 2$, then x belongs to both T_1' and T_2' since $d_{T_i'}(x) \leq \lceil d_{G'}(x)/2 \rceil$ for $i = 1$ and 2. Also note that y does not belong to either T_1' or T_2', say T_1'. Again let $T_1 = T_1' + xy$, $T_2 = T_2'$, and $H = H'$. We see that T_1 is a forest and $d_{T_1}(x) = 2 = \lceil 3/2 \rceil = \lceil d_G(x)/2 \rceil$. If $d_{G'}(x) = 1$, then x does not belong to T_1'. Let $T_1 = T_1' + xy$, $T_2 = T_2'$, and $H = H'$. We see that T_1 is a forest and $d_{T_1}(x) = 1 = \lceil d_G(x)/2 \rceil$. Furthermore, $d_{T_1}(y) = d_{T_1'}(y) + 1 \leq 3 < \lceil d_G(y)/2 \rceil$.

Case 2. There is a 2-alternating cycle $C = v_1 v_2 \cdots v_{2s} v_1$, $s \geq 2$, such that $d_G(v_1) = d_G(v_3) = \cdots = d_G(v_{2s-1}) = 2$.

Define $G' = G - E(C)$. Let $H = H'$, $T_1 = T_1' + \{v_1v_2, v_3v_4, \ldots, v_{2s-1}v_{2s}\}$ and $T_2 = T_2' + \{v_2v_3, v_4v_5, \ldots, v_{2s}v_1\}$. Note that both T_1 and T_2 are forests. Since $d_G = d_{G'} + 2$ for vertices x of the cycle C, we see that $d_{T_1}(v_j) = d_{T_2}(v_j) = 1 = d_G(v_j)/2$ for $j = 1, 3, \ldots, 2s-1$, and $d_{T_i}(v_j) = d_{T_i'}(v_j) + 1 \leq \lceil d_{G'}(v_j)/2 \rceil + 1 = \lceil d_G(v_j)/2 \rceil$ for $i = 1, 2$ and $j = 2, 4, \ldots, 2s$. □

The following is a direct consequence of Lemma 8.

Corollary 9. *Let G be a planar graph without* 5*-cycles or without* 6*-cycles. Then G can be edge-partitioned into two forests* T_1, T_2 *and a subgraph H such that* $\Delta(T_1) \leq \lceil \Delta(G)/2 \rceil$, $\Delta(T_2) \leq \lceil \Delta(G)/2 \rceil$ *and* $\Delta(H) \leq 4$.

Now we are ready to prove our first main result.

Theorem 10. *If G is a planar graph without* 5*-cycles or without* 6*-cycles, then* $la_2 \leq \lceil (\Delta(G)+1)/2 \rceil + 6$.

Proof. By Corollary 9, G has an edge-partition into two forests T_1, T_2 and a subgraph H with $\Delta(T_1) \leq \lceil \Delta(G)/2 \rceil$, $\Delta(T_2) \leq \lceil \Delta(G)/2 \rceil$, and $\Delta(H) \leq 4$. Combining Lemmas 5–7, we obtain the following sequence of inequalities.

$$\begin{aligned} la_2(G) &\leq la_2(T_1) + la_2(T_2) + la_2(H) \\ &\leq \lceil (\Delta(T_1)+1)/2 \rceil + \lceil (\Delta(T_2)+1)/2 \rceil + \Delta(H) \\ &\leq 2\lceil (\lceil \Delta(G)/2 \rceil + 1)/2 \rceil + 4 \\ &\leq \lceil (\Delta(G)+1)/2 \rceil + 6. \quad \square \end{aligned}$$

Lemma 11 ([12]). $\chi''_{\text{list}}(G) = \chi''(G)$ *for a graph G of the maximum degree* 2.

Our second main result is the following theorem.

Theorem 12. *Let* $\Delta \geq 8$ *and let G be a planar graph with maximum degree* $\Delta(G) \leq \Delta$. *If G contains no* 5*-cycles or contains no* 6*-cycles, then* $\chi'_{\text{list}}(G) = \chi'(G) = \Delta(G)$ *and* $\chi''_{\text{list}}(G) = \chi''(G) = \Delta(G) + 1$.

Proof. Let G be a minimal counterexample to Theorem 12 ($\Delta(G) \leq \Delta$). It is easy to see that $\delta(G) \geq 2$. Suppose first that G contains an edge $e = uw$ with $d(u) + d(w) \leq 9$. Without loss of generality, assume $d(u) \leq d(w)$. Then $d(u) \leq 4$. Color all edges and (if appropriate) vertices of $G - e$ from their lists. If we are coloring vertices, erase the color of u. There are now at least $\Delta - 7 \geq 1$ colors available to give to e, so color e with one of them. If we are coloring vertices, then there are now at least $\Delta + 1 - 2 \times 4 \geq 1$ colors available for u. Thus we can color all elements of G.

This contradiction shows that in fact $d(u) + d(w) \geq 10$ for every edge $e = uw$ of G. By Theorem 4, G must contain a 2-alternating cycle C. Remove the edges and 2-vertices of C from G, and color the remaining edges and (if appropriate) vertices of G from their lists, which is possible by the minimality of G as a counterexample. There are now at least two colors available for each edge of C, and so these edges can be colored by Lemma 11; and now (if we are coloring vertices) the vertices of C are easily colored. Thus G is not a counterexample, which is a contradiction. □

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