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Solitary traveling wave solutions of pressure equation of bubbly liquids with examination for viscosity and heat transfer

Mostafa M.A. Khater [a], Aly R. Seadawy [b,c,*], Dianchen Lu [a,*]

[a] Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, PR China
[b] Mathematics Department, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia
[c] Mathematics Department, Faculty of Science, Beni-Suef University, Egypt

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ABSTRACT

In this research, we investigate one of the most popular model in nature and also industrial which is the pressure equation of bubbly liquids with examination for viscosity and heat transfer which has many application in nature and engineering. Understanding the physical meaning of exact and solitary traveling wave solutions for this equation gives the researchers in this field a great clear vision of the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer and also dynamics of contrast agents in the blood flow at ultrasonic researches. To achieve our goal, we apply three different methods which are extended tanh-function method, extended simple equation method and a new auxiliary equation method on this equation. We obtained exact and solitary traveling wave solutions and we also discuss the similarity and difference between these three method and make a comparison between results that we obtained with another results that obtained with the different researchers using different methods. All of these results and discussion explained the fact that our new auxiliary equation method is considered to be the most general, powerful and the most result-oriented. These kinds of solutions and discussion allow for the understanding of the phenomenon and its intrinsic properties as well as the ease of way of application and its applicability to other phenomena.

Introduction

The characteristic of bubbles is one of the most important phenomena and characteristics of some fluids, whether they are liquid fluids or liquid fluids. Due to the Marangoni effect, bubbles may stand by intact when they reach the surface of the impressive material. Bubbles form, and coalesce, into globular shapes, because those forms are at a lower energy state. For the physics and chemistry caused it, we can observe the bubbles as it have a various refractive index (IR) than the surrounding substance. For instance, the IR of air is approximately 1.0003 and the IR of water is approximately 1.333. Snell's Law characterized how electromagnetic waves shift direction at the interface between two mediums with various IR; thus bubbles can be specified from the accompanying refraction and internal reflection even though both the immersed and immersing mediums are transparent. The fierce collapse of bubbles (cavitation) near solid surfaces and the resulting impinging jet shape the mechanism applied in ultrasonic cleaning. The same influence, but on a larger scale, is used in concentrated energy weapons such as the bazooka and the torpedo. Pistol shrimp also use a collapsing cavitation bubble as a weapon. The same influence is used to treat kidney stones in a lithotripter. The damage by bubble pointing and growth in body tissues is the mechanism of decompression sickness, which occurs when supersaturated dissolved inert gases leave solution as bubbles during decompression. The damage can be due to mechanical deformation of tissues due to bubble growth in situ, or by blocking blood vessels where the bubble has lodged. Injury by bubble formation and growth in body tissues is the mechanism of decompression sickness, which occurs when supersaturated dissolved inert gases leave solution as bubbles during decompression. The damage can be due to mechanical deformation of tissues due to bubble growth in situ, or by blocking blood vessels where the bubble has lodged. Arterial gas embolism can occur when a gas bubble is introduced to the circulatory system and it lodges in a blood vessel which is too small for it to pass through under the available pressure difference. This can occur as a result of decompression after hyperbaric exposure, a lung over expansion injury, during

* Corresponding authors at: Mathematics Department, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia (A.R. Seadawy) and Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, PR China (D. Lu).

E-mail addresses: Aly742001@yahoo.com (A.R. Seadawy), dclu@ujs.edu.cn (D. Lu).

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intravenous fluid administration, or during surgery. Arterial gas embolism can occur when a gas bubble is introduced to the circulatory system and it lodges in a blood vessel which is too small for it to pass through under the available pressure difference. This can occur as a result of decompression after hyperbaric exposure, a lung over expansion injury, during intravenous fluid administration, or during surgery.All of what mentioned above gives a great importance of studying the physical properties of these models like our model (the Kudryashov Sinelshchikov equation) and for serving our goal, the mathematician discover, invited and improve some powerful methods [1–15]. For our model we used three different methods (extended tanh-function method, extended simple equation method and a new auxiliary equation method (Khater method) [16–20]). The remnant of this paper is systematized as follows: In Section 'Formulation for pressure equation of bubbly liquids with examination for viscosity and heat transfer', we use these methods to get the exact solutions of (NLPDE.) pointed out above. In Section 'Discuss the results', we make the comparison between the obtained results with each other and that obtained by using different method. In Section 'Conclusion', conclusions are given. Table 1.

Formulation for pressure equation of bubbly liquids with examination for viscosity and heat transfer

The volume and mass of the gas in the unit of the mass mixture can be written as

$$V=\frac{4}{3}\pi R^3 N,\ X=V\rho_g. \tag{2.1}$$

We consider the long wavelength perturbations in a mixture of the liquid and the gas bubbles assuming that characteristic length of waves of perturbation more than distance between bubbles. We also assume, that distance between bubbles much more than the averaged radius of a bubble. We describe dynamics of a bubble using the Rayleigh-Lamb equation. We also take the equation of energy for a bubble and the state equation for the gas in a bubble into account. The system of equation for the description of the gas bubble takes the form [21,22].

$$\begin{cases}\rho_1\left(RR_{tt}+\frac{3}{2}R_t^2+\frac{4\upsilon}{3R}R_t\right)=Q_y-Q,\\ Q_{g,t}+\frac{3nQ_g}{R}R_t+\frac{3\chi_g Nu(n-1)}{2R^2}\left(T_g-T_1\right)=0,\\ T_g=\frac{T_0 Q_g}{Q_{g,0}}\left(\frac{R}{R_0}\right)^3.\end{cases} \tag{2.2}$$

The expression for the density of a mixture can be presented in the form [23]

$$\frac{1}{\rho}=\frac{1-\chi}{\rho_t}+V\Rightarrow\rho=\frac{\rho_t}{1-\chi+V_{\rho t}}. \tag{2.3}$$

Table 1
Terminology of all characters that used in our research.

1,0,0Arbitrary constant	1,0,0Definition
$R=R(x,t)$	Bubble radius
$\rho_g=\rho_g(x,t)$	The gas density
g	Gas phase
l	Liquid phase
$P(x,t)$	Gas–liquid mixture
P_g	Gas pressure.
$T_g\,\&\,T_l$	Temperature of liquid and gas accordingly
χ_g	A coefficient of the gas thermal conduction
N_u	The Nusselt number
n	The poly tropic exponent
v	The viscosity of a liquid
$u=u(x,t)$	A velocity of a flow of a gas–liquid mixture
$\alpha,\beta,\lambda,\mu,\sigma,C,b,a_i,b_1$	Arbitrary constants will be evaluated in research

Considering the small deviation of the bubble radius in comparison with the averaged radius of bubble, we have

$$R(x,t)=R_0+\eta(x,t),\ R_0=constant,\ \|\eta\|\ll R_0,\ R(x,0)=R_0. \tag{2.4}$$

Assume that the liquid temperature is constant and equal to the initial value:

$$T_l=T|_{t=0}=T_0,\ T_0=constant. \tag{2.5}$$

At the initial moment, we also have

$$t=0,\ Q=Q_g=Q_0,\ Q_0=constant,\ V=V_0=\frac{4}{3}\pi R_0^3 N. \tag{2.6}$$

Substituting Q_g and T_g from first and third equations into the second equation in the system (2.2) and taking relation (2.4) into account we have the pressure dependence of a mixture on the radius perturbation in the form:

$$\begin{aligned}&Q-Q_0+\frac{\eta}{R_0}Q+\frac{3n\chi}{R_0}Q\eta_t+\chi Q_t+\frac{\rho_1\left(3R_0^2+4\upsilon\chi\right)}{3R_0}\eta_{tt}\\&\quad+\frac{\rho_t\left(6R_0^2-4\upsilon\chi\right)}{3R_0^2}\eta\eta_{tt}+\frac{\rho_t\left(8\upsilon\chi(3n-1)+9R_0^2\right)}{6R_0^2}\eta_t^2\\&\quad+\frac{4\upsilon\rho_l}{3R_0}\eta_t+\frac{2Q_0}{R_0}\eta+\frac{3P_0}{R_0^2}\eta^2=0,\\&\chi=\frac{2R_0^2Q_0}{3\chi Nu(n-1)T_0}.\end{aligned} \tag{2.7}$$

From Eq. (2.3) we also have the dependence ρ on η using formula (2.4)

$$\begin{cases}\rho=\rho_0-\mu\eta+\mu_1\eta^2,\\ \rho_0=\frac{\rho_1}{1-\chi+V_0\rho_l},\\ \mu=\frac{3\rho_1^2V_0}{R_0(1-\chi+V_0\rho_l)^2},\\ \mu_1=\frac{6\rho_l^2V_0(2\rho_lV_0-1+\chi)}{R_0^2(1-\chi+V_0\rho_l)^3}.\end{cases} \tag{2.8}$$

Using the system of equations for description of the motion of a gas-liquid mixture flow in the form

$$\begin{cases}\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0,\\ \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial Q}{\partial x}=0.\end{cases} \tag{2.9}$$

where $u=u(x,t)$ is a velocity of a flow of a gas-liquid mixture. Eq. (2.7) together with Eqs. (2.5) and (2.6) can be applied for description of nonlinear waves in a gas-liquid medium. Consider the linear case of the system of Eqs. (2.5)–(2.7). Assuming, that pressure in a mixture is proportional to perturbation radius, we obtain the linear wave equation for the radius perturbations

$$\eta_{tt}=c_0^2\eta_{xx}\,\&\,c_0=\sqrt{\frac{3Q_0}{\mu R_0}}. \tag{2.10}$$

Let us introduce the following dimensionless variables

$$t=\frac{l}{c_0}t'\,\&\,x=lx'\,\&\,u=c_0u'\,\&\,\eta=R_0\eta'\,\&\,Q=Q_0Q'+Q_0, \tag{2.11}$$

where l is the characteristic length of wave. Using the dimensionless variables the system of Eqs. (2.5)–(2.7) can be reduced to the following (the primes of the variables are omitted)

$$\begin{cases}\eta_t-\frac{\rho_0}{\mu R_0}u_x+u\eta_x+\eta u_x-\frac{2\mu_1R_0}{\mu}\eta\eta_t=0,\\ \frac{-\rho_0}{\mu R_0}(u_t+uu_x)+\eta u_t-\frac{1}{3}Q_x=0,\\ Q+\chi_1Q_t+\eta Q+3n\chi_1\eta_tQ\\ =-(\beta_1+\beta_2)\eta_{tt}-(2\beta_2-\beta_1)\eta\eta_{tt}\\ -\left(\frac{3n-1}{2}\beta_1+\frac{3}{2}\beta_2\right)\eta_t^2-\lambda\eta_t-3\eta+3\eta^2.\end{cases} \tag{2.12}$$

where the parameters are determined by formulas

$$\lambda = \frac{4\,v\,\rho_l c_0}{3\,Q_0\,l} + 3\,n\,\chi_1,\; \beta_1 = \frac{4\,v\,\chi\,\rho_l c_0^2}{3\,Q_0\,l^2},\; \gamma = \frac{\chi\,\rho_1 R_0^2 c_0^3}{Q_0\,l^3},\; \chi_1 = \frac{\chi c_0}{l}. \tag{2.13}$$

We introduce a scale transformation of independent variables

$$\xi = \epsilon^m (x - t),\; \tau = \epsilon^{m+1} t,\; m > 0,\; \epsilon \ll 1,$$
$$\frac{\partial}{\partial x} = \epsilon^m \frac{\partial}{\partial \xi},\; \frac{\partial}{\partial t} = \epsilon^{m+1}\frac{\partial}{\partial \tau} - \epsilon^m \frac{\partial}{\partial \xi}. \tag{2.14}$$

Substituting (2.11) into(2.9) and dividing on ”m in first two equations we have the following system of equations

$$\epsilon\,\eta_\tau - \eta_\xi - \frac{\rho_0}{\mu R_0} u_\xi + \eta\,u_\xi + u\,\eta_\xi - \epsilon\,\frac{2\,\mu_1 R_0}{\mu}\,\eta\,\eta_\tau + \frac{2\,\mu_1 R_0}{\mu}\,\eta\,\eta_\xi = 0, \tag{2.15}$$

$$\epsilon\left(-\frac{\rho_0}{\mu R_0}\right) u_r + \frac{\rho_0}{\mu R_0} u_\xi + \epsilon\,\eta\,u_r - \eta\,u_\xi - \frac{\rho_0}{\mu R_0}\,u\,u_\xi - \frac{1}{3}\,Q_\xi = 0, \tag{2.16}$$

$$\begin{aligned} &Q + \epsilon^{m+1}\chi_1 Q_\tau - \epsilon^m \chi_1 Q_\xi + \eta\,Q + 3\,\epsilon^{m+1} n\,\chi_1\,\eta_\tau Q - 3\,\epsilon^m n\,\chi_1\,\eta_\xi Q \\ &= -\epsilon^{2m+2}(\beta_1+\beta_2)\,\eta_{\tau\tau} + 2\,\epsilon^{2m+1}(\beta_1+\beta_2)\,\eta_{\tau\xi} - \epsilon^{2m}(\beta_1+\beta_2)\,\eta_{\xi\xi} \\ &- \epsilon^{2m+2}(2\beta_2-\beta_1)\,\eta\,\eta_{\tau\tau} + 2\,\epsilon^{2m+1}(2\beta_2-\beta_1)\,\eta\,\eta_{\tau\xi} \\ &- \epsilon^{2m}(2\beta_2-\beta_1)\,\eta\,\eta_{\xi\xi} - \epsilon^{2m+2}\left(\frac{3n-1}{2}\beta_1 + \frac{3}{2}\beta_2\right)\eta_\tau^2 \\ &+ 2\,\epsilon^{2m+1}\left(\frac{3n-1}{2}\beta_1 + \frac{3}{2}\beta_2\right)\eta_\tau\,\eta_\xi \\ &- \epsilon^{2m}\left(\frac{3n-1}{2}\beta_1 + \frac{3}{2}\beta_2\right)\eta_\xi^2 - \epsilon^{m+1}\lambda\,\eta_\tau + \epsilon^m\lambda\,\eta_\xi - 3\,\eta + 3\,\eta^2. \end{aligned} \tag{2.17}$$

We now assume that the state variables u, η and Q can be represented asymptotically as series in powers of ϵ about an equilibrium state

$$u = \epsilon u_1 + \epsilon^2 u_2 + \ldots,\; \eta = \epsilon\,\eta_1 + \epsilon^2\eta_2 + \ldots,\; Q = \epsilon Q_1 + \epsilon^2 Q_2 + \ldots. \tag{2.18}$$

Substituting (2.18) into (2.15)–(2.17) and equating expressions at ϵ and ϵ^2 to zero, we obtain some equations with respect to $u_1, u_2, \eta_1, \eta_2, Q_1, Q_2$. Solving these equations with respect to Q_1 we have the equation for pressure Q_1 in the form:

$$\begin{aligned} &Q_{1\tau} + \Omega Q_1 Q_{1\xi} + \epsilon^{2m-1}\frac{\beta_1+\beta_2}{6} Q_{1\xi\xi\xi} - \epsilon^{2m}\frac{2\beta_2-\beta_1}{18} Q_1 Q_{1\xi\xi\xi} \\ &- \epsilon^{2m}\left(\frac{3n-2}{18}\beta_1 + \frac{5}{18}\beta_2\right) Q_{1\xi\xi} = \epsilon^{m-1}\left(\frac{\lambda}{6} - \frac{\chi_1}{2}\right) Q_{1\xi\xi} + \epsilon^m \frac{n\chi_1}{2}\left(Q_1 Q_{1\xi}\right)_\xi, \end{aligned} \tag{2.19}$$

where $\Omega = \frac{3\,\mu R_0}{\rho_0} - \frac{3\,\mu_1 R_0}{\mu} + \frac{2}{3}$. From Eq. (2.19) one can find some nonlinear evolution equations for description of waves in a mixture liquid and gas bubbles with consideration for the heat transfer and the viscosity. Assuming

$$m = 1,\; \frac{\beta_1+\beta_2}{6} = \beta\epsilon^\delta,\; \delta > 0,\; \beta \cong 1,\; \beta_1 = 2\beta_2,\; \lambda = 3\,\chi.$$

we have nonlinear evolution equation of the second order in the form

$$Q_\tau + \Omega\,Q\,Q_\xi + \alpha_1\,Q_{\xi\xi\xi} - \alpha_2\,Q_\xi\,Q_{\xi\xi} - \alpha_3\left(Q\,Q_\xi\right)_\xi = 0, \tag{2.20}$$

where $b = \left(\frac{\lambda}{6} - \frac{\chi_1}{2}\right)$, $c = \epsilon\,\frac{n\chi_1}{2}$, $\alpha_1 = \left(\epsilon^{2m-1}\,\frac{\beta_1+\beta_2}{6}\right)$, $\alpha_3 = \epsilon^{2m}\left(\frac{3n-2}{18}\beta_1 + \frac{5}{18}\beta_2\right)$, $\alpha_4 = \left(\epsilon^m\,\frac{n\chi_1}{2}\right)$. Using the traveling wave transformation $Q(\xi,\tau) = U(\zeta)$ where $\zeta = \xi + c\,\tau$ and integrating Eq. (2.20) with zero constant of integration, we obtain:

$$c\,U + \frac{\Omega}{2}u^2 + \alpha_1 U'' - \frac{\alpha_2}{2}U'^2 - \alpha_3\,U\,U' = 0. \tag{2.21}$$

This equation also called the Kudryashov-Sinelshchikov equation [21–25] which can described in the simplest case by the most famous model Korteweg-de Vries (KdV) equation [26] when $(\alpha_2 = \alpha_3 = 0) \Rightarrow (u_t + \alpha\,u\,u_x + u_{xxx} = 0)$. Not just that but also it describe the most famous model Burger Korteweg-de Vries (BKdV) when $(\alpha_2 = \alpha_3 = \lambda = 0)$. Kudryashov-Sinelshchikov Equation also is the general form of famous Kawahara equation [27]:

$$u_t + u\,u_x + u_{xxx} - u_{xxxxx} = 0.$$

That describe the motions of plasma waves, capillary-gravity water waves, water waves with surface tension. Kudryashov-Sinelshchikov used reductive perturbation method to obtain this general form of Kawahara equation to be in the form (2.21).

Balancing the the highest order derivative term and nonlinear term in Eq. (2.21) $\Rightarrow \left(U''\,\&\,U^2\right) \Rightarrow (N = 2)$. But we can not use this balance into this model and you can check by yourself by using any methods like modified simple equation method, extended tanh-function method, modified simple equation method, new auxiliary equation method,…and so on. so that, we will use the transformation $U(\zeta) = P^2(\zeta)$, we get:

$$c\,P^2 + \frac{\Omega}{2}P^4 + 2\,\alpha_1 P'^2 + 2\,\alpha_1 P P'' - 2\,\alpha_2 P^2 P'^2 - 2\,\alpha_3 P^3 P'' = 0. \tag{2.22}$$

Balancing the the highest order derivative term and nonlinear term in Eq. (2.21) $\Rightarrow \left(P P''\,\&\,P^4\right) \Rightarrow (N = 1)$.

Now, we will apply three different methods on our model to get closed form of solutions and solitary traveling wave solutions and we also represent in details the discussion and comparison between the three methods and the results that obtained by these methods and that obtained by another researchers who used a different methods.

The modified extended tanh-function method:

Implement modified extended tanh-function method on the Pressure equation of bubbly liquids with examination for viscosity and heat transfer. So that, according the suggested method, we can presume the solution of Eq. (2.22) in the following form:

$$p(\zeta) = \frac{b_1}{\phi(\zeta)} + a_0 + a_1\,\phi(\zeta). \tag{2.23}$$

Subrogate Eq. (2.23) and its derivative into Eq. (2.22) and gathering all term with the same power of ϕ^i where $(i = -6, \ldots, 6)$ we get suit of algebraic equations. Disbanding this suit by any computer program (Mathematica, Maple, Matlab,…, etc), we obtain:

$$\Omega = -\frac{-64\,\alpha_3 b_1^2 + 3\,c}{4\,a_1 b_1},\; b = \frac{b_1}{a_1},\; a_0 = 0,\; a_1 = a_1,\; \alpha_1 = \frac{c a_1}{16\,b_1},$$
$$\alpha_2 = -2\,\alpha_3.$$

So that, the exact traveling wave solutions:

$$P(\zeta) = \frac{b_1}{\phi(\zeta)} + a_1\,\phi(\zeta). \tag{2.24}$$

therefor, the solitary traveling wave solutions:

Case 1. If $b < 0$:

$$P(\zeta) = \frac{b_1 \sqrt{-b}}{b}\, coth(\sqrt{-b}\ \xi) - a_1 \sqrt{-b}\, tanh(\sqrt{-b}\ \xi), \tag{2.25}$$

$$U(\zeta) = \left(\frac{b_1 \sqrt{-b}}{b}\, coth(\sqrt{-b}\ \xi) - a_1 \sqrt{-b}\, tanh(\sqrt{-b}\ \xi)\right)^2. \tag{2.26}$$

Or

$$P(\zeta) = \frac{b_1 \sqrt{-b}}{b}\, tanh(\sqrt{-b}\ \xi) - a_1 \sqrt{-b}\, coth(\sqrt{-b}\ \xi), \tag{2.27}$$

$$U(\zeta) = \left(\frac{b_1 \sqrt{-b}}{b}\, tanh(\sqrt{-b}\ \xi) - a_1 \sqrt{-b}\, coth(\sqrt{-b}\ \xi)\right)^2. \tag{2.28}$$

Case 2. If $b > 0$:

$$P(\zeta) = \frac{b_1 \sqrt{b}}{b}\, cot(\sqrt{b}\ \xi) + a_1 \sqrt{b}\, tan(\sqrt{b}\ \xi), \tag{2.29}$$

$$U(\zeta) = \left(\frac{b_1 \sqrt{b}}{b}\, cot(\sqrt{b}\ \xi) + a_1 \sqrt{b}\, tan(\sqrt{b}\ \xi)\right)^2. \tag{2.30}$$

Or

$$P(\zeta) = \frac{b_1 \sqrt{b}}{b}\, tan(\sqrt{b}\ \xi) + a_1 \sqrt{b}\, cot(\sqrt{b}\ \xi), \tag{2.31}$$

$$U(\zeta) = \left(\frac{b_1 \sqrt{b}}{b}\, tan(\sqrt{b}\ \xi) + a_1 \sqrt{b}\, cot(\sqrt{b}\ \xi)\right)^2. \tag{2.32}$$

Case 3. If $b = 0$:

$$P(\zeta) = -b_1\, \xi - \frac{a_1}{\xi}, \tag{2.33}$$

$$U(\zeta) = \left(-b_1\, \xi - \frac{a_1}{\xi}\right)^2. \tag{2.34}$$

Extended simple equation method:

Implement an extended simple equation method on the Pressure equation of bubbly liquids with examination for viscosity and heat transfer. So that, according the suggested method, we can presume the solution of Eq. (2.22) in the following form:

$$P(\zeta) = \frac{a_{-1}}{\phi(\zeta)} + a_0 + a_1\, \phi(\zeta). \tag{2.35}$$

Subrogate Eq. (2.35)its derivative into Eq. (2.22)gathering all term with the same power of ϕ^i where $(i = -6, \ldots, 6)$ we get suit of algebraic equations. Disbanding this suit by any computer program (Mathematica, Maple, Matlab,…, etc), we obtain:

Case I.

$$\Omega = 4\lambda^2\alpha_2 + 4\alpha_3\lambda^2,\ \alpha = 0,\ c = -4\lambda^2\alpha_1,\ a_{-1} = \frac{\lambda a_0}{\mu},\ a_0 = a_0,\ a_1 = 0.$$

So that, the exact traveling wave solutions:

$$P(\zeta) = \frac{\lambda a_0}{\mu\, \phi(\zeta)} + a_0. \tag{2.36}$$

therefor, the solitary traveling wave solutions:

Case 1. When $\lambda > 0$

$$P(\zeta) = \frac{\lambda a_0 \left(1 - \mu e^{\lambda(\xi+C)}\right)}{\mu \left(\lambda e^{\lambda(\xi+C)}\right)} + a_0, \tag{2.37}$$

$$U(\zeta) = \left(\frac{\lambda a_0 \left(1 - \mu e^{\lambda(\xi+C)}\right)}{\mu \left(\lambda e^{\lambda(\xi+C)}\right)} + a_0\right)^2. \tag{2.38}$$

Case 2. When $\lambda < 0$

$$P(\zeta) = \frac{\lambda a_0 \left(1 + \mu e^{\lambda(\xi+C)}\right)}{\left(-\mu e^{\lambda(\xi+C)}\right)} + a_0, \tag{2.39}$$

$$U(\zeta) = \left(\frac{\lambda a_0 \left(1 + \mu e^{\lambda(\xi+C)}\right)}{\mu \left(-\mu e^{\lambda(\xi+C)}\right)} + a_0\right)^2. \tag{2.40}$$

Case II.

$$\Omega = \frac{8\alpha\alpha_3\left(\alpha a_0^2 - \mu a_{-1}^2\right)}{a_{-1}^2},\ c = \frac{-4\alpha_3\left(\alpha^2 a_0^4 - 2\alpha\mu a_{-1}^2 a_0^2 + a_{-1}^4\mu^2\right)}{a_{-1}^2},\ \lambda = \frac{2\alpha a_0}{a_{-1}},$$

$a_{-1} = a_{-1},\ a_0 = a_0,\ a_1 = 0,\ \alpha_1 = 0,\ \alpha_2 = -2\alpha_3.$

So that, the exact traveling wave solutions:

$$P(\zeta) = \frac{a_{-1}}{\phi(\zeta)} + a_0. \tag{2.41}$$

Thus, the solitary traveling wave solutions:

When $\lambda = 0$, we get:

Case 1. When $\alpha\mu > 0$

$$P(\zeta) = \frac{a_{-1}\sqrt{\alpha\mu}}{\alpha}\, cot(\sqrt{\alpha\mu}(\xi + C)) + a_0, \tag{2.42}$$

$$U(\zeta) = \left(\frac{a_{-1}\sqrt{\alpha\mu}}{\alpha}\, cot(\sqrt{\alpha\mu}(\xi + C)) + a_0\right)^2, \tag{2.43}$$

and

$$P(\zeta) = \frac{a_{-1}\sqrt{\alpha\mu}}{\alpha}\, tan(\sqrt{\alpha\mu}(\xi + C)) + a_0, \tag{2.44}$$

$$U(\zeta) = \left(\frac{a_{-1}\sqrt{\alpha\mu}}{\alpha}\, tan(\sqrt{\alpha\mu}(\xi + C)) + a_0\right)^2. \tag{2.45}$$

Case 2. When $\alpha\mu < 0$

$$P(\zeta) = \frac{-a_{-1}\sqrt{-\alpha\mu}}{\alpha}\, coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_0, \tag{2.46}$$

$$U(\zeta) = \left(\frac{-a_{-1}\sqrt{-\alpha\mu}}{\alpha}\, coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_0\right)^2, \tag{2.47}$$

and

$$P(\zeta) = \frac{-a_{-1}\sqrt{-\alpha\mu}}{\alpha}\, tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_0, \tag{2.48}$$

$$U(\zeta) = \left(\frac{-a_{-1}\sqrt{-\alpha\mu}}{\alpha}\, tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_0\right)^2. \tag{2.49}$$

When $\alpha = 0$, we get

Case 1. When $\lambda > 0$

$$P(\zeta) = \frac{a_{-1}\left(1 - \mu e^{\lambda(\xi+C)}\right)}{\lambda e^{\lambda(\xi+C)}} + a_0, \tag{2.50}$$

$$U(\zeta) = \left(\frac{a_{-1}\left(1 - \mu e^{\lambda(\xi+C)}\right)}{\lambda e^{\lambda(\xi+C)}} + a_0\right)^2. \tag{2.51}$$

Case 2. When $\lambda < 0$

$$P(\zeta) = -\frac{a_{-1}\left(1 + \mu e^{\lambda(\xi+C)}\right)}{\mu e^{\lambda(\xi+C)}} + a_0, \tag{2.52}$$

$$U(\zeta) = \left(\frac{a_{-1}\left(1+\mu e^{\lambda(\xi+C)}\right)}{\mu e^{\lambda(\xi+C)}} + a_0\right)^2. \tag{2.53}$$

General solutions will be in the follows form:

Case 1. When $4\alpha\mu > \lambda^2$ and $\mu > 0$

$$P(\zeta) = \frac{2a_{-1}}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + a_0, \tag{2.54}$$

$$U(\zeta) = \left(\frac{2a_{-1}}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + a_0\right)^2, \tag{2.55}$$

and

$$P(\zeta) = \frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + a_0, \tag{2.56}$$

$$U(\zeta) = \left(\frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + a_0\right)^2, \tag{2.57}$$

Case 2. When $4\alpha\mu > \lambda^2$ and $\mu < 0$

$$P(\zeta) = \frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda} + a_0, \tag{2.58}$$

$$U(\zeta) = \left(\frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda} + a_0\right)^2, \tag{2.59}$$

and

$$P(\zeta) = \frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda} + a_0, \tag{2.60}$$

$$U(\zeta) = \left(\frac{2a_{-1}\mu}{\sqrt{4\alpha\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda} + a_0\right)^2. \tag{2.61}$$

Case III.

$$\Omega = \frac{4\left(4\alpha a_1^2\alpha_3 - 3\mu\alpha_1\right)\mu}{a_1^2},\ c = 16\alpha\mu\alpha_1,\ a_{-1} = \frac{\alpha a_1}{\mu},$$
$$a_0 = 0,\ a_1 = a_1,\ \alpha_2 = -2\alpha_3.$$

So that, the exact traveling wave solutions:

$$v(\zeta) = \frac{\alpha a_1}{\mu\phi(\zeta)} + a_1\phi(\zeta). \tag{2.62}$$

therefor, the solitary traveling wave solutions:

When $\lambda = 0$, we get:

Case 1. When $\alpha\mu > 0$

$$P(\zeta) = \frac{a_1\sqrt{\alpha\mu}}{\mu}cot(\sqrt{\alpha\mu}(\xi+C)) + a_1\frac{\sqrt{\alpha\mu}}{\mu}tan(\sqrt{\alpha\mu}(\xi+C)), \tag{2.63}$$

$$U(\zeta) = \left(\frac{a_1\sqrt{\alpha\mu}}{\mu}cot(\sqrt{\alpha\mu}(\xi+C)) + a_1\frac{\sqrt{\alpha\mu}}{\mu}tan(\sqrt{\alpha\mu}(\xi+C))\right)^2, \tag{2.64}$$

and

$$P(\zeta) = \frac{a_1\sqrt{\alpha\mu}}{\mu}tan(\sqrt{\alpha\mu}(\xi+C)) + a_1\frac{\sqrt{\alpha\mu}}{\mu}cot(\sqrt{\alpha\mu}(\xi+C)), \tag{2.65}$$

$$U(\zeta) = \left(\frac{a_1\sqrt{\alpha\mu}}{\mu}tan(\sqrt{\alpha\mu}(\xi+C)) + a_1\frac{\sqrt{\alpha\mu}}{\mu}cot(\sqrt{\alpha\mu}(\xi+C))\right)^2, \tag{2.66}$$

Case 2. When $\alpha\mu < 0$

$$P(\zeta) = \frac{-a_1\sqrt{-\alpha\mu}}{\mu}coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_1\frac{\sqrt{-\alpha\mu}}{\mu}tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right), \tag{2.67}$$

$$U(\zeta) = \left(\frac{-a_1\sqrt{-\alpha\mu}}{\mu}coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_1\frac{\sqrt{-\alpha\mu}}{\mu}tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right)\right)^2, \tag{2.68}$$

and

$$P(\zeta) = \frac{-a_1\sqrt{-\alpha\mu}}{\mu}tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_1\frac{\sqrt{-\alpha\mu}}{\mu}coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right), \tag{2.69}$$

$$U(\zeta) = \left(\frac{-a_1\sqrt{-\alpha\mu}}{\mu}tanh\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right) + a_1\frac{\sqrt{-\alpha\mu}}{\mu}coth\left(\sqrt{-\alpha\mu}\,\xi \mp \frac{ln(C)}{2}\right)\right)^2. \tag{2.70}$$

When $\alpha = 0$, we get

Case 1. When $\lambda > 0$

$$P(\zeta) = \frac{\alpha a_1\left(1-\mu e^{\lambda(\xi+C)}\right)}{\lambda\mu e^{\lambda(\xi+C)}} + \frac{a_1\lambda e^{\lambda(\xi+C)}}{1-\mu e^{\lambda(\xi+C)}}, \tag{2.71}$$

$$U(\zeta) = \left(\frac{\alpha a_1\left(1-\mu e^{\lambda(\xi+C)}\right)}{\lambda\mu e^{\lambda(\xi+C)}} + \frac{a_1\lambda e^{\lambda(\xi+C)}}{1-\mu e^{\lambda(\xi+C)}}\right)^2. \tag{2.72}$$

Case 2. When $\lambda < 0$

$$P(\zeta) = -\frac{\alpha a_1\left(1+\mu e^{\lambda(\xi+C)}\right)}{\mu^2 e^{\lambda(\xi+C)}} - \frac{a_1\mu e^{\lambda(\xi+C)}}{1+\mu e^{\lambda(\xi+C)}}, \tag{2.73}$$

$$U(\zeta) = \left(\frac{\alpha a_1\left(1+\mu e^{\lambda(\xi+C)}\right)}{\mu^2 e^{\lambda(\xi+C)}} + \frac{a_1\mu e^{\lambda(\xi+C)}}{1+\mu e^{\lambda(\xi+C)}}\right)^2. \tag{2.74}$$

General solutions will be in the follows form:

Case 1. When $4\alpha\mu > \lambda^2$ and $\mu > 0$

$$P(\zeta) = \frac{2\alpha a_1}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + \frac{a_1}{2\mu}\left[\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda\right], \tag{2.75}$$

$$U(\zeta) = \left(\frac{2\alpha a_1}{\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda} + \frac{a_1}{2\mu}\left[\sqrt{4\alpha\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\alpha\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda\right]\right)^2, \tag{2.76}$$

and

$$P(\zeta)=\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda\right], \quad (2.77)$$

$$U(\zeta)=\left(\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)-\lambda\right]\right)^2, \quad (2.78)$$

Case 2. When $4\,\alpha\,\mu>\lambda^2$ and $\mu<0$

$$P(\zeta)=\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda\right]. \quad (2.79)$$

$$U(\zeta)=\left(\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,tan\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda\right]\right)^2, \quad (2.80)$$

and

$$P(\zeta)=\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda\right], \quad (2.81)$$

$$U(\zeta)=\left(\frac{2\,\alpha\,a_1}{\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda}+\frac{a_1}{2\,\mu}\left[\sqrt{4\,\alpha\,\mu-\lambda^2}\,cot\left(\frac{\sqrt{4\,\alpha\,\mu-\lambda^2}}{2}(\xi+C)\right)+\lambda\right]\right)^2. \quad (2.82)$$

New auxiliary equation method:

Implement new auxiliary equation method on the Pressure equation of bubbly liquids with examination for viscosity and heat transfer. So that, according the suggested method, we can presume the solution of Eq. (2.22) in the following form:

$$P(\zeta)=a_0+a_1\,a^{f(\zeta)}. \quad (2.83)$$

Subrogate Eq. (2.83)its derivative into Eq. (2.22) and gathering all term with the same power of ϕ^i where $(i=-6,\ldots,6)$ we get suit of algebraic equations. Disbanding this suit by any computer program (Mathematica, Maple, matlab,…,etc), we obtain:

Case I.

$\Omega=-4\,\beta^2\alpha_2,\ \sigma=0,\ a_0=\frac{\alpha\,a_1}{\beta},\ a_1=a_1,\ \alpha_1=-\frac{c}{4\,\beta^2},\ \alpha_3=-2\,\alpha_2.$

So that, the exact traveling wave solution:

$$P(\zeta)=\frac{\alpha\,a_1}{\beta}+a_1\,a^{f(\zeta)}. \quad (2.84)$$

Therefor, the solitary traveling wave solutions:
When $(\alpha=\sigma=0)$.

$$P(\zeta)=\frac{\alpha\,a_1}{\beta}+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{2(e^{4\,\beta\,\xi}+1)}}{e^{2\,\beta\,\xi}-1}\right], \quad (2.85)$$

$$U(\zeta)=\left(\frac{\alpha\,a_1}{\beta}+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{2(e^{4\,\beta\,\xi}+1)}}{e^{2\,\beta\,\xi}-1}\right]\right)^2, \quad (2.86)$$

or

$$P(\zeta)=\frac{\alpha\,a_1}{\beta}+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{e^{4\,\beta\,\xi}+6\,e^{2\,\beta\,\xi}+1}}{2\,e^{2\,\beta\,\xi}}\right], \quad (2.87)$$

$$U(\zeta)=\left(\frac{\alpha\,a_1}{\beta}+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{e^{4\,\beta\,\xi}+6\,e^{2\,\beta\,\xi}+1}}{2\,e^{2\,\beta\,\xi}}\right]\right)^2. \quad (2.88)$$

When $(\beta=k,\ \alpha=2\,k,\ \sigma=0)$.

$$P(\zeta)=\frac{\alpha\,a_1}{\beta}+a_1\left[e^{k\,\xi}-1\right], \quad (2.89)$$

$$U(\zeta)=\left(\frac{\alpha\,a_1}{\beta}+a_1\left[e^{k\,\xi}-1\right]\right)^2. \quad (2.90)$$

When $(\sigma=0)$.

$$P(\zeta)=\frac{\alpha\,a_1}{\beta}+a_1\left[e^{\beta\,\xi}-\frac{\alpha}{2\,\beta}\right], \quad (2.91)$$

$$U(\zeta)=\left(\frac{\alpha\,a_1}{\beta}+a_1\left[e^{\beta\,\xi}-\frac{\alpha}{2\,\beta}\right]\right)^2. \quad (2.92)$$

Case II.

$\Omega=4\,\beta^2\alpha_2+4\,\beta^2\alpha_3,\ \alpha=\frac{\beta\,a_0}{a_1},\ \sigma=0,\ a_0=a_0,\ a_1=a_1,\ \alpha_1=-\frac{c}{4\,\beta^2}.$

So that, the exact traveling wave solution:

$$P(\zeta)=a_0+a_1\,a^{f(\zeta)}. \quad (2.93)$$

Therefor, the solitary traveling wave solutions:
When $(\alpha=\sigma=0)$.

$$P(\zeta)=a_0+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{2(e^{4\,\beta\,\xi}+1)}}{e^{2\,\beta\,\xi}-1}\right], \quad (2.94)$$

$$U(\zeta)=\left(a_0+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{2(e^{4\,\beta\,\xi}+1)}}{e^{2\,\beta\,\xi}-1}\right]\right)^2, \quad (2.95)$$

or

$$P(\zeta)=a_0+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{e^{4\,\beta\,\xi}+6\,e^{2\,\beta\,\xi}+1}}{2\,e^{2\,\beta\,\xi}}\right], \quad (2.96)$$

$$U(\zeta)=\left(a_0+a_1\left[\frac{-(1+e^{2\,\beta\,\xi})\pm\sqrt{e^{4\,\beta\,\xi}+6\,e^{2\,\beta\,\xi}+1}}{2\,e^{2\,\beta\,\xi}}\right]\right)^2. \quad (2.97)$$

When $(\beta=k,\ \alpha=2\,k,\ \sigma=0)$.

$$P(\zeta) = a_0 + a_1\left[e^{k\,\xi} - 1\right], \tag{2.98}$$

$$U(\zeta) = \left(a_0 + a_1\left[e^{k\,\xi} - 1\right]\right)^2. \tag{2.99}$$

When $(\sigma = 0)$.

$$P(\zeta) = a_0 + a_1\left[e^{\beta\,\xi} - \frac{\alpha}{2\,\beta}\right], \tag{2.100}$$

$$U(\zeta) = \left(a_0 + a_1\left[e^{\beta\,\xi} - \frac{\alpha}{2\,\beta}\right]\right)^2. \tag{2.101}$$

Case 3.

$$\Omega = \frac{2\,\beta^2 c}{a_0^2(4\,\alpha\,\sigma - \beta^2)},\, a_0 = a_0,\, a_1 = \frac{2\,\sigma\,a_0}{\beta},\, \alpha_1 = 0,$$
$$\alpha_2 = \frac{2\,c\beta^2}{a_0^2(16\,\alpha^2\sigma^2 - 8\,\alpha\,\beta^2\sigma + \beta^4)},$$

$$\alpha_3 = -\frac{c\beta^2}{a_0^2(16\,\alpha^2\sigma^2 - 8\,\alpha\,\beta^2\sigma + \beta^4)}.$$

So that, the exact traveling wave solution:

$$P(\zeta) = a_0 + \frac{2\,\sigma\,a_0}{\beta}\,a^{f(\zeta)}. \tag{2.102}$$

therefor, the solitary traveling wave solutions:

When $(\beta^2 - \alpha\,\sigma < 0\,\&\,\sigma \neq 0)$.

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha\,\sigma)}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right), \tag{2.103}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha\,\sigma)}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right)^2, \tag{2.104}$$

or

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha\,\sigma)}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right), \tag{2.105}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha\,\sigma)}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right)^2. \tag{2.106}$$

When $(\beta^2 - \alpha\,\sigma > 0\,\&\,\sigma \neq 0)$.

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\sqrt{(\beta^2 - \alpha\,\sigma)}\,tanh\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right), \tag{2.107}$$

$$U(\zeta) = \left(a_0 + \frac{2\,a_0}{\beta}\sqrt{(\beta^2 - \alpha\,\sigma)}\,tanh\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right)^2, \tag{2.108}$$

or

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\sqrt{(\beta^2 - \alpha\,\sigma)}\,coth\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right), \tag{2.109}$$

$$U(\zeta) = \left(a_0 + \frac{2\,a_0}{\beta}\sqrt{(\beta^2 - \alpha\,\sigma)}\,coth\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right)^2. \tag{2.110}$$

When $(\beta^2 + \alpha^2 > 0\,\&\,\sigma \neq 0\,\&\,\sigma = -\alpha)$.

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\frac{\sqrt{\beta^2 + \alpha^2}}{}tanh\left(\frac{\sqrt{\beta^2 + \alpha^2}}{2}\,\xi\right), \tag{2.111}$$

$$U(\zeta) = \left(a_0 + \frac{2\,a_0}{\beta}\frac{\sqrt{\beta^2 + \alpha^2}}{}tanh\left(\frac{\sqrt{\beta^2 + \alpha^2}}{2}\,\xi\right)\right)^2, \tag{2.112}$$

or

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\frac{\sqrt{\beta^2 + \alpha^2}}{}coth\left(\frac{\sqrt{\beta^2 + \alpha^2}}{2}\,\xi\right), \tag{2.113}$$

$$U(\zeta) = \left(-a_0 - \frac{2\,a_0}{\beta}\frac{\sqrt{\beta^2 + \alpha^2}}{}coth\left(\frac{\sqrt{\beta^2 + \alpha^2}}{2}\,\xi\right)\right)^2. \tag{2.114}$$

When $(\beta^2 + \alpha^2 < 0\,\&\,\sigma \neq 0\,\&\,\sigma = -\alpha)$.

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 + \alpha^2)}\,tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right), \tag{2.115}$$

$$U(\zeta) = \left(a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 + \alpha^2)}\,tan\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)^2, \tag{2.116}$$

or

$$P(\zeta) = -a_0 - \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 + \alpha^2)}\,cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right), \tag{2.117}$$

$$U(\zeta) = \left(-a_0 - \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 + \alpha^2)}\,cot\left(\frac{\sqrt{-(\beta^2 + \alpha^2)}}{2}\,\xi\right)\right)^2. \tag{2.118}$$

When $(\beta^2 - \alpha^2 < 0\,\&\,\sigma = \alpha)$.

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha^2)}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha^2)}}{2}\,\xi\right), \tag{2.119}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha^2)}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha^2)}}{2}\,\xi\right)\right)^2, \tag{2.120}$$

or

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha^2)}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha^2)}}{2}\,\xi\right), \tag{2.121}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{-(\beta^2 - \alpha^2)}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha^2)}}{2}\,\xi\right)\right)^2. \tag{2.122}$$

When $(\beta^2 - \alpha^2 > 0 \,\&\, \sigma = \alpha)$.

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{\beta^2 - \alpha^2}\,tanh\left(\frac{\sqrt{\beta^2 - \alpha^2}}{2}\,\xi\right), \tag{2.123}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{\beta^2 - \alpha^2}\,tanh\left(\frac{\sqrt{\beta^2 - \alpha^2}}{2}\,\xi\right)\right)^2, \tag{2.124}$$

or

$$P(\zeta) = -a_0 + \frac{2\,a_0}{\beta}\sqrt{\beta^2 - \alpha^2}\,coth\left(\frac{\sqrt{\beta^2 - \alpha^2}}{2}\,\xi\right), \tag{2.125}$$

$$U(\zeta) = \left(-a_0 + \frac{2\,a_0}{\beta}\sqrt{\beta^2 - \alpha^2}\,coth\left(\frac{\sqrt{\beta^2 - \alpha^2}}{2}\,\xi\right)\right)^2. \tag{2.126}$$

When $(\beta^2 = \alpha\,\sigma)$.

$$P(\zeta) = a_0 - \frac{2\,\sigma\,a_0\,\alpha\,(\beta\,\xi + 2)}{\beta^3\,\xi}, \tag{2.127}$$

$$U(\zeta) = \left(a_0 - \frac{2\,\sigma\,a_0\,\alpha\,(\beta\,\xi + 2)}{\beta^3\,\xi}\right)^2. \tag{2.128}$$

When $(\beta = k,\ \sigma = 2\,k,\ \alpha = 0)$.

$$P(\zeta) = a_0 + \frac{2\,\sigma\,a_0\,e^{k\,\xi}}{\beta\,(1 - e^{k\,\xi})}, \tag{2.129}$$

$$U(\zeta) = \left(a_0 + \frac{2\,\sigma\,a_0\,e^{k\,\xi}}{\beta\,(1 - e^{k\,\xi})}\right)^2. \tag{2.130}$$

When $(2\,\beta = \alpha + \sigma)$.

$$v(\zeta) = a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{1 - \alpha\,e^{\frac{1}{2}(\alpha-\sigma)\xi}}{1 - \sigma\,e^{\frac{1}{2}(\alpha-\sigma)\xi}}\right], \tag{2.131}$$

$$U(\zeta) = \left(a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{1 - \alpha\,e^{\frac{1}{2}(\alpha-\sigma)\xi}}{1 - \sigma\,e^{\frac{1}{2}(\alpha-\sigma)\xi}}\right]\right)^2, \tag{2.132}$$

or

$$P(\zeta) = a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{\alpha\,e^{\frac{1}{2}(\alpha-\sigma)\xi} + 1}{-\sigma\,e^{\frac{1}{2}(\alpha-\sigma)\xi} - 1}\right], \tag{2.133}$$

$$U(\zeta) = \left(a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{\alpha\,e^{\frac{1}{2}(\alpha-\sigma)\xi} + 1}{-\sigma\,e^{\frac{1}{2}(\alpha-\sigma)\xi} - 1}\right]\right)^2. \tag{2.134}$$

When $(-2\,\beta = \alpha + \sigma)$.

$$v(\zeta) = a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{e^{\frac{1}{2}(\alpha-\sigma)\,\xi} + \alpha}{e^{\frac{1}{2}(\alpha-\sigma)\,\xi} + \sigma}\right], \tag{2.135}$$

$$U(\zeta) = \left(a_0 + \frac{2\,\sigma\,a_0}{\beta}\left[\frac{e^{\frac{1}{2}(\alpha-\sigma)\,\xi} + \alpha}{e^{\frac{1}{2}(\alpha-\sigma)\,\xi} + \sigma}\right]\right)^2. \tag{2.136}$$

When $(\alpha = 0)$.

$$P(\zeta) = a_0 + \frac{2\,\sigma\,a_0\,e^{\beta\,\xi}}{1 + \frac{\sigma}{2}\,e^{\beta\,\xi}}, \tag{2.137}$$

$$U(\zeta) = \left(a_0 + \frac{2\,\sigma\,a_0\,e^{\beta\,\xi}}{1 + \frac{\sigma}{2}\,e^{\beta\,\xi}}\right)^2. \tag{2.138}$$

When $(\beta = \alpha = \sigma \neq 0)$.

$$P(\zeta) = a_0 - \frac{2\,\sigma\,a_0(\alpha\,\xi + 2)}{\beta\,\alpha\,\xi}, \tag{2.139}$$

$$U(\zeta) = \left(a_0 - \frac{2\,\sigma\,a_0(\alpha\,\xi + 2)}{\beta\,\alpha\,\xi}\right)^2. \tag{2.140}$$

Case IV.

$$\Omega = \frac{2\,\sigma\,c}{\alpha\,a_1^2 - \sigma\,a_0^2},\ \beta = \frac{2\,\sigma\,a_0}{a_1},\ a_0 = a_0,\ a_1 = a_1,\ \alpha_1 = 0,$$
$$\alpha_2 = \frac{c{a_1}^2}{2\,(\alpha\,a_1^2 - \sigma\,a_0^2)^2},\ \alpha_3 = -\frac{c{a_1}^2}{4\,(\alpha\,a_1^2 - \sigma\,a_0^2)^2}.$$

So that, the exact traveling wave solution:

$$P(\zeta) = a_0 + a_1\,a^{f(\zeta)}. \tag{2.141}$$

Thus, the solitary traveling wave solutions:
When $(\beta^2 - \alpha\,\sigma < 0 \,\&\, \sigma \neq 0)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{\sigma}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right], \tag{2.142}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{\sigma}\,tan\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right]\right)^2, \tag{2.143}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{\sigma}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right], \tag{2.144}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{\sigma}\,cot\left(\frac{\sqrt{-(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right]\right)^2. \tag{2.145}$$

When $(\beta^2 - \alpha\,\sigma > 0 \,\&\, \sigma \neq 0)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\sigma} - \frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{\sigma}\,tanh\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right], \tag{2.146}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\sigma} - \frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{\sigma}\,tanh\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right]\right)^2, \tag{2.147}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\sigma} - \frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{\sigma}\,coth\left(\frac{\sqrt{(\beta^2 - \alpha\,\sigma)}}{2}\,\xi\right)\right], \tag{2.148}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\sigma} - \frac{\sqrt{(\beta^2-\alpha\sigma)}}{\sigma}\, coth\left(\frac{\sqrt{(\beta^2-\alpha\sigma)}}{2}\,\xi\right)\right]\right)^2. \tag{2.149}$$

When $(\beta^2+\alpha^2>0\,\&\,\sigma\neq 0\,\&\,\sigma=-\alpha)$.

$$P(\zeta) = a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\, tanh\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\,\xi\right)\right], \tag{2.150}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\, tanh\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\,\xi\right)\right]\right)^2, \tag{2.151}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\, coth\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\,\xi\right)\right], \tag{2.152}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{\beta^2+\alpha^2}}{\alpha}\, coth\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\,\xi\right)\right]\right)^2. \tag{2.153}$$

When $(\beta^2+\alpha^2<0\,\&\,\sigma\neq 0\,\&\,\sigma=-\alpha)$.

$$P(\zeta) = a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\alpha}\, tan\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\,\xi\right)\right], \tag{2.154}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\alpha}\, tan\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\,\xi\right)\right]\right)^2, \tag{2.155}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\alpha}\, cot\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\,\xi\right)\right], \tag{2.156}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{\beta}{\alpha} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\alpha}\, cot\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\,\xi\right)\right]\right)^2. \tag{2.157}$$

When $(\beta^2-\alpha^2<0\,\&\,\sigma=\alpha)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\alpha}\, tan\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\,\xi\right)\right], \tag{2.158}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\alpha}\, tan\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\,\xi\right)\right]\right)^2, \tag{2.159}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\alpha}\, cot\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\,\xi\right)\right], \tag{2.160}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\alpha}\, cot\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\,\xi\right)\right]\right)^2. \tag{2.161}$$

When $(\beta^2-\alpha^2>0\,\&\,\sigma=\alpha)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{\beta^2-\alpha^2}}{\alpha}\, tanh\left(\frac{\sqrt{\beta^2-\alpha^2}}{2}\,\xi\right)\right], \tag{2.162}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{\beta^2-\alpha^2}}{\alpha}\, tanh\left(\frac{\sqrt{\beta^2-\alpha^2}}{2}\,\xi\right)\right]\right)^2, \tag{2.163}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{\beta^2-\alpha^2}}{\alpha}\, coth\left(\frac{\sqrt{\beta^2-\alpha^2}}{2}\,\xi\right)\right], \tag{2.164}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-\beta}{\alpha} + \frac{\sqrt{\beta^2-\alpha^2}}{\alpha}\, coth\left(\frac{\sqrt{\beta^2-\alpha^2}}{2}\,\xi\right)\right]\right)^2. \tag{2.165}$$

When $(\alpha\sigma<0\,\&\,\sigma\neq 0\,\&\,\beta=0)$.

$$P(\zeta) = a_0 + a_1\left[\sqrt{\frac{-\alpha}{\sigma}}\, tanh\left(\frac{\sqrt{-\alpha\sigma}}{2}\,\xi\right)\right], \tag{2.166}$$

$$U(\zeta) = \left(a_0 + a_1\left[\sqrt{\frac{-\alpha}{\sigma}}\, tanh\left(\frac{\sqrt{-\alpha\sigma}}{2}\,\xi\right)\right]\right)^2, \tag{2.167}$$

or

$$P(\zeta) = a_0 + a_1\left[\sqrt{\frac{-\alpha}{\sigma}}\, coth\left(\frac{\sqrt{-\alpha\sigma}}{2}\,\xi\right)\right], \tag{2.168}$$

$$U(\zeta) = \left(a_0 + a_1\left[\sqrt{\frac{-\alpha}{\sigma}}\, coth\left(\frac{\sqrt{-\alpha\sigma}}{2}\,\xi\right)\right]\right)^2. \tag{2.169}$$

When $(\beta=0,\ \&\,\alpha=-\sigma)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-(1+e^{2\alpha\xi}) \pm \sqrt{2(e^{4\alpha\xi}+1)}}{e^{2\alpha\xi}-1}\right], \tag{2.170}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-(1+e^{2\alpha\xi}) \pm \sqrt{2(e^{4\alpha\xi}+1)}}{e^{2\alpha\xi}-1}\right]\right)^2, \tag{2.171}$$

or

$$P(\zeta) = a_0 + a_1\left[\frac{-(1+e^{2\alpha\xi}) \pm \sqrt{e^{4\alpha\xi}+6e^{2\alpha\xi}+1}}{2e^{2\alpha\xi}}\right], \tag{2.172}$$

$$U(\zeta) = \left(a_0 + a_1\left[\frac{-(1+e^{2\alpha\xi}) \pm \sqrt{e^{4\alpha\xi}+6e^{2\alpha\xi}+1}}{2e^{2\alpha\xi}}\right]\right)^2. \tag{2.173}$$

When $(\alpha=\sigma=0)$.

$$P(\zeta) = a_0 + a_1\left[\frac{-(1+e^{2\beta\xi}) \pm \sqrt{2(e^{4\beta\xi}+1)}}{e^{2\beta\xi}-1}\right], \tag{2.174}$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{-\left(1+e^{2\beta\xi}\right) \pm \sqrt{2(e^{4\beta\xi}+1)}}{e^{2\beta\xi}-1}\right]\right)^2, \quad (2.175)$$

or

$$P(\zeta) = a_0 + a_1 \left[\frac{-\left(1+e^{2\beta\xi}\right) \pm \sqrt{e^{4\beta\xi}+6e^{2\beta\xi}+1}}{2e^{2\beta\xi}}\right], \quad (2.176)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{-\left(1+e^{2\beta\xi}\right) \pm \sqrt{e^{4\beta\xi}+6e^{2\beta\xi}+1}}{2e^{2\beta\xi}}\right]\right)^2. \quad (2.177)$$

When ($\beta^2 = \alpha\sigma$).

$$P(\zeta) = a_0 + a_1 \left[\frac{-\alpha(\beta\xi+2)}{\beta^2\xi}\right], \quad (2.178)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{-\alpha(\beta\xi+2)}{\beta^2\xi}\right]\right)^2. \quad (2.179)$$

When ($\beta = k,\ \alpha = 2k,\ \sigma = 0$).

$$P(\zeta) = a_0 + a_1 \left[e^{k\xi} - 1\right], \quad (2.180)$$

$$U(\zeta) = \left(a_0 + a_1 \left[e^{k\xi} - 1\right]\right)^2. \quad (2.181)$$

When ($\beta = k,\ \sigma = 2k,\ \alpha = 0$).

$$P(\zeta) = a_0 + a_1 \left[\frac{e^{k\xi}}{1-e^{k\xi}}\right], \quad (2.182)$$

$$U(\zeta) = \left(a_0 + \frac{2\sigma a_0}{\beta}\left[\frac{e^{k\xi}}{1-e^{k\xi}}\right]\right)^2. \quad (2.183)$$

When ($2\beta = \alpha + \sigma$).

$$P(\zeta) = a_0 + a_1 \left[\frac{1-\alpha e^{\frac{1}{2}(\alpha-\sigma)\xi}}{1-\sigma e^{\frac{1}{2}(\alpha-\sigma)\xi}}\right], \quad (2.184)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{1-\alpha e^{\frac{1}{2}(\alpha-\sigma)\xi}}{1-\sigma e^{\frac{1}{2}(\alpha-\sigma)\xi}}\right]\right)^2, \quad (2.185)$$

or

$$P(\zeta) = a_0 + a_1 \left[\frac{\alpha e^{\frac{1}{2}(\alpha-\sigma)\xi}+1}{-\sigma e^{\frac{1}{2}(\alpha-\sigma)\xi}-1}\right], \quad (2.186)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{\alpha e^{\frac{1}{2}(\alpha-\sigma)\xi}+1}{-\sigma e^{\frac{1}{2}(\alpha-\sigma)\xi}-1}\right]\right)^2. \quad (2.187)$$

When ($-2\beta = \alpha + \sigma$).

$$P(\zeta) = a_0 + a_1 \left[\frac{e^{\frac{1}{2}(\alpha-\sigma)\xi}+\alpha}{e^{\frac{1}{2}(\alpha-\sigma)\xi}+\sigma}\right], \quad (2.188)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{e^{\frac{1}{2}(\alpha-\sigma)\xi}+\alpha}{e^{\frac{1}{2}(\alpha-\sigma)\xi}+\sigma}\right]\right)^2. \quad (2.189)$$

When ($\alpha = 0$).

$$P(\zeta) = a_0 + a_1 \left[\frac{\beta e^{\beta\xi}}{1+\frac{\sigma}{2}e^{\beta\xi}}\right], \quad (2.190)$$

$$U(\zeta) = \left(a_0 + \frac{2\sigma a_0}{\beta}\left[\frac{\beta e^{\beta\xi}}{1+\frac{\sigma}{2}e^{\beta\xi}}\right]\right)^2. \quad (2.191)$$

When ($\beta = \alpha = \sigma \neq 0$).

$$P(\zeta) = a_0 + a_1 \left[\frac{-(\alpha\xi+2)}{\alpha\xi}\right], \quad (2.192)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{-(\alpha\xi+2)}{\alpha\xi}\right]\right)^2. \quad (2.193)$$

When ($\beta = \sigma = 0$).

$$P(\zeta) = a_0 + a_1 \left[\frac{\alpha}{2}\xi\right], \quad (2.194)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{\alpha}{2}\xi\right]\right)^2. \quad (2.195)$$

When ($\beta = \alpha = 0$).

$$P(\zeta) = a_0 + a_1 \left[\frac{-2}{\sigma\xi}\right], \quad (2.196)$$

$$U(\zeta) = \left(a_0 + a_1 \left[\frac{-2}{\sigma\xi}\right]\right)^2. \quad (2.197)$$

When ($\beta = 0,\ \alpha = \sigma$).

$$P(\zeta) = a_0 + a_1 \left[tan\left(\frac{\alpha\xi+C}{2}\right)\right], \quad (2.198)$$

$$U(\zeta) = \left(a_0 + a_1 \left[tan\left(\frac{\alpha\xi+C}{2}\right)\right]\right)^2. \quad (2.199)$$

When ($\sigma = 0$).

$$P(\zeta) = a_0 + a_1 \left[e^{\beta\xi} - \frac{\alpha}{2\beta}\right], \quad (2.200)$$

$$U(\zeta) = \left(a_0 + a_1 \left[e^{\beta\xi} - \frac{\alpha}{2\beta}\right]\right)^2. \quad (2.201)$$

- **Note that:**
  All the obtained results have been checked with Maple 2017 by putting them back into the original equation and found correct.

Discuss the results

In our discussion, we will concentrate on three items which are:

1. Methods (extended tanh-function method, extended simple equation method and new auxiliary equation method).
2. Results that obtained by the methods that pointed above.
3. Our results and an other results that obtained by different researchers who apply different methods on this model.

Now, we explain and discuss previous items in details in the following steps:

1. Methods (extended tanh-function method, extended simple equation method and new auxiliary equation method). we can see each of these methods assumed the exact solution form for each model

$$P(\zeta) = \begin{cases} a_0 + \sum_{i=1}^{m}\left(a_i\ \phi^i(\zeta) + b_i\ \phi^{-i}(\zeta)\right), \\ \sum_{i=-N}^{N} a_i\,\phi^i(\zeta), \\ \sum_{i=-N}^{N} a_i\,a^{i f(\zeta)}. \end{cases} \quad (3.1)$$

where $\phi(\zeta), f(\zeta)$ satisfy the following second order non linear ordinary differential equation(LODE) in the following order:

$$\begin{cases} \phi'(\zeta) = b + \phi^2(\zeta), \\ \phi'(\zeta) = \alpha + \lambda\,\phi(\zeta) + \mu\,\phi^2(\zeta), \\ f'(\zeta) = \frac{1}{ln(a)}\left(\alpha\,a^{-f(\zeta)} + \beta + \sigma\,a^{f(\zeta)}\right). \end{cases} \tag{3.2}$$

With simple precision, we can conclude that each of three methods is based on Riccati equation and also we can get equivalent of three methods and that happened when:

- $\left[\phi(\zeta) = a^{f(\zeta)},\ \beta = 0,\ \sigma = 1,\ b = \alpha\right]$, so that extended tanh-function method is equal to new auxiliary equation method.
- $\left[\phi(\zeta) = a^{f(\zeta)},\ \beta = \lambda,\ \sigma = \mu\right]$, so that extended simple equation method is equal to new auxiliary equation method.

However, this equivalent between three methods but we can say that both of extended tanh-function method and extended simple equation method are special cases of our new auxiliary equation method and that's because of the results that obtained by new auxiliary equation method can cover all results obtained by these two methods and not just like that but it gives also the other form of solutions such that it is one of the few methods that obtained a very large number of solutions for every model.

We can also see the superiority of our new auxiliary equation method on Riccati method itself and many of methods such that: We obtain (6) exact and solitary traveling wave solutions by applying (extended tanh-function method), (25) exact and solitary traveling wave solutions by applying (extended simple equation method), (61) exact and solitary traveling wave solutions by our new auxiliary equation (Khater method). By applying these methods on pressure equation of bubbly liquids with examination for viscosity and heat transfer that makes the study of the physical properties of this model is very comfortable, easy and interesting.

2. Results that obtained by the methods that pointed above.

   In this part of our research, we will show some of equivalent between our solutions in the following steps:

   - Eqs. (2.25), (2.27) are equal to Eqs. (2.67), (2.69) when: $(b_1 = -a_1,\ \alpha = C = 1,\ \mu = -b)$.
   - Eqs. (2.29), (2.31) are equal to Eqs. (2.63), (2.65) when: $(a_1 = b_1,\ b = \mu,\ \alpha = 1,\ C = 0)$.
   - Eqs. (2.42), (2.44) are equal to Eqs. (2.103), (2.105) when: $\left(C = 0,\ \beta = \sqrt{\alpha\,(4\,\mu - \sigma)} = 4\,\alpha\right)$.
   - Eqs. (2.46), (2.48) are equal to Eqs. (2.107), (2.109) when: $(C = 1,\ \alpha = \beta,\ a_1 = 2\,a_0)$.

We showed some few of equivalence between our solutions and you can complete it by the same manner. Using this manner helps us to see how our new auxiliary equation method covered both of extended tanh-function method and extended simple equation method.

3. Our results and an other results that obtained by different researchers who apply different methods on this model:

   - Firstly: we Will compare our solutions with that obtained by Yun-Mei Zhao who used the F-expansion method, and the extended version of F-expansion method [27]:
     In this research the author try to find the balance of Eq. (11, [27]) and he obtained it (n = 2), however the probability of finding the balance of this equation is impossible such that the mathematical formula of the balance is:

$$D\left(\frac{d^q\,u}{d\,\xi^q}\right) = m + q,\ D\left(u^p\left(\frac{d^q\,u}{d\,\xi^q}\right)^s\right) = m\,p + s\,(m + q).$$

   That thing make us think again how the author get the balance between $\phi\,\phi''\,\&\,\phi^2\ \Rightarrow\ n + n + 2 = 2n$. So that, this relation is wrong. and even if he take the balance between $\phi\,\phi''\,\&\,\phi'^2\ \Rightarrow\ n + n + 2 = 2(n+1)$. So that, this relation is wrong too. For this reason, we strongly recommend the author have to think again of this research.

   - Secondly: We will compare our solutions with that obtained by Yinghui He, Shaolin Li, and Yao Long who used the Multiple (G'/G)-Expansion Method [28]:
     In this research the author try to find the balance of Eq. (14, [28]) which is equal to Eq. (11, [24]) and they obtained the value of the balance (n = 1). So that, for the same equation two different balance once equal one and another one equal one. As we show that is the mistake in calculating the value of the balance that make all of these papers are wrong. For this reason, we strongly recommend the author have to think again of this research.
   - Thirdly: We will compare our solutions with that obtained by Bin He, Qing Meng, Yao Long who used the bifurcation method of dynamical systems and the method of phase portraits analysis [31] in the following steps:(a) Eq. (2.49) is equal to Eq. (16, [31]) when:

$$\omega = \sqrt{-\alpha\,\mu},\ \alpha = -2\,a_0\,a_{-1}\,\omega\,tanh(\omega\,\zeta),\ a_{-1} = \sqrt{\frac{\alpha\,(\phi_2 - \phi_M)}{\mu}},\ a_0 = \sqrt{-\phi_M}.$$

   (b) Eq. (2.167) is equal to Eq. (18, [31]) when:

$$2\,\omega = \sqrt{-\alpha\,\sigma},\ \sigma = -4\,a_0^2\,a_1^2\,\alpha\,tanh(2\,\omega\,\zeta),\ a_1 = \sqrt{\frac{\sigma\,(\phi_1 - \phi_m)}{\alpha}},\ a_0 = \sqrt{-\phi_m}.$$

Conclusion

In this research, we succeed in apply three different methods on one of the most important model in fluid mechanic which is pressure equation of bubbly liquids with examination for viscosity and heat transfer which also has the name the Kudryashov-Sinelshchikov equation. That equation which describe the pressure waves in the liquid with gas bubbles taking into account the heat transfer and viscosity. We get many forms of exact and solitary traveling wave solutions. we present a good comparison between the methods and results. All of these improve, how our new auxiliary equation method is one of the most powerful, direct, accuracy, efficiency, versatility method in the field of nonlinear partial differential equation.

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