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The toroidal crossing number of $K_{4,n}$

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ABSTRACT

In this paper, we show that the toroidal crossing number of $K_{4,n}$, $K_{1,3,n}$, $K_{2,2,n}$, $K_{1,1,2,n}$ and $K_{1,1,1,1,n}$ is $\lfloor\frac{n}{4}\rfloor[2n-4(1+\lfloor\frac{n}{4}\rfloor)]$. In addition, a new lower bound for the toroidal crossing number of $K_{m,n}$ has been obtained. We also discuss about the crossing number of $K_{4,n}$ in the general surfaces.

1. Introduction

Guy and Jenkyns [1] showed that the toroidal crossing number of $K_{3,n}$ is $\lfloor(n-3)^2/12\rfloor$. Richter and Širáň [5] generalized their result and obtained that the crossing number of $K_{3,n}$ in a surface with the Euler genus ϵ is $\lfloor n/(2\epsilon+2)\rfloor(n-(\epsilon+1))1+\lfloor n/(2\epsilon+2)\rfloor$. In [2], it is shown that the crossing number of $K_{4,n}$ on the projective plane is $\lfloor\frac{n}{3}\rfloor[2n-3(1+\lfloor\frac{n}{3}\rfloor)]$. In this paper, we prove that:

Theorem 1.1. *The toroidal crossing number of $K_{4,n}$ is*

$$\left\lfloor\frac{n}{4}\right\rfloor\left[2n-4\left(1+\left\lfloor\frac{n}{4}\right\rfloor\right)\right].$$

Moreover, the toroidal crossing numbers of $K_{1,3,n}$, $K_{2,2,n}$, $K_{1,1,2,n}$ and $K_{1,1,1,1,n}$ are found to be equal to the toroidal crossing number of $K_{4,n}$. In addition we apply arguments similar to those in [1] to obtain a new lower bound for the toroidal crossing number of $K_{m,n}$.

2. Definitions

In this paper, we assume that all the drawings of a graph are good, i.e. no two edges have more than one point in common, such a common point is either a vertex or is a crossing, and no more than two edges cross at a point.

We will denote a_i, a_j, a_k (b_i, b_j, b_k respectively) to be the vertices on the "n-side" ("4-side" respectively) of $K_{4,n}$. In a drawing D of $K_{4,n}$ on the torus, denote the drawing of $K_{4,n-1}$ yielded by deleting the vertex a_i in the drawing D by $D-a_i$. We denote by $cr_D(a_i,a_k)$ the number of crossings of edges, one incident to a_i, the other to a_k, and by $cr_D(a_i)$ the number of crossings on edges incident to a_i, that is, $cr_D(a_i)=\sum_{k=1}^{n}cr_D(a_i,a_k)$. Since D is good, $cr_D(a_i,a_i)=0$ for all i. We define the *toroidal crossing number of* D, $cr_1(D)$, to be: $cr_1(D)=\sum_{i=1}^{n}\sum_{k=i+1}^{n}cr_D(a_i,a_k)$. From this, we have $cr_1(D)=\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n}cr_D(a_i,a_k)$. The *toroidal crossing number* of $K_{4,n}$, $cr_1(K_{4,n})$, is defined to be the minimum crossing number among all good drawings of $K_{4,n}$ on the torus. Finally, let $f(n)$=$\lfloor\frac{n}{4}\rfloor[2n-4(1+\lfloor\frac{n}{4}\rfloor)]$.

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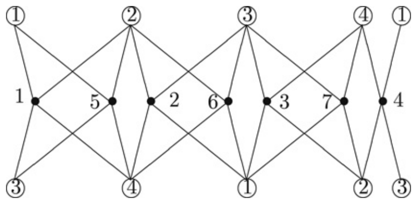


Fig. 1. Drawing of $K_{4,7}$.

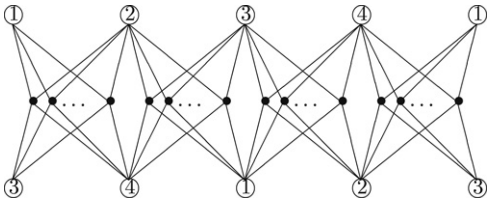


Fig. 2. Drawing of $K_{4,n}$.

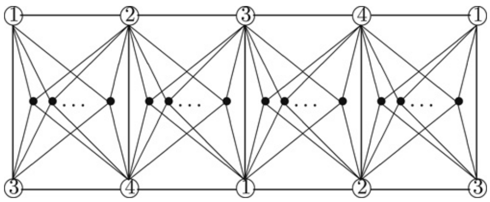


Fig. 3. Drawing of $K_{1,1,1,1,n}$.

3. Some general remarks

We note that in a crossing-free drawing of a (connected) subgraph of $K_{4,n}$ every circuit has an even number of vertices and in particular every region into which the edges divide the surface is bounded by an even circuit. So if t_j is the number of regions with j bounding edges, F the number of regions, E the number of edges, and V the number of vertices, then $t_j = 0$ if j is odd, $F = t_4 + t_6 + t_8 + \cdots$, and $2E = 4t_4 + 6t_6 + 8t_8 + \cdots$, and by Euler's theorem for the torus,

$$V \geq E - F, \quad (1)$$

$$V \geq t_4 + 2t_6 + 3t_8 + \cdots \geq F. \quad (2)$$

Suppose we have a drawing of $K_{4,n}$ on the torus with $cr_1(K_{4,n})$ crossings, and that by removing $cr_1(K_{4,n})$ edges, a crossing-free drawing is produced. Then (1) and (2) give $E - V = (4n - cr_1(K_{4,n})) - (4 + n) \leq F \leq V = 4 + n$, so

$$cr_1(K_{4,n}) \geq 2n - 8. \quad (3)$$

Fig. 1 and (3) show that

$$cr_1(K_{4,n}) = f(n), \quad n \leq 7. \quad (4)$$

Fig. 2 is a generalization of Fig. 1. It contains four vertices and $n = 4q + r (0 \leq r \leq 3)$ vertices represented by black dots, in four groups of $q + \epsilon_i$, $1 \leq i \leq 4$, $\epsilon_1 = \epsilon_2 = \cdots = \epsilon_r = 1$, $\epsilon_{r+1} = \cdots = \epsilon_4 = 0$. From this it follows that $cr_1(K_{4,n}) \leq rq(q + 1) + (4 - r)q(q - 1)$. Therefore, we have:

Lemma 3.1. $cr_1(K_{4,n}) \leq f(n)$.

In addition, Fig. 3 is a drawing of $K_{1,1,1,1,n}$ on the torus. This gives $cr_1(K_{1,1,1,1,n}) \leq f(n)$. It is clear that the toroidal crossing numbers of $K_{1,3,n}$, $K_{2,2,n}$, $K_{1,1,2,n}$ are not greater than that of $K_{1,1,1,1,n}$. Therefore, we have:

Lemma 3.2. $cr_1(K_{1,3,n}), cr_1(K_{2,2,n}), cr_1(K_{1,1,2,n}), cr_1(K_{1,1,1,1,n}) \leq f(n)$.

It is also obvious that the toroidal crossing numbers of $K_{1,3,n}$, $K_{2,2,n}$, $K_{1,1,2,n}$ and $K_{1,1,1,1,n}$ are not less than $K_{4,n}$. Therefore, by Theorem 1.1 and Lemma 3.2 we immediately have:

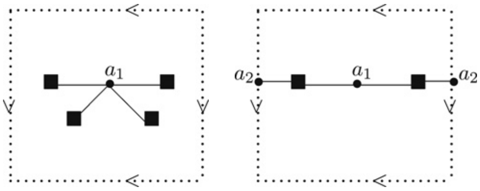


Fig. 4. (i) Drawing of the edges a_1b_1, a_1b_2, a_1b_3 and a_1b_4. (ii) Non-contractible cycle.

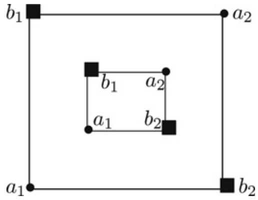


Fig. 5. After cutting along the non-contractible cycle.

Theorem 3.1.

$$cr_1(K_{1,3,n}) = cr_1(K_{2,2,n}) = cr_1(K_{1,1,2,n}) = cr_1(K_{1,1,1,1,n}) = f(n).$$

To prove Theorem 1.1, we need the following lemma from [5]:

Lemma 3.3. *If D is a drawing of $K_{m,n}$ on the torus such that, for some $k < n$, some $K_{m,k}$ is optimally drawn on the torus, then*

$$cr_1(D) \geq cr_1(K_{m,k}) + (n-k)(cr_1(K_{m,k+1}) - cr_1(K_{m,k})) + cr_1(K_{m,n-k}).$$

The proof can be found in [5]. We also need the following:

Lemma 3.4. *For any drawing D of $K_{4,n}$ on the torus, let A be the matrix defined by $A_{ij} = cr_D(a_i, a_j)$. Then it is impossible for the following to hold for some distinct i_j, $1 \leq j \leq 5$:*

$$\begin{pmatrix} A_{i_1i_1} & A_{i_1i_2} & A_{i_1i_3} & A_{i_1i_4} & A_{i_1i_5} \\ A_{i_2i_1} & A_{i_2i_2} & A_{i_2i_3} & A_{i_2i_4} & A_{i_2i_5} \\ A_{i_3i_1} & A_{i_3i_2} & A_{i_3i_3} & A_{i_3i_4} & A_{i_3i_5} \\ A_{i_4i_1} & A_{i_4i_2} & A_{i_4i_3} & A_{i_4i_4} & A_{i_4i_5} \\ A_{i_5i_1} & A_{i_5i_2} & A_{i_5i_3} & A_{i_5i_4} & A_{i_5i_5} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (5)$$

Proof. To prove Lemma 3.4 by contradiction, we may assume that there exists a drawing D of $K_{4,5}$ such that (5) holds with $a_{i_j} = a_j$ for $1 \leq j \leq 5$.

To consider a drawing of $K_{4,5}$, we use ■ and • to denote the vertices in "4-side" and "5-side" respectively. Note that the torus can be viewed as a square with its opposite sides identified. By deformation of the edges without changing the crossings, we can assume that the edges a_1b_1, a_1b_2, a_1b_3 and a_1b_4 are drawn as in Fig. 4(i).

By (5), the $K_{2,4}$ containing a_1 and a_2 must have a region whose boundary contains all b_i (otherwise, $cr_D(a_i, a_j) \geq 1$ for some $i = 1, 2$ and $j = 3, 4, 5$). Then there must exist distinct i, j such that the cycle $a_1b_ia_2b_j$ is not contractible (since there is no planar drawing of $K_{2,4}$ such that all b_i are lying in one region). Therefore, we may assume that the cycle $a_1b_1a_2b_2$ is not contractible, which is drawn as in Fig. 4(ii).

In Fig. 4(ii), if we cut the torus along the cycle $a_1b_1a_2b_2$, we can obtain a surface which is isomorphic to the torus with the inner and outer boundaries being the cycle $a_1b_1a_2b_2$, as shown in Fig. 5.

Since $cr_D(a_1, a_2) = 0$ by (5), the edges a_1b_i and a_2b_j do not cross. Therefore, there are 8 ways of drawing the edges a_1b_3 and a_2b_3 in Fig. 5 (as shown in Fig. 6(i)–(viii)), depending on which a_1 and a_2 in Fig. 5 that b_3 are connected to.

By exactly the same reason, there are 8 ways of drawing the edges a_1b_4 and a_2b_4 in Fig. 5 (as shown in Fig. 6(i)–(viii) by replacing b_3 by b_4). Hence, combining these different ways of drawing edges a_1b_3, a_2b_3, a_1b_4 and a_2b_4, we can obtain all the possible drawings of the $K_{2,4}$ containing a_1 and a_2 which satisfies $cr_D(a_1, a_2) = 0$ (see Appendix of [3] for all the drawings). However, as we have mentioned, the $K_{2,4}$ containing a_1 and a_2 must have a region whose boundary contains all b_i. Therefore, some of the drawings are forbidden (e.g. Fig. 7(ix) and (x)). Up to symmetry, all possible drawings of the $K_{2,4}$ containing a_1 and a_2 are shown in Fig. 7(i)–(viii) (One can check that Fig. 7(i)–(vii) exhaust all the cases by looking at the drawings in the Appendix of [3]).

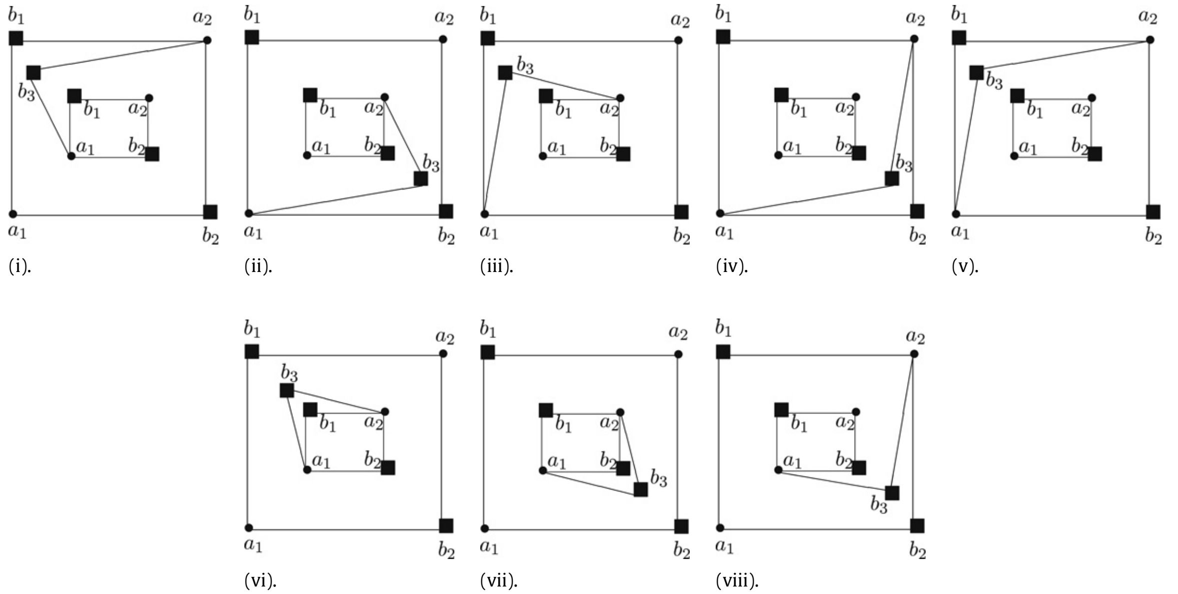


Fig. 6. Possible drawings of the edges a_1b_3 and a_2b_3.

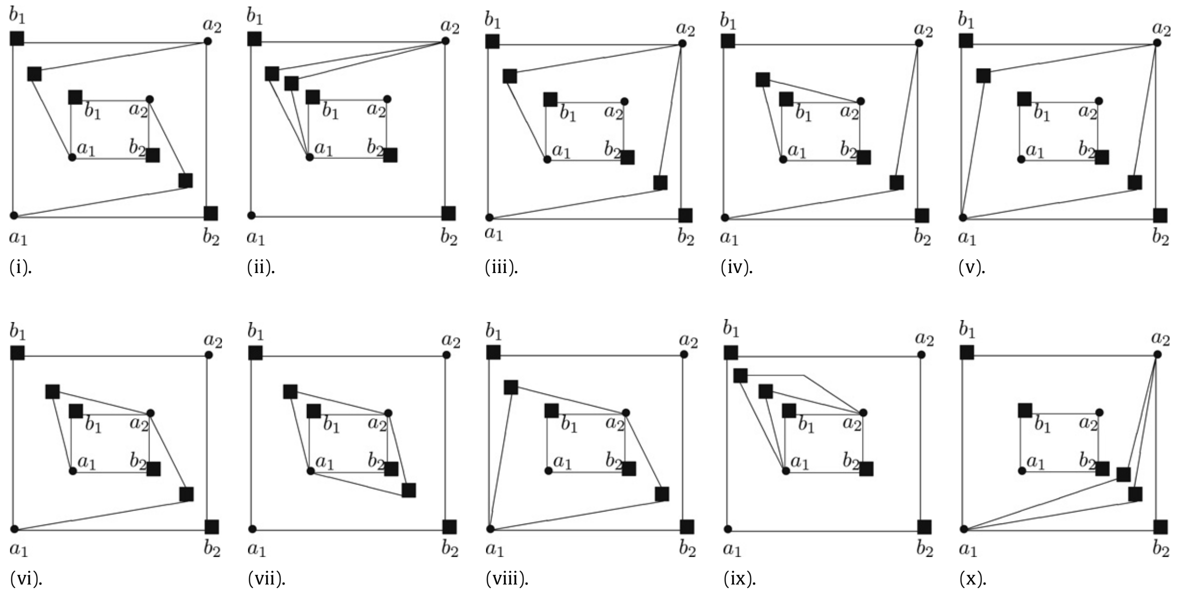


Fig. 7. Drawings of $K_{2,4}$.

By (5), $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$. This implies that a_3 must be drawn in a region of Fig. 7(i)–(viii) such as the region whose boundary contains all b_i. By checking Fig. 7(i)–(viii), it can be checked that the region whose boundary contains all b_i must be in the form of Fig. 8(i)–(iii), which is unique up to renaming the vertices b_i (see the following table).

7(i)	7(ii)	7(iii)	7(iv)	7(v)	7(vi)	7(vii)	7(viii)
8(ii)	8(iii)	8(iii)	8(i)	8(i)	8(iii)	8(i)	8(ii)

Note that the thick lines in Fig. 8(i)–(iii) denote the boundary of that region which is formed by some edges in the form of a_ib_j, $1 \leq i \leq 2$, $1 \leq j \leq 4$.

First, we consider Fig. 8(i). If a_3 lies in the region in the form of Fig. 8(i), by (5), $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$. This implies that the edges a_3b_i cannot cross the thick lines. (Remember the thick lines represent the edges formed by a_1b_j and a_2b_i for some $1 \leq i, j \leq 4$.) Therefore the edges a_3b_i where $1 \leq i \leq 4$ must be drawn as in Fig. 9(i), (ii) or (iii), where $\bullet$ represents a_3.

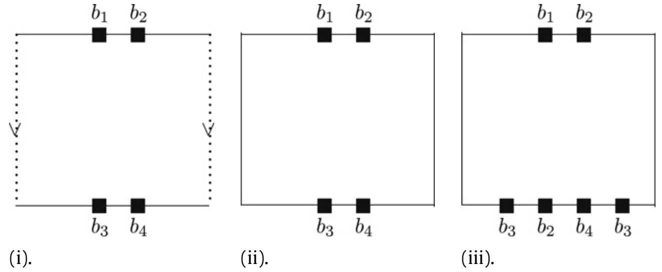


Fig. 8. The region whose boundary contains all b_i.

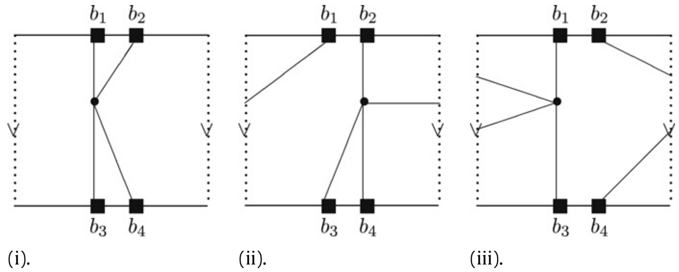


Fig. 9. Drawing the edges a_3b_i in Fig. 8(i).

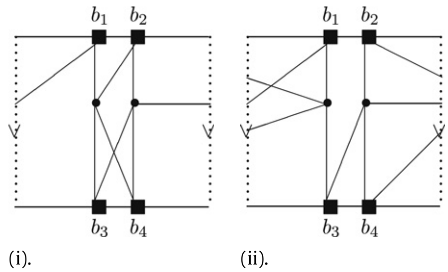


Fig. 10. Drawing the edges a_3b_i and a_4b_i in Fig. 8(i).

Note that Fig. 7(iv), 7(v) and 7(vii), which contains a region as in Fig. 8(i), has only one region whose boundary contains all b_i. Therefore a_4, a_5 must lie in a region in Fig. 9(i), 9(ii) or 9(iii) (otherwise, $cr_D(a_i, a_j) \geq 1$ for some $i = 1, 2, j = 3, 4, 5$ which contradicts (5)).

By (5), $cr_D(a_i, a_4) = 0$ for $i = 1, 2$. This implies that the edges a_4b_i where $1 \leq i \leq 4$ cannot cross the thick lines in Fig. 9(i), (ii) and (iii). Moreover, $cr_D(a_3, a_4) = 1$ by (5). One can check that $1 \leq i \leq 4$ must be drawn as in Fig. 10(i) or (ii) where a_3 and a_4 are represented by •.

However, in Fig. 10(i) and (ii), no matter which region a_5 lies in, one of the following happens: (1) $cr_D(a_i, a_5) \geq 1$ for some $i = 1, 2$, (2) $cr_D(a_i, a_5) \geq 2$ for some $i = 3, 4$. This contradicts (5).

Next, we consider Fig. 8(ii). Note that Fig. 7(i) and (viii), which contains a region as in Fig. 8(ii), has exactly two regions whose boundary contains all b_i, and these regions are in the form of Fig. 8(ii). By (5), a_3, a_4, a_5 must lie in one of these two regions (otherwise, $cr_D(a_i, a_j) \geq 1$ for some $i = 1, 2, j = 3, 4, 5$). Therefore, we may assume that a_3 and a_4 lie in the same region in the form of Fig. 8(ii).

Since $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$ by (5), the edges a_3b_j where $1 \leq j \leq 4$ cannot cross the thick lines in Fig. 8(ii). (Remember the thick lines represent the edges formed by a_1b_j and a_2b_i for some $1 \leq i, j \leq 4$.) Therefore, a_3b_j where $1 \leq j \leq 4$ must be drawn as in Fig. 11 where • represents a_3.

However, if a_4 lies in Fig. 11, then one of the following happens: (1) $cr_D(a_i, a_5) \geq 1$ for some $i = 1, 2$, (2) $cr_D(a_3, a_4) \geq 2$ for some $i = 1, 2$. This contradicts (5).

Finally we consider Fig. 8(iii). If a_3 lies in the region in the form of Fig. 8(iii), then by the fact that $cr_D(a_1, a_3) = cr_D(a_2, a_3) = 0$, a_3b_j cannot cross the thick lines. Therefore, a_3b_i where $1 \leq i \leq 4$ must be drawn as in Fig. 12(i) and (ii) where • represents a_3.

Note that Fig. 7(ii), (iii) and (vi), which contains a region as in Fig. 8(iii), has only one region whose boundary contains all b_i. Therefore, by (5), a_4, a_5 must lie in a region of Fig. 12(i) or (ii) (otherwise, $cr_D(a_i, a_j) \geq 1$ for some $i = 1, 2, j = 4, 5$).

By (5), $cr_D(a_1, a_4) = cr_D(a_2, a_4) = 0$, $cr_D(a_3, a_4) = 1$. Therefore, if a_4 is drawn in Fig. 12(i) or (ii), a_4b_i where $1 \leq i \leq 4$ must be drawn as in Fig. 13 where a_3 and a_4 are represented by •.

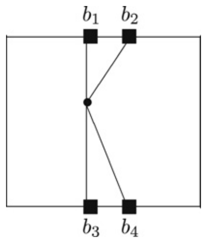
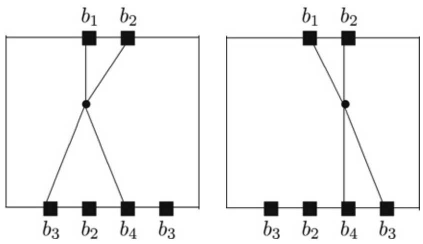


Fig. 11. Drawing the edges a_3b_i in Fig. 8(ii).



(i). (ii).

Fig. 12. Drawing the edges a_3b_i in Fig. 8(iii).

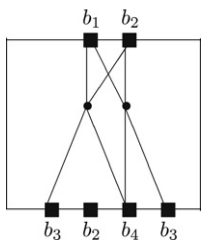


Fig. 13. Drawing the edges a_3b_i and a_4b_i in Fig. 8(iii).

However, in Fig. 13, no matter which region a_5 lies in, one of the following happens: (1) $cr_D(a_i, a_5) \geq 1$ for some $i = 1, 2$ (2) $cr_D(a_i, a_5) \geq 2$ for some $i = 3, 4$. This contradicts (5). □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 is by induction on n. The base of the induction is $n \leq 7$ and has been obtained in (4).

Now, consider $n \geq 8$. Let $n = 4q + r$, $0 \leq r \leq 3$. By Lemma 3.1, we only need to show $cr_1(K_{4,n}) \geq f(n)$. Let D be any good drawing of $K_{4,n}$ on the torus. We will show that $cr_1(D) \geq f(n)$ by considering two cases. The two cases will depend on whether there is a $K_{4,4}$ in D drawn without crossings.

In the first case, suppose that there is a $K_{4,4}$ in D drawn without crossings. From (4), we have $cr_1(K_{4,5}) = 2$ and by the inductive assumption, $cr_1(K_{4,n-4}) \geq f(n-4)$. Thus, by applying Lemma 3.3 with $m = 4$ and $k = 4$, the number of crossings in D is at least $2(n-4) + f(n-4) = f(n)$.

In the second case, suppose that D is a drawing of $K_{4,n}$ in the torus such that no $K_{4,4}$ is drawn without crossings. Note that $K_{4,n}$ contains n subgraphs $K_{4,n-1}$, each of which contains at least $f(n-1)$ crossings by the induction hypothesis. A crossing arises from two of the n vertices, so a crossing will have been counted $n-2$ times. Hence

$$cr_1(D) \geq \frac{n}{n-2} f(n-1). \tag{6}$$

From (6), we have

$$cr_1(D) \geq \begin{cases} q(4q-2) - \dfrac{3}{2} - \dfrac{3}{2(4q-1)} & \text{if } n-1 = 4q, \\ q(4q) - 1 & \text{if } n-1 = 4q+1, \\ q(4q+2) - \dfrac{1}{2} + \dfrac{1}{2(4q+1)} & \text{if } n-1 = 4q+2, \\ q(4q+4) & \text{if } n-1 = 4q+3. \end{cases} \tag{7}$$

Since the crossing number is an integer, if $n-1=4q+2$ or $4q+3$, $cr_1(D)\geq f(n)$. Therefore we only need to deal with the following cases:

Case 1. $n-1=4q$;

Case 2. $n-1=4q+1$.

4.1. Case 1. $n-1=4q$

Since $n\geq 8$, $q\geq 2$ and the crossing number is an integer, by (7), $cr_1(K_{4,n})=f(n)$ or $f(n)-1$. By way of contradiction, we suppose that $cr_1(K_{4,n})=f(n)-1$ and D is the drawing of $K_{4,n}$ on the torus such that $cr_1(D)=f(n)-1$. Then by the induction assumption, $cr_1(D-a_i)\geq f(n-1)$ for all i. Therefore, $cr_D(a_i)+cr_1(D-a_i)=cr_1(D)$ implies that

$$cr_D(a_i)\leq f(n)-1-f(n-1)=2q-1. \tag{8}$$

By the same argument which yields (6), we have

$$cr_1(K_{4,n+1})\geq \frac{n+1}{n-1}cr_1(K_{4,n})=\frac{n+1}{n-1}(f(n)-1),$$

which gives $cr_1(K_{4,n+1})\geq q(4q)-2-\frac{1}{2q}$. Since the crossing number is an integer, we have

$$cr_1(K_{4,n+1})\geq f(n+1)-2. \tag{9}$$

Now, draw a new vertex a_{n+1} near the vertex a_{i_0} where $cr_D(a_{i_0})=\min_{1\leq i\leq n}cr_D(a_i)$. Then by connecting the vertices of "4-side" with a_{n+1} in the same way as a_{i_0} connects the vertices of "4-side", we obtain a drawing of $K_{4,n+1}$. Then we have $cr_D(a_j,a_{n+1})=cr_D(a_j,a_{i_0})$ for $j\neq i_0$ and $cr_D(a_{i_0},a_{n+1})\leq 2$. Therefore we obtain a drawing of $K_{4,n+1}$ on the torus whose crossing number is at most $cr_1(D)+2+cr_D(a_{i_0})=(f(n)-1)+2+cr_D(a_{i_0})=q(4q-2)+1+\min_{1\leq i\leq n}cr_D(a_i)$ which is at least $f(n+1)-2$ by (9). This gives

$$\min_{1\leq i\leq n}cr_D(a_i)\geq 2q-3. \tag{10}$$

By (8) and (10), for $1\leq i\leq n$, we have $cr_D(a_i)=2q-3, 2q-2$ or $2q-1$. Let t and s be the numbers of a_i such that $cr_D(a_i)=2q-1$ and $cr_D(a_i)=2q-2$ respectively. Thus the number of a_i such that $cr_D(a_i)=2q-3$ is $n-t-s$. By definition, $\frac{1}{2}\sum_{i=1}^{n}cr_D(a_i)=cr_1(D)=q(4q-2)-1$ which implies $(2q-1)t+(2q-2)s+(2q-3)(n-t-s)=2q(4q-2)-2$, which gives

$$2t+s=6q+1. \tag{11}$$

Since $t+s\leq n=4q+1$, from (11),

$$t\geq 2q. \tag{12}$$

Claim 1. If $i\neq j$ and $cr_D(a_i)=cr_D(a_j)=2q-1$, then $cr_D(a_i,a_j)>0$.

To prove Claim 1, suppose that there exists $i_0\neq j_0$ such that $cr_D(a_{i_0})=cr_D(a_{j_0})=2q-1$ and $cr_D(a_{i_0},a_{j_0})=0$. Denote the drawing of $K_{4,n-2}$ by deleting the vertices a_i and a_j by $D-\{a_i,a_j\}$, then we have $cr_1(D-\{a_{i_0},a_{j_0}\})=cr_1(D)-cr_D(a_{i_0})-cr_D(a_{j_0})+cr_D(a_{i_0},a_{j_0})=q(4q-2)-1-2(2q-1)=4q^2-6q+1$. But this is a contradiction since by the induction assumption, the number of crossings of $D-\{a_i,a_j\}$ is at least $cr_1(K_{4,n-2})=f(n-2)=4q^2-6q+2$. This proves Claim 1.

Then we must have

$$t=2q. \tag{13}$$

(Otherwise, if $t\neq 2q$, then by (12), we have $t>2q$. Thus there must exist $i\neq j$ such that $cr_D(a_i,a_j)=0$ and $cr_D(a_i)=cr_D(a_j)=2q-1$, which contradicts Claim 1.) By Claim 1, we also have

$$cr_D(a_i,a_j)=1 \quad \text{for } cr_D(a_i)=cr_D(a_j)=2q-1. \tag{14}$$

By (11) and (13), we have

$$s=2q+1. \tag{15}$$

Therefore, by renaming the vertices, we may assume that $cr_D(a_i)=2q-1$ for $i\leq 2q$ and $cr_D(a_i)\leq 2q-2$ for $i\geq 2q+1$. By Claim 1, we also have $cr_D(a_i,a_j)=0$ if $1\leq i\leq 2q$ and $2q+1\leq j\leq n$. Now consider the matrix A defined by $A_{ij}=cr_D(a_i,a_j)$. By our assumption, A is of the form:

$$\begin{pmatrix} J_{2q}-I_{2q} & O_{(2q)\times(2q+1)} \\ O^T_{(2q)\times(2q+1)} & \widetilde{A} \end{pmatrix}, \tag{16}$$

where J_{2q} is the square matrix of size $2q$ with all entries equaling 1; I_{2q} is the identity matrix of size of $2q$; $O_{(2q)\times(2q+1)}$ is the zero matrix of size $(2q)\times(2q+1)$; and $\widetilde{A}$ is the square matrix of size $2q+1$ and the sum of each column of $\widetilde{A}$ is $2q-2$.

Now, consider the first two columns of $\widetilde{A}$. Let $\widetilde{A}_{ij}$ be the (i,j)-entry of $\widetilde{A}$. Therefore $\widetilde{A}_{ij} = A_{2q+i\,2q+j}$. Since all entries of $\widetilde{A}$ are non-negative integers and $2q+1=(2q-2)+3$, there should be at least two distinct i such that $\widetilde{A}_{i1}=0$. By renaming the vertices if necessary, we can assume that

$$\widetilde{A}_{ij} = 0 \quad \text{for } 1 \le i, j \le 2. \tag{17}$$

Then by (16) and (17), considering the drawing of $K_{4,5}$ in D which contains $a_1, a_2, a_3, a_{2q+1}, a_{2q+2}$, (5) must hold for A, which contradicts Lemma 3.4.

4.2. Case 2. $n-1=4q+1$

By (7), $cr_1(K_{4,n}) = f(n)$ or $f(n)-1$. By way of contradiction, suppose that $cr_1(K_{4,n}) = f(n)-1$ and D is a drawing of $K_{4,n}$ on the torus such that $cr_1(D) = f(n)-1$.

Claim 1. *$cr_1(D-a_i) = f(n-1)$ for all i, or equivalently, $cr_D(a_i) = cr_1(D) - cr_1(D-a_i) = f(n)-1-f(n-1) = 2q-1$ for all i.*

To prove Claim 1, suppose on the contrary, there exists i_0 such that $cr_1(D-a_{i_0}) > f(n-1)$. By the same argument which yields (6), we have $(n-2)cr_1(D) \ge \sum_{i=1}^{n} cr_1(D-a_i)$. Moreover, by induction assumption, we have $cr_1(D-a_i) \ge f(n-1)$ for all i. Hence, we have $(n-2)cr_1(D) \ge \sum_{i\ne i_0}^{n} cr_1(D-a_i) + cr_1(D-a_{i_0}) > nf(n-1)$, which implies that $cr_1(D) > nf(n-1)/(n-2) = f(n)-1$. This contradicts our assumption that $cr_1(D) = f(n)-1$ and this proves Claim 1.

Claim 2. *For all i, j, we have $cr_D(a_i, a_j) \le 2$.*

To prove Claim 2, suppose there exists $i_0 \ne j_0$ such that $cr_D(a_{i_0}, a_{j_0}) \ge 3$. Then removing the vertex a_{i_0} from D, we obtain a drawing of $K_{4,n-1}$. Note that

$$\begin{aligned} &cr_{D-a_{i_0}}(a_{j_0}) = cr_D(a_{j_0}) - cr_D(a_{i_0}, a_{j_0}) \le 2q-1-3 = 2q-4 \\ &cr_1(D-a_{i_0}) = cr_1(D) - cr_D(a_{i_0}) = f(n)-1-(2q-1) = q(4q-2). \end{aligned} \tag{18}$$

Now, we add a vertex a near a_{j_0} and connecting the vertices of “4-side” with a in the same way as a_{j_0} connects the vertices of “4-side”, we obtain a new drawing of $K_{4,n}$ and denote it by D'. Note that

$$\begin{aligned} &cr_{D'}(a_i, a_j) = cr_D(a_i, a_j) \text{ for } i, j \ne i_0, j_0; \\ &cr_{D'}(a_i, a) = cr_D(a_{j_0}, a_j) \text{ for } i, j \ne i_0, j_0; \\ &cr_{D'}(a_{j_0}, a) \le 2. \end{aligned} \tag{19}$$

Hence, (18) and (19) give $cr_1(D') \le cr_1(D-a_{i_0}) + 2 + \sum_{k\ne i_0} cr_{D-a_{i_0}}(a_k, a_{j_0}) = cr_1(D-a_{i_0}) + 2 + cr_{D-a_{i_0}}(a_{j_0}) \le q(4q-2)+2+2q-4 = 4q^2-2$, which is impossible since we have assumed that $cr_1(K_{4,n}) = f(n)-1 = 4q^2-1$. This proves Claim 2.

Claim 3. *If $cr_D(a_i, a_j) = 2$, then $cr_D(a_i, a_k) = cr_D(a_j, a_k)$ for all $k \ne i, j$.*

To prove Claim 3, we suppose that $cr_D(a_i, a_j) = 2$ and $cr_D(a_i, a_k) \ne cr_D(a_j, a_k)$ for some $k \ne i, j$. We may suppose that $cr_D(a_i, a_k) > cr_D(a_j, a_k)$. Now delete the vertex a_i, we obtain a drawing of $K_{4,n-1}$, denote it by $D-a_i$. Then, we have

$$\begin{aligned} &cr_{D-a_i}(a_k) = cr_D(a_k) - cr_D(a_i, a_k) < 2q-1-cr_D(a_j, a_k); \\ &cr_{D-a_i}(a_j) = cr_D(a_i) - cr_D(a_i, a_j) = 2q-1-2 = 2q-3; \\ &cr_1(D-a_i) = cr_1(D) - cr_D(a_i) = f(n)-1-(2q-1) = q(4q-2). \end{aligned} \tag{20}$$

By adding a vertex a near a_j, and connecting the vertices of “4-side” with a in the same way as a_j connects the vertices of “4-side”, we obtain a new drawing of $K_{4,n}$ and denote the drawing by D'. Note that

$$\begin{aligned} &cr_{D'}(a_j, a) \le 2 \\ &cr_{D'}(a_l, a) = cr_{D-a_i}(a_l, a_j) \text{ for } l \ne j. \end{aligned} \tag{21}$$

From (20) and (21), $cr_1(D') \le cr_1(D-a_i) + 2 + cr_{D-a_i}(a_j) = q(4q-2)+2+2q-3 = f(n)-1$. On the other hand, $cr_{D'}(a_k) = cr_{D-a_i}(a_k) + cr_D(a_j, a_k) < cr_{D-a_i}(a_k) + cr_D(a_i, a_k) = cr_D(a_k) = 2q-1$. By the argument in Claim 1, the crossing number of D' must be at least $f(n)$. This proves Claim 3.

Now consider the matrix A defined by $A_{ij} = cr_D(a_i, a_j)$. By the property of $cr_D(a_i, a_j)$ and the assumption on D, A is an $n \times n$ symmetric matrix such that the diagonal elements are zero. By Claim 1, the sum of each column of A is $2q - 1$. By Claim 2, A_{ij} is 0, 1 or 2. Moreover, A has no zero 4×4 submatrix in the form:

$$\begin{pmatrix} A_{i_1i_1} & A_{i_1i_2} & A_{i_1i_3} & A_{i_1i_4} \\ A_{i_2i_1} & A_{i_2i_2} & A_{i_2i_3} & A_{i_2i_4} \\ A_{i_3i_1} & A_{i_3i_2} & A_{i_3i_3} & A_{i_3i_4} \\ A_{i_4i_1} & A_{i_4i_2} & A_{i_4i_3} & A_{i_4i_4} \end{pmatrix}. \tag{22}$$

Otherwise D has a $K_{4,4}$ drawn without crossings.

Now, consider the first two columns of A. Since all entries are non-negative integers and $n = 4q + 2 = 2(2q - 1) + 4$, there should be at least four i such that $A_{i1} = A_{i2} = 0$. By renaming the vertices if necessary, we can assume that $A_{ij} = 0$ for $1 \le i, j \le 2$. By the same reason, there must exist $i > 2$ such that $A_{i1} = A_{i2} = 0$. By renaming the vertices if necessary, we may assume that

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{23}$$

is zero.

Since D has no $K_{4,4}$ drawn without crossings, the submatrix

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{1i} \\ A_{21} & A_{22} & A_{23} & A_{2i} \\ A_{31} & A_{32} & A_{33} & A_{3i} \\ A_{i1} & A_{i2} & A_{i3} & A_{ii} \end{pmatrix} \tag{24}$$

must be non-zero for all $i \ge 4$. That is to say. $\sum_{k=1}^{3} A_{ik} \ge 1$ for each $i > 3$. On the other hand, there exists $i > 3$ such that the sum of $\sum_{k=1}^{3} A_{ik} \le 1$ (otherwise the sum of first three columns is at least $2(n - 3) = 2(4q - 1) > 3(2q - 1)$). These imply that there exists $i > 3$ such that $\sum_{k=1}^{3} A_{ik} = 1$. By renaming the vertices, we may assume that

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{25}$$

Note that it is also impossible for $\sum_{k=1}^{4} A_{ik} \ge 2$ for each $i \ge 5$ (otherwise, the sum of first four columns is at least $2 + 2(n - 4) = 8q - 2 > 4(2q - 1)$). Therefore, there exists i_0 such that $\sum_{k=1}^{4} A_{i_0k} \le 1$, by renaming the vertices, we can assume $i_0 = 5$. Moreover, it is impossible for $\sum_{k=1}^{4} A_{5k} = 0$, otherwise, A has a zero matrix in the form of (22). Therefore $\sum_{k=1}^{4} A_{5k} = 1$ which implies that exactly one of the A_{5k}, $1 \le k \le 4$, is 1. Note also that it is impossible for $A_{53} = 1$ or $A_{54} = 1$, otherwise, A has a zero matrix in the form of (22). Therefore $A_{51} = 1$ or $A_{52} = 1$. By renaming the vertices if necessary, we may assume that

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Since $cr_1(K_{4,6}) = 4$ by (4), we have $\sum_{k=1}^{5} A_{ik} \ge 2$ for each $i > 5$. Since the sum of the first column of A is $2q - 1$, the number of $i > 5$ such that A_{i1} is non-zero is at most $2q - 1$. Therefore the number of $i > 5$ such that $A_{i1} = 0$ is at least $n - 5 - (2q - 1) = 2q - 2$. We summarize all of these by stating the following:

Fact 1. *For each $i > 5$, we have $\sum_{k=1}^{5} A_{ik} \ge 2$. The number of $i > 5$ such that $A_{i1} \ne 0$ is at most $2q - 1$; and the number of $i > 5$ such that $A_{i1} = 0$ is at least $2q - 2$.*

We have the following:

Claim 4. *If $i > 5$ and $A_{i1} = 0$, it is impossible for $A_{ik} \le 1$ for all $2 \le k \le 5$ and $\sum_{k=2}^{5} A_{ik} \le 2$.*

We prove Claim 4 by contradiction. We assume that $A_{ik} \le 1$ for all $2 \le k \le 5$ and $\sum_{k=2}^{5} A_{ik} \le 2$ for some $i > 5$ with $A_{i1} = 0$. Then by Fact 1, we have $\sum_{k=2}^{5} A_{ik} = 2$. Since $A_{ik} \le 1$ for all $2 \le k \le 5$, we must have $(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = (0, 1, 1, 0, 0)$, $(0, 1, 0, 1, 0)$, $(0, 1, 0, 0, 1)$, $(0, 0, 1, 1, 0)$, $(0, 0, 1, 0, 1)$ or $(0, 0, 0, 1, 1)$. Note that it is impossible for $(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = (0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 0, 1, 0, 1)$ or $(0, 0, 0, 1, 1)$, otherwise A has a zero 4×4 submatrix of the form (22).

Therefore, $(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = (0, 1, 0, 0, 1)$ or $(0, 0, 1, 1, 0)$. Then, if $(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = (0, 1, 0, 0, 1)$, then by considering the drawing of $K_{4,5}$ in D which contains a_1, a_2, a_3, a_5, a_i, (5) holds which contradicts Lemma 3.4. Similarly, if $(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = (0, 0, 1, 1, 0)$, then by considering the drawing of $K_{4,5}$ in D which contains a_1, a_2, a_3, a_4, a_i, (5) holds which contradicts Lemma 3.4. This proves Claim 4.

By Claim 4, for all $i > 5$ such that $A_{i1} = 0$, we must have $\sum_{k=2}^{5} A_{ik} \geq 3$ or $A_{ik} = 2$ for some k, $2 \leq k \leq 5$. From Claim 3, we have

$$(A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}) = \begin{cases} (0, 2, 0, 0, 1) & \text{if } A_{i2} = 2, \\ (0, 0, 2, 1, 0) & \text{if } A_{i3} = 2, \\ (0, 0, 1, 2, 0) & \text{if } A_{i4} = 2, \\ (0, 1, 0, 0, 2) & \text{if } A_{i5} = 2. \end{cases} \tag{27}$$

From (27) and Claim 4, we have:

Fact 2. *If $i > 5$ such that $A_{i1} = 0$, then $\sum_{k=2}^{5} A_{ik} \geq 3$.*

Combining (26), Fact 1 and Fact 2, we have that the sum of first five columns is at least $4+2(2q-1)+3(2q-2) = 10q-4$, but this is a contradiction since the sum of first five columns is exactly $5(2q-1) = 10q-5$.

5. Lower bound for $cr_1(K_{m,n})$

In this section, we apply a similar argument in [1] to obtain a new lower bound for the toroidal crossing number of $K_{m,n}$.

If $p \leq m, q \leq n$, then $K_{m,n}$ contains $\binom{m}{p}\binom{n}{q}$ subgraphs $K_{p,q}$. If we count the minimum number of crossings in these, noting that each crossing arises from just two vertices among the m and just two among the n, so that it is counted $\binom{m-2}{p-2}\binom{n-2}{q-2}$ times,

$$cr_1(K_{m,n}) \geq \binom{m}{p}\binom{n}{q} cr_1(K_{p,q}) \Big/ \binom{m-2}{p-2}\binom{n-2}{q-2}, \text{ or}$$

$$cr_1(K_{m,n}) \geq \frac{mn(m-1)(n-1)}{pq(p-1)(q-1)} cr_1(K_{p,q}). \tag{28}$$

Hence, we have the following

Theorem 5.1. *$cr_1(K_{m,n}) \geq \frac{1}{6}\binom{m}{2}\lfloor\frac{n}{4}\rfloor[2n - 4(1 + \lfloor\frac{n}{4}\rfloor)]$ for $m \geq 4$.*

Proof. By taking $p = 4$ and $q = n$ in (28) and applying Theorem 1.1, the result follows. □

In [1], it has been shown that the toroidal crossing number of $K_{m,n}$ lies between $\frac{1}{15}\binom{m}{2}\binom{n}{2}$ and $\frac{1}{6}\binom{m-1}{2}\binom{n-1}{2}$ for sufficiently large m and n. Therefore Theorem 5.1 improved this lower bound for the toroidal crossing number of $K_{m,n}$.

6. Concluding remark

We conclude this paper by posing the following:

Conjecture 6.1. *The crossing number of $K_{4,n}$ on Σ_ϵ is given by*

$$\left\lfloor \frac{n}{\epsilon+2} \right\rfloor \left[2n - (\epsilon+2)\left(1 + \left\lfloor \frac{n}{\epsilon+2} \right\rfloor\right)\right],$$

where Σ_ϵ denotes the surface with the Euler genus ϵ. (The Euler genus of a surface Σ is $2h$ if Σ is the sphere with h handles and k if Σ is the sphere with k crosscaps.)

From [4,2] and Theorem 1.1, we know that Conjecture 6.1 is true for a sphere, projective plane and torus. We believe that one can use a similar technique in this paper to prove that these conjectures are true. Actually it is expected that the same technique in this paper can be used to show that the crossing number of $K_{4,n}$ on the Klein bottle is $f(n)$ by replacing some lemmas carefully. However, the corresponding Lemma 3.4 for the Klein bottle is not true:

There exists a drawing D of $K_{4,n}$ on the Klein bottle such that if A is the matrix defined by $A_{ij} = \widetilde{cr}_D(a_i, a_j)$, then (5) holds for some distinct i_j, $1 \leq j \leq 5$.

To see this, one can refer to Fig. 14, the drawing of $K_{4,5}$ on the Klein bottle with $a_j = a_{i_j}$ where ▲ represents the vertices a_3, a_4 and a_5. Therefore one may need some new techniques to prove that Conjecture 6.1 is true for the Klein bottle.

Moreover, we have:

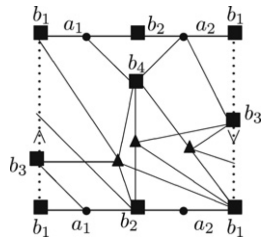


Fig. 14. Drawing of $K_{4,5}$ on the Klein bottle.

Proposition 6.1. *Denote the crossing number of $K_{4,n}$ in Σ_ϵ by $cr_{\Sigma_\epsilon}(K_{4,n})$. We have*

$$cr_{\Sigma_\epsilon}(K_{4,n}) \leq \left\lfloor \frac{n}{\epsilon+2} \right\rfloor \left[2n - (\epsilon+2)\left(1 + \left\lfloor \frac{n}{\epsilon+2} \right\rfloor\right)\right].$$

To prove this, we need the following lemma from [5] and its proof can be found in [5]:

Lemma 6.1. *If D is a drawing of $K_{m,n}$ in Σ_ϵ such that, for some $k < n$, some $K_{m,k}$ is optimally drawn in Σ_ϵ, then*

$$cr_{\Sigma_\epsilon}(D) \geq cr_{\Sigma_\epsilon}(K_{m,k}) + (n-k)(cr_{\Sigma_\epsilon}(K_{m,k+1}) - cr_{\Sigma_\epsilon}(K_{m,k})) + cr_{\Sigma_\epsilon}(K_{m,n-k}).$$

Proof of Proposition 6.1. By [6,7], we know that $K_{4,\epsilon+2}$ can be embedded in Σ_ϵ. Proposition 6.1 follows from Lemma 6.1 by taking $m = 4$ and $h = \epsilon + 2$. □

Moreover, by the same technique to obtain (1)–(3), one can show that, by using Euler's theorem for Σ_ϵ instead,

$$cr_{\Sigma_\epsilon}(K_{4,n}) \geq 2n - 4 - 2\epsilon. \tag{29}$$

Combining Proposition 6.1 and (29), we know that Conjecture 6.1 is true for $n \leq 2\epsilon + 4$, that is,

Corollary 6.1. *For $n \leq 2\epsilon + 4$, the crossing number of $K_{4,n}$ on Σ_ϵ is given by*

$$cr_{\Sigma_\epsilon}(K_{4,n}) = \left\lfloor \frac{n}{\epsilon+2} \right\rfloor \left[2n - (\epsilon+2)\left(1 + \left\lfloor \frac{n}{\epsilon+2} \right\rfloor\right)\right].$$

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