Results in Physics 8 (2018) 239–242

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

Physically elastic analysis of a cylindrical ring as a unit cell of a complete composite under applied stress in the complex plane using cubic polynomials

Vahid Monfared

Young Researchers and Elite Club, Zanjan Branch, Islamic Azad University, Zanjan, Iran

ARTICLE INFO

Article history:
Received 12 August 2017
Received in revised form 27 November 2017
Accepted 30 November 2017
Available online 11 December 2017

Keywords:
Elastic analysis
Cylindrical ring
Complex plane
Composite
Unit cell

ABSTRACT

Elastic analysis is analytically presented to predict the behaviors of the stress and displacement components in the cylindrical ring as a unit cell of a complete composite under applied stress in the complex plane using cubic polynomials. This analysis is based on the complex computation of the stress functions in the complex plane and polar coordinates. Also, suitable boundary conditions are considered and assumed to analyze along with the equilibrium equations and bi-harmonic equation. This method has some important applications in many fields of engineering such as mechanical, civil and material engineering generally. One of the applications of this research work is in composite design and designing the cylindrical devices under various loadings. Finally, it is founded that the convergence and accuracy of the results are suitable and acceptable through comparing the results.

Introduction

In the engineering and scientific problems, complex variable method presents an influential approach for solution of many scientific problems in applied mechanics such as plasticity and elasticity problems theoretically. The approach may be used in the anisotropic, thermo-plastic, visco-plastic, plasticity and creep problems or other mathematical aspects like computation of inappropriate integrals analytically. The complex variable method enables the numerous problems to be analyzed and solved that are inflexible and intractable by other schemes like usual and classic methods. The elastic deformation of the fibrous composites is an extremely important characteristic of materials which are under the mechanical loadings.

Widespread and extended studies were previously conducted to make use of the complex variables in elasticity and mathematical problems analytically [1–22]. The complex variable theory was initially applied and formulated in the elasticity problems by Kolosov in 1909 [1].

General and widespread texts on complex variable method were developed by Muskhelishvili [2], Milne-Thomson [3], Green and Zerna [4] and England [5]. Also, extra concise references of information can be found in Sokolnikoff and Little [6,7]. Supplementary information on complex variables can be found in the mathematical texts by Churchill and Kreyszig [8,9].

Some classes of bounded mappings and pure mathematics theorems were investigated using theory of applied complex functions [10–13]. For example, Gao [11] discussed both the displacement function and the Airy stress function methods which are based on an extended version of Green's theory. Obtaining the stress and displacement fields in arbitrary bounded or unbounded areas under to a given surface tractions is one of the most noteworthy problems in theory of plane elasticity. As an interesting and important research work, a study has been done on several properties of strongly starlike mappings of order α $(0 < \alpha < 1)$ and bounded convex mappings on domain B^n [12]. Also, a researcher determined the form of polynomially bounded solutions to the Loewner differential equation that was satisfied by univalent subordination chains of the form of exponential function [13].

The analysis of the stresses and the displacements in the annular domains such as a elastic cylindrical ring is essential in designing, manufacturing, and optimizing industrial tools and devices. For instance, in the cold rolling process of thin strips, observed that the rollers are under elastic deformations in the roll-bite zone. Computing these elastic deformations (or all displacement components) precisely in the roll-bite zone affects the final anticipated values of the total rolling load [23–25] and qualities of productions. Additionally, recently, an investigator studied the topic of energy and transmissibility in nonlinear viscous base isolators considering

E-mail addresses: vahid_monfared@alum.sharif.edu, vahid_monfared_57@yahoo.com

https://doi.org/10.1016/j.rinp.2017.11.036
2211-3797/

creeping phenomenon [26]. In addition, recently, two applied and important research works have been done about the improvement of the mechanical properties of the reduced graphene oxide (RGO)/Cu composites [28], and broadband dielectric/electric properties of multiwalled carbon nanotubes (MWCNT) polymethylmetacrylate composites [29]. Also, newly two applied and important research works have been carried out on the subject of composites experimentally [30,31].

In this present paper, the elastic analysis of a cylindrical ring as a unit cell of a complete composite under applied stress in the complex plane is done utilizing cubic polynomials theoretically. An important aspect of the research is in application of it in the pure and applied sciences and engineering problems for calculating and predicting the stress and displacement behaviors elastically. The significant aspects and advantages of the method are totally explained and emphasized in [2] in order for clarifying the application of the method for engineering and scientific problems.

Material and method

To solve and analyze the mentioned problem, a cylindrical ring shown in Fig. 1 is considered and assumed as the following. That is, a cylindrical ring is supposed as a unit cell, representative of a complete composite, under applied stress in the complex plane schematically.

Analysis and solution of the various problems of the plane stress and plane strain in absence of body forces resulted to solution of the biharmonic equation. Consequently, the below equation is clearly resulted [1–10],

$$\nabla^4\varphi = 0 \tag{1}$$

In which, "φ" and "∇^4" are stress function and biharmonic operator respectively. Moreover, the stress components in the Cartesian coordinates are defined as below,

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2} \tag{2}$$

$$\sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2} \tag{3}$$

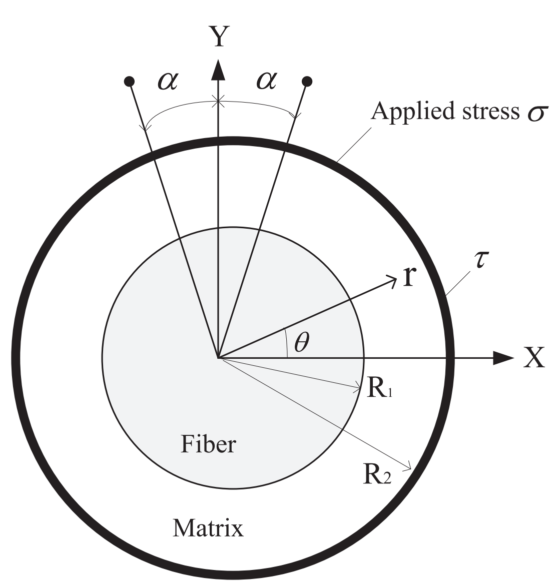

Fig. 1. Cylindrical ring as a unit cell representative with general loadings on boundaries.

$$\tau_{xy}(=\sigma_{xy}) = -\frac{\partial^2\varphi}{\partial x\partial y} \tag{4}$$

Also, the stress components in the polar coordinates are defined by the following relations,

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} \tag{5}$$

$$\tau_{r\theta}(=\sigma_{r\theta}) = \frac{\partial}{\partial r}\left(-\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} \tag{7}$$

Eq. (1) may be expressed employing the complex variables as the following [1–10],

$$\frac{\partial^4\varphi}{\partial^2 z\partial^2\bar{z}} = 0 \tag{8}$$

Here z indicates the complex variable on the plane $z = x + iy$, $z \cong r\left[1 + \theta i - \theta^2/2 - i\theta^3/6\right]$ and $\theta = \tan^{-1}(y/x)$ and $\bar{z}$ is conjugate of z ($\bar{z} = x - iy$, $\bar{z} \cong r\left[1 - \theta i - \theta^2/2 + i\theta^3/6\right]$). Integrating the above equation, stress function in the terms of the complex variable can be expressed as the below form,

$$\varphi(z,\bar{z}) = \mathrm{Re}(\chi(z) + \bar{z}\gamma(z)) \tag{9}$$

where γ and χ are random functions of the indicated variables, and *Re* indicates the real part of the complex functions. Therefore, φ must be a real function. So, these demonstrations explain that Airy function can be formulated in the terms of two functions of the complex variable. The solution of all problems consists of the evaluation of the complex potential functions $\gamma(z)$ and $\chi(z)$, in the terms of σ_{xx}, σ_{yy} and σ_{xy} (τ_{xy}), which are obtained as the complex series by the following well-known expressions [1–5],

$$\sigma_{xx} + \sigma_{yy} = 2\left(\gamma'(z) + \overline{\gamma'(z)}\right) \tag{10}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\tau_{xy} = 2(\chi(z) + \bar{z}\gamma''(z)) \tag{11}$$

And, also with combinations of the mentioned equations have the below formulation in the polar coordinates,

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{xx} + \sigma_{yy} \tag{12a}$$

$$\sigma_{\theta\theta} - \sigma_{rr} + 2i\tau_{r\theta} = \left[1 + 2\theta i - 2\theta^2 - \frac{4i}{3}\right](\sigma_{yy} - \sigma_{xx} + 2i\tau_{xy}) \tag{12b}$$

By some operations have the below formulation,

$$\sigma_{rr} - i\tau_{r\theta} = \gamma'(z) + \overline{\gamma'(z)} - \left[1 + 2\theta i - 2\theta^2 - \frac{4i}{3}\right](\chi(z) + \bar{z}\gamma''(z)) \tag{13}$$

The complex displacement defined by $U = u_r + iu_\theta$. To compute and determine displacements from the stress function, the below equations are used [2–6],

$$(u_r + iu_\theta) = \frac{1}{2\mu}\left[1 - \theta i - \frac{\theta^2}{2} + \frac{i\theta^3}{6}\right]\left(\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\gamma'(z)}\right) \tag{14}$$

Here κ is the Kolosov coefficient [1] which is $3 - 4v$ for the plane strain and for the plane stress is $(3 - v)/(1 + v)$. Also, v and μ are respectively Poisson ratio and shear modulus. For simplicity of the formulations, it is assumed that $\gamma'(z) = \psi(z)$. The center of the ring is considered to be coincident with the origin of the coordinates. Using complex Fourier transformation, the boundary conditions may be expressed in the following form [2,3],

$$\sigma_{rr} - i\tau_{r\theta} = \sum_{-\infty}^{+\infty} A'_k \left[1 + k\theta i - \frac{k^2\theta^2}{2} - \frac{k^3\theta^3 i}{6}\right], \ (r = R_1) \tag{15}$$

$$\sigma_{rr} - i\tau_{r\theta} = \sum_{-\infty}^{+\infty} A''_k \left[1 + k\theta i - \frac{k^2\theta^2}{2} - \frac{k^3\theta^3 i}{6}\right], \ (r = R_2) \tag{16}$$

In above equations, A'_k and A''_k are unknown coefficients which are determined by the Fourier transformation along the rollers' boundaries. With combinations of the mentioned equations have the below formulation,

$$\overline{\psi(z)} + \psi'(z) - (\bar{z}\psi'(z) + \chi(z))\left[1 + 2\theta i - 2\theta^2 - \frac{4i}{3}\right] = \sum_{-\infty}^{+\infty} A'_k \left[1 + k\theta i - \frac{k^2\theta^2}{2} - \frac{k^3\theta^3 i}{6}\right], \ r = R_1 \tag{17a}$$

$$\overline{\psi(z)} + \psi'(z) - (\bar{z}\psi'(z) + \chi(z))\left[1 + 2\theta i - 2\theta^2 - \frac{4i}{3}\right] = \sum_{-\infty}^{+\infty} A''_k \left[1 + k\theta i - \frac{k^2\theta^2}{2} - \frac{k^3\theta^3 i}{6}\right], \ r = R_2 \tag{17b}$$

To predict the stress function in the terms of the complex functions $\gamma(z)$ and $\chi(z)$, they are defined as follows by a new function with order of three $\psi(z)$,

$$\psi(z) = A\left[(z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3}\right] + \sum_{-\infty}^{+\infty} a_k z^k \tag{18a}$$

$$\chi(z) = \sum_{-\infty}^{+\infty} a'_k z^k \tag{18b}$$

In Eqs. (18a, b), A is a real constant which appears to generalize the stress function in different zones [2,5]. Now, by combination of the mentioned equations have,

$$(u_r + iu_\theta) = \frac{1}{2\mu}\left[1 - i\theta - \frac{\theta^2}{2} + \frac{i\theta^3}{6}\right](2\pi i((\kappa + 1)Az + \kappa a_{-1} + a'_{-1})) \tag{19}$$

where the right side of above equation indicates the enhancing by the expression in parenthesis for the anti-clockwise boundary of the inner contour (inner surface of the hollow cylinder). Therefore, for the condition of single-value of the displacement, the left side of the above equation should be zero. All above functions are analytical.

Consequently, the stress and displacement components for cylinder rollers can be obtained by an analytical solution. It is to be noted that if the region had another shape it should be transformed to a circular one by mapping functions and then solving them for the new region.

Results and discussions

Here, the formulations obtained from previous section are used and applied to calculate the stress and displacement components in the cylindrical ring. It is recommendable to be noticed that the method can be used for many other engineering problems. The pressure distribution on the rings is a second order function given by the mentioned equations, according to many numerical and analytical results as below,

$$\sigma_{rr} - i\sigma_{r\theta} = P_1\left(1 - \left(\frac{\theta}{\alpha}\right)^2\right) - iP_2\left(\frac{\theta}{\alpha}\right)^2 \tag{20}$$

The radial displacement has the maximum value in $\theta = 0$ for any A. However, its maximum is reduced with increasing A.

In neighborhood of $\theta = \alpha$ and $A = 0$, the radial displacement is negative and it has different attitude versus θ. For $A = 0$ to $A = 0.2$, the variation of radial displacement with respect to r is much higher than that of $A = 0.2$ to $A = 1$. The following results and diagrams (Figs. 2 and 3) are the average value of the displacements. Also, the behavior of the displacement is stable nearing $\theta = 4$.

In Fig. 2 and according to the obtained results, it can be seen that with the increase of α, the tangential displacement (u_θ) increases.

That is, the gradient and behavior are positive and ascending respectively. According to the results have,

$$u_\theta = -0.0013\theta^6 + 0.0179\theta^5 - 0.0914\theta^4 + 0.2218\theta^3 - 0.2524\theta^2 + 0.1059\theta + 0.0001 \tag{21}$$

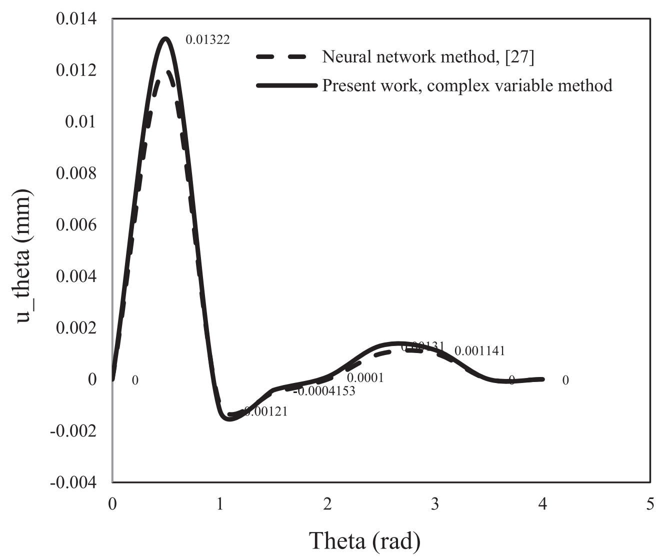

Fig. 2. Tangential displacement versus θ (theta). (See above-mentioned references for further information.)

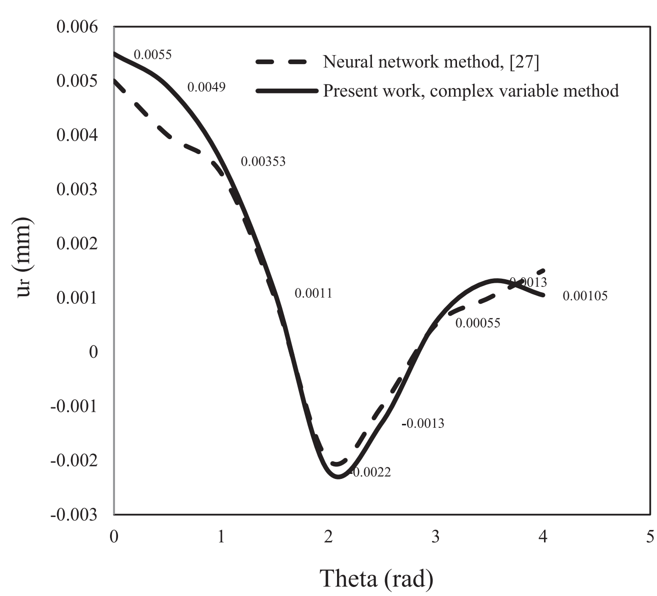

Fig. 3. Radial displacement versus θ (theta). (See above-mentioned references for further information.)

In this paper, by the method of try and error, sinusoidal, trigonometric and cubic functions are considered and assumed for analytical functions of $\gamma(z)$, $\chi(z)$ and $\psi(z)$ for some calculations. In which, they are compatible and suitable with the problem conditions. The radial displacement (u_r) has the maximum value in $\theta = 0$ for any *A*.

However, its maximum is reduced with increasing *A*. In neighborhood of $\theta = \alpha$ and *A = 0*, the radial displacement is negative and it has different attitude versus θ. For *A = 0* to *A = 0.2*, the variation of radial displacement with respect to *r* is much higher than that of *A = 0.2* to *A = 1*. In Fig. 3, as we increase the loading angle (θ) it leads to an increase in the maximum radial displacement (u_r). According to the results have,

$$u_r = 0.0003\theta^6 - 0.004\theta^5 + 0.018\theta^4 - 0.0354\theta^3 + 0.028\theta^2 - 0.0086\theta + 0.0055 \quad (22)$$

Conclusion

According to this research and based on the determined results of the displacement and stress behaviors, it is concluded that they are drastically functions of the loading angle and location in the radial direction interestingly. With an increase of the loading angle, the maximum amounts of the stresses and displacements increase.

In the neighborhood of the inner edge, all components of the stress and displacement are very sensitive to variations of the radius. However, away from inner edge to outer edge, its variations are decreased rapidly. Furthermore, none of the stress and displacement components in the neighbor of the end points of the loading angle have precise enough results because of existing of the singular points. Hence, it is very obvious, especially for tangential stress and displacement for those regions, calculated results are a little higher than they could be. Nevertheless, this results very reliable and acceptable or their deviations are little and ignorable, because of satisfying the conditions of the equilibrium and compatibility which are the bases of this method.

So, the prediction of the elastic behavior of a cylindrical ring as a unit cell of a composite under applied stress in the complex plane was performed using complex variable method analytically. Finally, one of the significant applications of the present research is in the pure and applied sciences such as prediction of the elastic behavior of the fibrous composites by a cylindrical ring as a unit cell. Also, we can approximately control the elastic behavior of a cylindrical ring, representative of a complete composite, in order for preventing undesired and dangerous events by cubic polynomials appropriately.

References

[1] Kolosov GV. On the application of complex function theory to a plane problem of the mathematical theory of elasticity. Diss Dorpat Univ 1909.
[2] Muskhelishvili NI. Some basic problems of the mathematical theory of elasticity, [English Trans] Noordhoff, Groningen, The Netherlands, 1963.
[3] Milne-Thomson LM. Plane elastic systems. 1st ed. Berlin: Springer Verlag; 1960.
[4] Green AE, Zerna W. Theoretical elasticity. 2nd ed. London: Oxford University Press; 1968.
[5] England AH. Complex variable methods in elasticity. 1st ed. New York: John Wiley; 1971.
[6] Sokolnikoff IS. Mathematical theory of elasticity. 1st ed. New York: McGraw-Hill; 1956.
[7] Little RW. Elasticity. 1st ed. New Jersey: Prentice Hall, Englewood Cliffs; 1973.
[8] Churchill RV. Complex variables and applications. 2nd ed. New York: McGraw-Hill; 1960.
[9] Kreyszig E. Advanced engineering mathematics. 8th ed. New York: John Wiley; 1999.
[10] Trjitzinsky WJ. Some general developments in the theory of functions of a complex variable. Acta Math 1939;70(1):63–163.
[11] Gao XL. Alternative derivation of Marguerre's displacement solution in plane isotropic elasticity. ASME J Appl Mech 2000;67:419–21.
[12] Hamada H, Kohr G. On some classes of bounded univalent mappings in several complex variables. Manuscripta Math 2010;131(3–4):487–502.
[13] Hamada H. Polynomially bounded solutions to the Loewner differential equation in several complex variables. J Math Anal Appl 2011;381(1):179–86.
[14] Chou SI, Shamas-Ahmadi M. Complex variable boundary element method for torsion of hollow shafts. Nucl Eng Des 1992;136:255–63.
[15] Mogilevskaya SG. The universal algorithm based on complex hypersingular integral equation to solve plane elasticity problems. Comput Mech 1996;18:127–38.
[16] Hui CY, Mukherjee S. Evaluation of hypersingular integrals in the boundary element method by complex variable techniques. Int J Solids Struct 1997;34(2):203–21.
[17] Mogilevskaya SG, Linkov AM. Complex fundamental solutions and complex variables boundary element method in elasticity. Comput Mech 1998;22:88–92.
[18] Mogilevskaya SG, Crouch SL. A Galerkin boundary integral method for multiple circular elastic inclusions. Int J Numer Methods Eng 2001;52:1069–106.
[19] Wang J, Crouch SL, Mogilevskaya SG. A complex boundary integral method for multiple circular holes in an infinite plane. Eng Anal Bound Elem 2003;27:789–802.
[20] Mogilevskaya SG, Crouch SL. A Galerkin boundary integral method for multiple circular elastic inclusions with uniform interphase layers. Int J Solids Struct 2004;41:1285–311.
[21] Liew KM, Feng C, Cheng Y, Kitipornchai S. Complex variable moving least – squares method: a meshless approximation technique. Int J Numer Methods Eng 2007;70:46–70.
[22] Voorhees A, Millwater HR, Bagley RL. Complex variable methods for shape sensitivity of finite element models. Finite Elem Anal Des 2011;47:1146–56.
[23] Hossford WF, Caddel RM. Metal forming mechanics & metallurgy. Cambridge University Press; 2011.
[24] Lenard JG. Primer on flat rolling. Amsterdam: Elsevier; 2007.
[25] Dbouk T, Montmitonnet P, Matsumoto H, Suzuki N, Takahama Y, Legrand N, Ngo T. Advanced roll bite models for cold and temper rolling processes, In: Proceeding of 9th International & 6th European Rolling Conference, Venice, Italy 2013.
[26] Markou Athanasios A, Manolis George D. Energy and transmissibility in nonlinear viscous base isolators. J Theor Appl Mech 2016;46(3):17–34.
[27] Monfared V. Neural network based simulation of the micro creeping fibrous composites for predicting plastic behavior with application in sensors, nonlinear optics, and optical devices analysis. J Theor Appl Mech Sofia 2017;47(1):36–48.
[28] Li Mei-Xia, Xie Juan, Li Yong-De, Xu Huan-Huan. Reduced graphene oxide dispersed in copper matrix composites: facile preparation and enhanced mechanical properties. Phys Status Solidi (a) 2015;212(10):2154–61.
[29] Kranauskaite Ieva, Macutkevic Jan, Banys Juras, Kuznetsov Vladimir L, Moseenkov Sergey I, Rudyna Nina A, Krasnikov Dmitriy V. Length-dependent broadband electric properties of PMMA composites filled with carbon nanotubes. Phys Status Solidi (a) 2016;213(4):1025–33.
[30] Fayomi OSI, Popoola API, Kanyane LR, Monyai T. Development of reinforced in-situ anti-corrosion and wear Zn-TiO2/ZnTiB2 coatings on mild steel. Res Phys 2017;7:644–50.
[31] Tang Xiaobing, Zhou Leping, Xiaoze Du, Yang Yongping. Enhancing absorption properties of composite nanosphere and nanowire arrays by localized surface plasmon resonance shift. Res Phys 2017;7:87–94.