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Resolvable group divisible designs with block size four and general index

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ABSTRACT

In this paper, we investigate the existence of resolvable group divisible designs (RGDDs) with block size four, group-type h^n and general index λ. The necessary conditions for the existence of such a design are $n \geq 4$, $hn \equiv 0 \pmod 4$ and $\lambda h(n-1) \equiv 0 \pmod 3$. These necessary conditions are shown to be sufficient for all $\lambda \geq 2$, with the definite exceptions of $(\lambda, h, n) \in \{(3, 2, 6)\} \cup \{(2j+1, 2, 4) : j \geq 1\}$. The known existence result for $\lambda = 1$ is also improved.

1. Introduction

Let K be a set of positive integers and let λ be a positive integer. A *group divisible design* (GDD), denoted by (K, λ)-GDD, is a triple $(X, \mathcal{G}, \mathcal{B})$ where:

1. X is a finite set of *points*,
2. $\mathcal{G}$ is a set of subsets of X, called *groups*, which partition X,
3. $\mathcal{B}$ is a collection of subsets of X with sizes from K, called *blocks*, such that every pair of points from distinct groups occurs in exactly λ blocks, and
4. no pair of points belonging to a group occurs in any block.

The *group-type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. An "exponential" notation is usually used to describe the group-type: a type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, etc. When $K = \{k\}$, we write (K, λ)-GDD as (k, λ)-GDD. Further, we denote $(K, 1)$-GDD as K-GDD and $(k, 1)$-GDD as k-GDD.

A (K, λ)-GDD is said to be *resolvable* and denoted by (K, λ)-RGDD if its blocks can be partitioned into *parallel classes* each of which partitions the set of points.

Resolvable group divisible designs have been instrumental in the construction of other types of designs. Many researchers have been involved in investigating the existence of resolvable group divisible designs. Simple counting arguments show that if there is a (k, λ)-RGDD of type h^n, then

$$n \geq k,$$
$$hn \equiv 0 \pmod k \quad \text{and}$$
$$\lambda h(n-1) \equiv 0 \pmod{k-1}.$$

The above necessary conditions for the existence of a (k, λ)-RGDD of type h^n have been proved to be sufficient for $k = 3$ (see [1,17,19]), with the definite exception of $(3, \lambda)$-RGDDs of type h^n for $(\lambda, h, n) \in \{(1, 2, 6), (1, 6, 3)\} \cup \{(2j+1, 2, 3), (4j+2, 1, 6) : j \geq 0\}$. However, the case for $k = 4$ remains open despite the effort of many authors (see [7–9, 11–15,18,20,23–25]), and we have the following known results.

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Theorem 1.1. *The necessary conditions for the existence of a* $(4, 1)$*-RGDD of type* h^n*, namely,* $n \geq 4$*,* $hn \equiv 0 \pmod 4$ *and* $h(n-1) \equiv 0 \pmod 3$*, are also sufficient except for* $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:*

1. $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$*;* $h = 10$ *and* $n \in \{4, 34, 52, 94\}$*;* $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.
2. $h = 6$ *and* $n \in \{6, 54, 68\}$*;* $h = 18$ *and* $n \in \{18, 38, 62\}$.
3. $h = 9$ *and* $n = 44$.
4. $h = 12$ *and* $n = 27$*;* $h = 24$ *and* $n = 23$*;* $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

Theorem 1.2. *The necessary conditions for the existence of a* $(4, 3)$*-RGDD of type* h^n*, namely,* $n \geq 4$ *and* $hn \equiv 0 \pmod 4$*, are also sufficient except for* $(h, n) \in \{(2, 4), (2, 6)\}$ *and possibly excepting* $(h, n) = (2, 54)$.

In this paper, the main focus of our attention will be on the existence of $(4, \lambda)$-RGDDs of type h^n with general index λ. We will show that the necessary conditions for the existence of such designs are also sufficient for all $\lambda \geq 2$, with the definite exceptions of $(\lambda, h, n) \in \{(3, 2, 6)\} \cup \{(2j+1, 2, 4) : j \geq 1\}$. We will also improve the known existence result for $(4, 1)$-RGDDs.

2. Updating the cases for $\lambda = 1, 3$

In this section, we shall improve the known results on the existence of $(4, 1)$-RGDDs and $(4, 3)$-RGDDs.

Theorem 2.1. *The necessary conditions for the existence of a* $(4, 1)$*-RGDD of type* h^n*, namely,* $n \geq 4$*,* $hn \equiv 0 \pmod 4$ *and* $h(n-1) \equiv 0 \pmod 3$*, are also sufficient except for* $(h, n) \in \{(2, 4), (2, 10), (3, 4), (6, 4)\}$ *and possibly excepting:*

1. $h = 2$ *and* $n \in \{34, 46, 52, 70, 82, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$*;* $h = 10$ *and* $n \in \{4, 34, 52, 94\}$*;* $h \in [14, 454] \cup \{478, 502, 514, 526, 614, 626, 686\}$ *and* $n \in \{10, 70, 82\}$.
2. $h = 6$ *and* $n \in \{6, 54, 68\}$*;* $h = 18$ *and* $n \in \{18, 38, 62\}$.
3. $h = 9$ *and* $n = 44$.
4. $h = 24$ *and* $n = 23$*;* $h = 36$ *and* $n \in \{11, 14, 15, 18, 23\}$.

Proof. By Theorem 1.1, we only need to construct a $(4, 1)$-RGDD of type 12^{27}. Let the point set be $(Z_{104} \cup \{x, y, z, w\}) \times Z_3$, and let the group set be $\{\{j, j+26, j+52, j+78\} \times Z_3 : j = 0, \ldots, 25\} \cup \{\{x, y, z, w\} \times Z_3\}$. Below are the required base blocks.

$\{(25,0),(18,2),(13,1),(x,0)\}$,	$\{(3,0),(7,1),(90,2),(y,0)\}$,
$\{(52,0),(88,1),(98,2),(z,0)\}$,	$\{(96,0),(20,1),(54,2),(w,0)\}$,
$\{(0,0),(34,2),(45,0),(53,2)\}$,	$\{(11,0),(35,2),(77,1),(99,2)\}$,
$\{(102,0),(64,0),(78,2),(84,0)\}$,	$\{(71,0),(72,2),(33,2),(56,2)\}$,
$\{(47,0),(101,2),(38,1),(43,0)\}$,	$\{(73,0),(19,2),(27,0),(58,1)\}$,
$\{(14,0),(75,0),(86,0),(100,1)\}$,	$\{(2,0),(12,2),(39,2),(61,2)\}$,
$\{(10,0),(29,2),(82,2),(1,0)\}$,	$\{(9,0),(59,0),(83,0),(6,2)\}$,
$\{(28,0),(57,2),(16,2),(26,2)\}$,	$\{(92,0),(44,1),(23,0),(17,0)\}$,
$\{(97,0),(85,0),(74,1),(37,0)\}$,	$\{(62,0),(21,2),(70,0),(69,0)\}$,
$\{(31,0),(48,2),(51,1),(15,1)\}$,	$\{(67,0),(76,1),(89,2),(46,0)\}$,
$\{(80,0),(55,0),(49,1),(36,2)\}$,	$\{(65,0),(5,2),(40,1),(60,0)\}$,
$\{(66,0),(32,0),(103,0),(68,2)\}$,	$\{(95,0),(93,0),(63,1),(4,0)\}$,
$\{(8,0),(22,0),(50,0),(79,1)\}$,	$\{(42,0),(81,1),(24,2),(41,2)\}$,
$\{(94,0),(91,0),(87,1),(30,1)\}$.	

Here, we first develop these blocks $(-, \bmod 3)$ to get a parallel class. Then, we develop this parallel class $(\bmod 104, -)$ to obtain the $(4, 1)$-RGDD of type 12^{27} as required. □

Theorem 2.2. *The necessary conditions for the existence of a* $(4, 3)$*-RGDD of type* h^n*, namely,* $n \geq 4$ *and* $hn \equiv 0 \pmod 4$*, are also sufficient except for* $(h, n) \in \{(2, 4), (2, 6)\}$.

Proof. By Theorem 1.2, we only need to construct a $(4, 3)$-RGDD of type 2^{54}. Let the point set be $Z_{106} \cup \{x, y\}$, and let the group set be $\{\{j, j+53\} : j = 0, \ldots, 52\} \cup \{\{x, y\}\}$. Below are the required base blocks.

$\{x, 51, 23, 47\}$,	$\{y, 81, 38, 7\}$,	$\{0, 80, 19, 25\}$,
$\{58, 67, 73, 105\}$,	$\{87, 11, 82, 103\}$,	$\{54, 84, 102, 94\}$,
$\{17, 74, 69, 14\}$,	$\{61, 3, 68, 43\}$,	$\{30, 22, 34, 55\}$,
$\{9, 60, 96, 20\}$,	$\{15, 27, 83, 4\}$,	$\{36, 32, 88, 97\}$,
$\{6, 16, 62, 45\}$,	$\{72, 89, 79, 63\}$,	$\{2, 41, 21, 75\}$,
$\{35, 40, 18, 42\}$,	$\{95, 93, 24, 46\}$,	$\{50, 86, 66, 44\}$,
$\{52, 65, 8, 85\}$,	$\{29, 77, 64, 5\}$,	$\{104, 92, 78, 1\}$,
$\{91, 59, 90, 48\}$,	$\{10, 56, 28, 13\}$,	$\{76, 12, 39, 53\}$,
$\{26, 57, 70, 49\}$,	$\{71, 99, 37, 98\}$,	$\{100, 31, 101, 33\}$.

Here, the above base blocks form a parallel class. Then, we develop this parallel class +1 mod 106 to obtain a (4, 3)-RGDD of type 2^{54} as required. □

3. Recursive constructions

To describe our recursive constructions, we need the following auxiliary designs. For more detailed information on some of these related combinatorial structures, the reader is referred to [2,3,26].

A (K, λ)-*frame* is a GDD $(X, \mathcal{G}, \mathcal{B})$ in which the collection of blocks $\mathcal{B}$ can be partitioned into *holey parallel classes* each of which partitions $X \setminus G$ for some $G \in \mathcal{G}$. A *uniform* frame is a frame in which all groups are of the same size. The *group-type* (or *type*) of the frame is the multiset $\{|G| : G \in \mathcal{G}\}$. As with GDDs we shall use an "exponential" notation to describe the group-type. The following results are known.

Theorem 3.1 ([4,6,10,13,16,20,28]). *There exists a* $(4, 1)$*-frame of type* h^u *if and only if* $u \geq 5$, $h \equiv 0 \pmod 3$ *and* $h(u-1) \equiv 0 \pmod 4$, *except possibly where:*

1. $h = 36$ *and* $u = 12$;
2. $h \equiv 6 \pmod{12}$ *and*
   (a) $h = 6$ *and* $u \in \{7, 23, 27, 35, 39, 47\}$;
   (b) $h = 30$ *or* $h \in \{n : 66 \leq n \leq 2190\}$ *and* $u \in \{7, 23, 27, 39, 47\}$;
   (c) $h \in \{42, 54\} \cup \{n : 2202 \leq n \leq 11238\}$ *and* $u \in \{23, 27\}$;
   (d) $h = 18$ *and* $u \in \{15, 23, 27\}$.

A *transversal design* (TD) $\mathrm{TD}_\lambda(k, n)$ is a GDD of group-type n^k with block size k and index λ. A $\mathrm{TD}_\lambda(k, n)$ is *resolvable* if the corresponding GDD is resolvable. When $\lambda = 1$, we write $\mathrm{TD}_\lambda(k, n)$ as $\mathrm{TD}(k, n)$. A resolvable $\mathrm{TD}(k, n)$ (denoted by $\mathrm{RTD}(k, n)$) is equivalent to a $\mathrm{TD}(k+1, n)$. It is well known that the existence of a $\mathrm{TD}(k, n)$ is equivalent to the existence of $k-2$ mutually orthogonal Latin squares (MOLS) of order n. In this paper, we mainly employ the following known results on RTDs.

Lemma 3.2 ([3]). *An* $\mathrm{RTD}(4, n)$ *exists for all* $n \geq 4$ *except for* $n = 6$ *and possibly excepting* $n = 10$.

To obtain our main results, we shall use the following basic constructions. The proofs for these can be found in [5].

Construction 3.3 (*Breaking up Groups*). *If there exist a* (k, λ)*-RGDD of type* $(hm)^u$ *and a* (k, λ)*-RGDD of type* h^m*, then there exists a* (k, λ)*-RGDD of type* h^{mu}.

Construction 3.4 (*Weighting*). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a GDD, and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X*. Suppose that for each block* $B \in \mathcal{B}$*, there exists a* (k, λ)*-frame of type* $\{w(x) : x \in B\}$*. Then there is a* (k, λ)*-frame of type* $\{\sum_{x \in G} w(x) : G \in \mathcal{G}\}$.

Construction 3.5 (*Inflating RGDDs by RTDs*). *If there exist a* (k, λ)*-RGDD of type* h^u *and an* $\mathrm{RTD}_\mu(k, m)$*, then there exists a* $(k, \lambda\mu)$*-RGDD of type* $(mh)^u$.

Construction 3.6 (*Frame Constructions*). *Suppose there is a* (k, λ)*-frame with type* $T = \{t_i : i = 1, 2, \ldots, n\}$*. Suppose also that* $t|t_i$ *and that there exists a* (k, λ)*-RGDD of type* $t^{1+t_i/t}$ *for* $i = 1, 2, \ldots, n$*. Then there exists a* (k, λ)*-RGDD of type* t^u *where* $u = 1 + \sum_{i=1}^{n} \frac{t_i}{t}$.

4. (4, 2)-RGDDs of type h^n

In this section, we deal with the case of $\lambda = 2$. It is easy to see that the necessary conditions for the existence of (4, 2)-RGDDs of type h^n are the same as those of (4, 1)-RGDDs. Since the existence of (4, 1)-RGDDs simply implies that of (4, 2)-RGDDs, we only need to deal with the cases undetermined in Theorem 2.1.

Lemma 4.1. *There exists a* $(4, 2)$*-RGDD of type* 2^4.

Proof. Let the point set be $I_4 \times Z_2$, and let the group set be $\{\{j\} \times Z_2 : j = 0, 1, 2, 3\}$. Below are the required base blocks.

$$\begin{array}{c}
\{(0,0), (1,0), (2,0), (3,0)\},\\
\{(0,0), (1,0), (2,1), (3,1)\},\\
\{(0,1), (1,0), (2,1), (3,0)\},\\
\{(0,0), (1,1), (2,1), (3,0)\}.
\end{array}$$

Here, the single base block in each row gives a parallel class when it is developed by $(-, \bmod 2)$. In total, we have 4 such parallel classes. □

Lemma 4.2. *There exists a* $(4, 2)$*-RGDD of type* 2^{10}.

Proof. Let the point set be $(Z_6 \times I_3) \cup \{x, y\}$, and let the group set be $\{\{i, i+3\} \times \{j\} : i = 0, 1, 2, j = 0, 1, 2\} \cup \{\{x, y\}\}$. Below are the required base blocks.

$\{x, (0, 0), (0, 1), (0, 2)\}$,	$\{x, (0, 0), (4, 1), (2, 2)\}$,
$\{y, (1, 0), (1, 1), (1, 2)\}$,	$\{y, (1, 0), (5, 1), (3, 2)\}$,
$\{(2, 0), (3, 0), (4, 1), (5, 1)\}$,	$\{(2, 0), (4, 0), (1, 1), (3, 1)\}$,
$\{(4, 0), (5, 0), (2, 2), (3, 2)\}$,	$\{(3, 0), (5, 0), (0, 2), (4, 2)\}$,
$\{(2, 1), (3, 1), (4, 2), (5, 2)\}$,	$\{(0, 1), (2, 1), (1, 2), (5, 2)\}$.

Here, both the 5 base blocks listed in the left-hand column and the right-hand column form a parallel class. Then, we develop these two parallel classes (mod 6,—) to obtain the RGDD as required. □

Lemma 4.3. *There exists a* $(4, 2)$*-RGDD of type* 3^4.

Proof. Let the point set be $I_3 \times I_4$, and let the group set be $\{I_3 \times \{j\} : j = 0, 1, 2, 3\}$. Below are the required blocks.

$\{(0, 0), (0, 1), (0, 2), (0, 3)\}$,	$\{(1, 0), (1, 1), (1, 2), (1, 3)\}$,	$\{(2, 0), (2, 1), (2, 2), (2, 3)\}$,
$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$,	$\{(1, 0), (1, 1), (2, 2), (2, 3)\}$,	$\{(2, 0), (2, 1), (0, 2), (0, 3)\}$,
$\{(0, 0), (1, 1), (0, 2), (2, 3)\}$,	$\{(1, 0), (2, 1), (1, 2), (0, 3)\}$,	$\{(2, 0), (0, 1), (2, 2), (1, 3)\}$,
$\{(0, 0), (1, 1), (2, 2), (0, 3)\}$,	$\{(1, 0), (2, 1), (0, 2), (1, 3)\}$,	$\{(2, 0), (0, 1), (1, 2), (2, 3)\}$,
$\{(0, 0), (2, 1), (1, 2), (2, 3)\}$,	$\{(1, 0), (0, 1), (2, 2), (0, 3)\}$,	$\{(2, 0), (1, 1), (0, 2), (1, 3)\}$,
$\{(0, 0), (2, 1), (2, 2), (1, 3)\}$,	$\{(1, 0), (0, 1), (0, 2), (2, 3)\}$,	$\{(2, 0), (1, 1), (1, 2), (0, 3)\}$.

Here, the blocks in each row form a parallel class. □

Lemma 4.4. *There exists a* $(4, 2)$*-RGDD of type* 6^4.

Proof. Let the point set be $I_4 \times Z_6$, and let the group set be $\{\{j\} \times Z_6 : j = 0, 1, 2, 3\}$. Below are the required base blocks.

$\{(0, 0), (1, 0), (2, 0), (3, 0)\}$,	$\{(0, 0), (1, 2), (2, 3), (3, 0)\}$,
$\{(0, 1), (1, 1), (2, 2), (3, 2)\}$,	$\{(0, 1), (1, 4), (2, 4), (3, 5)\}$,
$\{(0, 2), (1, 3), (2, 1), (3, 5)\}$,	$\{(0, 2), (1, 5), (2, 1), (3, 4)\}$,
$\{(0, 3), (1, 2), (2, 5), (3, 4)\}$,	$\{(0, 3), (1, 1), (2, 5), (3, 1)\}$,
$\{(0, 4), (1, 5), (2, 4), (3, 3)\}$,	$\{(0, 4), (1, 0), (2, 2), (3, 3)\}$,
$\{(0, 5), (1, 4), (2, 3), (3, 1)\}$,	$\{(0, 5), (1, 3), (2, 0), (3, 2)\}$.

Here, both of the 6 base blocks listed in the left-hand column and the right-hand column form a parallel class. Then, we develop these two parallel classes (—, mod 6) to obtain the RGDD as required. □

Lemma 4.5. *There exists a* $(4, 2)$*-RGDD of type* 6^6.

Proof. Let the point set be Z_{36}, and let the group set be $\{\{j, j+6, j+12, j+18, j+24, j+30\} : j = 0, 1, 2, 3, 4, 5\}$. Below are the required base blocks.

$\{2, 28, 30, 35\}, \{3, 7, 24, 32\}, \{13, 17, 33, 34\}$,
$\{3, 5, 8, 30\}$,
$\{0, 1, 10, 23\}$.

Here, all the base blocks are developed by $+1$ mod 36. The blocks in the first row generate one parallel class when developed by $+12$ mod 36 and give in total 12 such classes. Each of the blocks in the second row and the third row generates one parallel class when developed by $+4$ mod 36. □

To get a conclusive result on the existence of $(4, 2)$-RGDDs, we need the existence results for $(4, 2)$-frames. To establish the existence results for $(4, 2)$-frames, we need the concept of skew Room frames.

Let X be a set, and let $\{H_1, \ldots, H_n\}$ be a partition of X. An $\{H_1, \ldots, H_n\}$*-Room frame* is an $|X| \times |X|$ array, F, indexed by X, which satisfies the properties:

1. every cell either is empty or contains an unordered pair of symbols of X,
2. the subarrays H_k^2 are empty, for $1 \le k \le n$ (these subarrays are referred to as *holes*),
3. each symbol of $X \setminus H_k$ occurs precisely once in row (or column) r, where $r \in H_k$,
4. the pairs occurring in F are precisely those $\{i, j\}$ where $(i, j) \in X^2 \setminus \cup_{k=1}^{n} H_k^2$.

A *skew Room frame* is a Room frame in which cell (i, j) is occupied if and only if cell (j, i) is empty.

The *type* of an $\{H_1, \ldots, H_n\}$-Room frame F will be the multiset $\{|H_1|, \ldots, |H_n|\}$. We will say that F has type $t_1^{u_1} \cdots t_k^{u_k}$ provided there are u_j H_i's of cardinality t_j, for $1 \le j \le k$.

From a skew Room frame of type h^n one can get a 4-GDD of type $(6h)^n$ (see [21]). The 4-GDD has groups $H_i \times Z_6$, $1 \le i \le n$. The block set $\mathcal{B}$ contains all blocks $\{(a, j), (b, j), (c, 1+j), (r, 4+j)\}$, where $j \in Z_6$, $\{a, b\} \in F$, $\{a, b\}$ occurs in column c and row r.

If all the quadruples (a, b, c, r) can be partitioned into sets such that each set forms a partition of $X \setminus H_i$ for some i, and each H_i corresponds to $2h$ of the sets, we call the skew Room frame *partitionable*.

Skew Room frames have played an important role in the constructions of BIBDs and GDDs with block size four (see [21]) and the resolution of the existence problem for weakly 3-chromatic BIBDs with block size four (see [22]). Partitionable skew Room frames were introduced by Colbourn, Stinson and Zhu in [4] to construct $(4, 1)$-frames and employed by Zhang and Ge in [27] to construct super-simple $(4, 2)$-frames. Here, we restate their construction as follows.

Lemma 4.6 (*[27, Lemma 3.2]*). *If there exists a partitionable skew Room frame of type h^n, then there exists a $(4, 2)$-frame of type $(3h)^n$.*

For the existence of partitionable skew Room frames of type h^n, we have the following known results.

Theorem 4.7 (*[27,28]*). *The necessary conditions for the existence of partitionable skew Room frames of type h^n, namely, $n \ge 5$ and $h(n-1) \equiv 0 \pmod 4$, are also sufficient except for $h^n \in \{1^5, 1^9\}$, and possibly when:*

1. $h \equiv 1 \pmod 2$ *and*
   (a) $h = 1$ *and* $n \in \{45, 57, 69, 77, 93\}$;
   (b) $h \in A = \{17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 79, 83\}$ *and* $n \in \{5, 9, 45, 57, 69, 77, 93\}$;
   (c) $h \equiv 1, 5 \pmod 6$ *and* $h \notin \{1\} \cup A$, *or* $h = 9$, *and* $n \in \{9, 57\}$;
2. $h \equiv 2 \pmod 4$ *and*
   (a) $h = 2$ *and* $n \in \{23, 27, 33, 39\}$;
   (b) $h = 6$ *and* $n \in \{9, 17, 27\}$;
   (c) $h = 18$ *and* $n \in \{23, 27\}$;
   (d) $h \equiv 2, 10 \pmod{12}$, $h \ge 10$, *and* $n \in \{23, 27, 39\}$;
   (e) $h \equiv 6 \pmod{12}$, $h \ge 30$, *and* $n = 27$;
3. $h \equiv 0 \pmod 4$
   (a) $h = 4$ *and* $n \in B = \{12, 14, 15, 16, 18, 20, 22, 24, 27, 28, 32, 34\}$;
   (b) $h = 8$ *and* $n \in \{8, 12\}$;
   (c) $h \in C = \{4k : k \in A\}$ *and* $n \in B \setminus \{15\}$;
   (d) $h \equiv 4, 20 \pmod{24}$, $h \ge 20$ *and* $h \notin C$, *or* $h = 36$, *and* $n \in \{12, 14\}$;
   (e) $h \equiv 8, 12, 16 \pmod{24}$, $h \notin \{8, 36, 40\} \cup C$, *and* $n = 12$.

Now we are in a position to state our result on $(4, 2)$-frames.

Theorem 4.8. *There exists a $(4, 2)$-frame of type h^u if and only if $u \ge 5$, $h \equiv 0 \pmod 3$ and $h(u-1) \equiv 0 \pmod 4$, except possibly where:*

1. $h = 36$ *and* $u = 12$;
2. $h \equiv 6 \pmod{12}$ *and*
   (a) $h = 6$ *and* $u \in \{23, 27\}$;
   (b) $h = 18$ *and* $u = 27$;
   (c) $h \in \{n : 30 \le n \le 11238\}$ *and* $u \in \{23, 27\}$.

Proof. Since the existence of $(4, 1)$-frames simply implies that of $(4, 2)$-frames, we only need to deal with the cases undetermined in Theorem 3.1. Combining Lemma 4.6 and Theorem 4.7, we have $(4, 2)$-frames of types 6^7, 6^{35}, 6^{47}, 18^{15} and 18^{23}. By Theorem 2.1, there exists a 4-RGDD of type 12^5. Completing all of its parallel classes gives a 5-GDD of type $12^5 16^1$. Applying Construction 3.4 with weight 3 gives a $(4, 2)$-frame of type $36^5 48^1$. Adjoin 6 infinite points and fill in the holes with $(4, 2)$-frames of types 6^7 and 6^9 to obtain a $(4, 2)$-frame of type 6^{39}. Finally, inflate the $(4, 2)$-frames of type 6^u constructed previously with an RTD$(4, t)$ for odd $t \ge 5$ to obtain the $(4, 2)$-frames of type $(6t)^u$ as desired. □

Lemma 4.9. *There exists a $(4, 2)$-RGDD of type 2^n for each $n \in \{34, 46, 52, 94, 100, 118, 130, 142, 178, 184, 202, 214, 238, 250, 334, 346\}$.*

Proof. For each given n, we write $n = 3u + 1$ with $u \in \{11, 15, 17, 31, 33, 39, 43, 47, 59, 61, 67, 71, 79, 83, 111, 115\}$. By Theorem 4.8, we have a $(4, 2)$-frame of type 6^u. Adjoining 2 infinite points and applying Construction 3.6 give a $(4, 2)$-RGDD of type 2^{3u+1}. Here, we need a $(4, 2)$-RGDD of type 2^4 as the input design, which comes from Lemma 4.1. □

Lemma 4.10. *There exist $(4, 2)$-RGDDs of types 2^{70} and 2^{82}.*

Proof. By Theorem 2.1, we have a $(4, 1)$-RGDD of type 20^7, which simply gives a $(4, 2)$-RGDD of type 20^7. Applying Construction 3.3 with a $(4, 2)$-RGDD of type 2^{10} coming from Lemma 4.2 gives a $(4, 2)$-RGDD of type 2^{70} as desired. On the other hand, we have a $(4, 2)$-frame of type 18^9 by Theorem 4.8. Adjoining 2 infinite points and applying Construction 3.6 with a $(4, 2)$-RGDD of type 2^{10} as the input design give a $(4, 2)$-RGDD of type 2^{82}. □

Lemma 4.11. *There exist $(4, 2)$-RGDDs of types 6^{54} and 6^{68}.*

Proof. By Theorem 2.1, we have (4, 1)-RGDDs of types 36^9 and 24^{17}, which simply give (4, 2)-RGDDs of types 36^9 and 24^{17}. Applying Construction 3.3 with a (4, 2)-RGDD of type 6^6 coming from Lemma 4.5 and a (4, 2)-RGDD of type 6^4 coming from Lemma 4.4 gives the desired (4, 2)-RGDDs of types 6^{54} and 6^{68} respectively. □

Lemma 4.12. *There exist* (4, 2)*-RGDDs of types* 9^{44}, 18^{18}, 18^{38}, 18^{62}, 36^{11}, 36^{14}, 36^{15}, 36^{18} *and* 36^{23}.

Proof. By Theorem 2.1, we have (4, 1)-RGDDs of types 3^{44}, 6^{18}, 6^{38}, 6^{62}, 12^{11}, 12^{14}, 12^{15}, 12^{18} and 12^{23}. Applying Construction 3.5 with an $RTD_2(4, 3)$ coming from Lemma 4.3 gives the desired (4, 2)-RGDDs. □

Theorem 4.13. *The necessary conditions for the existence of a* (4, 2)*-RGDD of type* h^n*, namely,* $n \geq 4$, $hn \equiv 0 \pmod 4$ *and* $h(n-1) \equiv 0 \pmod 3$*, are also sufficient.*

Proof. Combining Theorem 2.1, Lemmas 4.1–4.5 and Lemmas 4.9–4.12, the conclusion then follows. □

5. (4, λ)-RGDDs of type h^n for $\lambda \geq 4$

In this section, we discuss the existence of (4, λ)-RGDDs of type h^n for $\lambda \geq 4$. We begin with the following nonexistence result.

Lemma 5.1. *There does not exist a* (4, λ)*-RGDD of type* 2^4 *for any odd* λ.

Proof. Suppose there exists a (4, λ)-RGDD of type 2^4. Let the point set be $\mathcal{X} = \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$, the group set be $\{\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}, \{d_1, d_2\}\}$ and the block set be $\mathcal{B}$. For each block $A \in \mathcal{B}$, we denote $\mathcal{X} \setminus A$ as $\bar{A}$. Since $|\mathcal{X}| = 8$, the (4, λ)-RGDD has the property that $A \bigcup \bar{A}$ forms a parallel class for any block $A \in \mathcal{B}$, and we have in total 2λ such classes. Then, the block set $\mathcal{B}$ can be partitioned into the following 2 parts, each of which has 2λ blocks.

Part 1: Each block contains the element a_1.

In this part, we have in total 2λ blocks, where λ of them contain the element b_1 and the other λ blocks contain the element b_2. Denote the set of the former λ blocks as $\mathcal{B}_{11}$ and the set of the latter λ blocks as $\mathcal{B}_{12}$. Suppose there are x blocks in $\mathcal{B}_{11}$ containing the element c_1, where $0 \leq x \leq \lambda$. That is, we have x blocks containing the elements $\{a_1, b_1, c_1\}$ simultaneously. Now, we look at the pair $\{a_1, c_1\}$. It is easy to deduce that we have in total $\lambda - x$ blocks in $\mathcal{B}_{12}$ containing $\{a_1, b_2, c_1\}$, since the pair $\{a_1, c_1\}$ appears in λ blocks of the RGDD. Hence, we have $\lambda - (\lambda - x) = x$ blocks in $\mathcal{B}_{12}$ containing $\{a_1, b_2, c_2\}$.

Part 2: Each block contains the element a_2.

Similarly to the case of Part 1, we have in total 2λ blocks, where λ of them contain the element b_1 and the other λ blocks contain the element b_2. Denote the set of the former λ blocks as $\mathcal{B}_{21}$ and the set of the latter λ blocks as $\mathcal{B}_{22}$. Suppose there are y blocks in $\mathcal{B}_{22}$ containing the element c_2, where $0 \leq y \leq \lambda$. That is, we have y blocks containing the elements $\{a_2, b_2, c_2\}$ simultaneously. Now, we look at the pair $\{a_2, c_2\}$. It is easy to deduce that we have in total $\lambda - y$ blocks in $\mathcal{B}_{21}$ containing $\{a_2, b_1, c_2\}$, since the pair $\{a_2, c_2\}$ appears in λ blocks of the RGDD. Hence, we have $\lambda - (\lambda - y) = y$ blocks in $\mathcal{B}_{21}$ containing $\{a_2, b_1, c_1\}$.

Now, we look at the blocks containing the pair $\{b_1, c_1\}$. We have x blocks in $\mathcal{B}_{11}$ and y blocks in $\mathcal{B}_{21}$. Hence, we have $\lambda = x + y$, since the pair $\{b_1, c_1\}$ appears in λ blocks of the RGDD. Furthermore, it is easy to see that: if a block A contains $\{a_1, b_1, c_1\}$, then $\bar{A}$ contains $\{a_2, b_2, c_2\}$, and vice versa. Then, we have that the number of blocks containing $\{a_1, b_1, c_1\}$ is the same as that of blocks containing $\{a_2, b_2, c_2\}$. Consequently, we have $x = y$, which implies that λ must be even. □

With a similar proof to that of Lemma 5.1, we can get the following more general theorem.

Theorem 5.2. *For each integer* $k \geq 3$ *and any odd* λ*, there does not exist a* (k, λ)*-RGDD of type* 2^k.

Now, we deal with the case having 6 groups.

Lemma 5.3. *There exists a* (4, 6)*-RGDD of type* 2^6.

Proof. Let the point set be Z_{12}, and let the group set be $\{\{j, j+6\} : j = 0, 1, 2, 3, 4, 5\}$. Below are the required base blocks.

$$\begin{array}{l}\{0, 1, 2, 4\}, \{3, 7, 8, 11\}, \{5, 6, 9, 10\},\\ \{0, 1, 3, 10\},\\ \{0, 2, 5, 7\}.\end{array}$$

Here, all the base blocks are developed by +1 mod 12. The blocks in the first row form one parallel class and generate in total 12 parallel classes when they are developed by +1 mod 12. Each of the block in the second row and the third row generates one parallel class when developed by +4 mod 12. □

Lemma 5.4. *There exists a* (4, 9)*-RGDD of type* 2^6.

Proof. Let the point set be $(Z_2 \times Z_5) \cup \{x, y\}$, and let the group set be $\{Z_2 \times \{j\} : j = 0, 1, 2, 3, 4\} \cup \{\{x, y\}\}$. Below are the required base blocks.

$$\begin{array}{lll} \{x, (0, 0), (0, 1), (0, 2)\}, & \{y, (0, 3), (1, 0), (1, 4)\}, & \{(0, 4), (1, 1), (1, 2), (1, 3)\}, \\ \{x, (0, 4), (0, 0), (0, 2)\}, & \{y, (0, 1), (1, 2), (1, 3)\}, & \{(0, 3), (1, 4), (1, 0), (1, 1)\}, \\ \{x, (0, 0), (0, 2), (1, 3)\}, & \{y, (0, 3), (1, 4), (1, 1)\}, & \{(0, 1), (0, 4), (1, 2), (1, 0)\}. \end{array}$$

Here, the blocks in each row form a parallel class. First, we develop these three parallel classes (mod 2, −) to get 6 parallel classes. Then, we develop the resultant 6 parallel classes (−, mod 5) to obtain the RGDD as required. □

The following lemma is simple but useful.

Lemma 5.5. *If there exist both a $(4, \lambda)$-RGDD of type h^n and a $(4, \mu)$-RGDD of type h^n, then there exists a $(4, x\lambda + y\mu)$-RGDD of type h^n for any nonnegative integers x, y.*

Now, we are in a position to state our main result of this section.

Theorem 5.6. *The necessary conditions for the existence of a $(4, \lambda)$-RGDD of type h^n with $\lambda \geq 4$, namely, $n \geq 4$, $hn \equiv 0 \pmod 4$ and $\lambda h(n-1) \equiv 0 \pmod 3$, are also sufficient except for $(\lambda, h, n) = (2j + 1, 2, 4)$ with $j \geq 2$.*

Proof. For $n = 4$, $h = 2$ and odd λ, the nonexistence result is proved in Lemma 5.1. For $n = 4$, $h = 2$ and even λ, we can simply make copies of the $(4, 2)$-RGDD of type 2^4 coming from Lemma 4.1 to obtain the desired RGDDs.

For $n = 6$ and $h = 2$, we necessarily have that $\lambda \equiv 0 \pmod 3$ and $\lambda \geq 6$. The existence of a $(4, 6)$-RGDD of type 2^6 and a $(4, 9)$-RGDD of type 2^6 has been shown in Lemmas 5.3 and 5.4 respectively. For the other values of $\lambda \equiv 0 \pmod 3$ and $\lambda \geq 12$, we can write $\lambda = 6x + 9y$ with $x \geq 0$ and $y \geq 0$. Applying Lemma 5.5 gives the desired RGDDs.

For the remaining parameters of n, h and λ, we can employ Lemma 5.5 again as follows: If $\lambda \equiv 0 \pmod 3$, we can simply make copies of the $(4, 3)$-RGDDs of type h^n coming from Theorem 2.2. If $\lambda \not\equiv 0 \pmod 3$, we can write $\lambda = 2x + 3y$ with $x \geq 1$ and $y \geq 0$. The conclusion then follows by combining the existence results for $(4, 2)$-RGDDs of type h^n coming from Theorem 4.13 and $(4, 3)$-RGDDs of type h^n coming from Theorem 2.2. □

6. Concluding remarks

Now, we are in a position to state our main result of this paper.

Theorem 6.1. *The necessary conditions for the existence of a $(4, \lambda)$-RGDD of type h^n with $\lambda \geq 2$, namely, $n \geq 4$, $hn \equiv 0 \pmod 4$, and $\lambda h(n-1) \equiv 0 \pmod 3$, are also sufficient with the definite exceptions of $(\lambda, h, n) \in \{(3, 2, 6)\} \cup \{(2j + 1, 2, 4) : j \geq 1\}$.*

Proof. The conclusion follows from Theorems 2.2, 4.13 and 5.6. □

In this paper, we investigate the existence of resolvable group divisible designs with block size four, group-type h^n and general index λ. We give a complete solution for the cases of $\lambda \geq 2$. We also improve slightly the known result for the case of $\lambda = 1$. However, to complete the existence problem of resolvable group divisible designs with block size four and index unity, much work remains to be done.

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