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Travelling wave solutions and conservation laws for the Korteweg-de Vries-Bejamin-Bona-Mahony equation

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ABSTRACT

In this work we study the Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation, which describes the two-way propagation of waves. Using Lie symmetry method together with Jacobi elliptic function expansion and Kudryashov methods we construct its travelling wave solutions. Also, we derive conservation laws of the KdV-BBM equation using the variational derivative approach. In this method, we begin by computing second-order multipliers for the KdV-BBM equation followed by a derivation of the respective conservation laws for each multiplier.

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Introduction

Over the years nonlinear partial differential equations (NLPDEs) have proven indispensable in the modelling of diverse nonlinear multidimensional systems which are manifest in countless and varied natural phenomena. Studying NLPDEs is fundamental to understanding the complex behaviours of these systems and many researchers continue to explore this avenue.

Many methods have been developed for obtaining exact solutions of NLPDEs. These include the homogeneous balance method [1], the ansatz method [2], the inverse scattering transform method [3], the Bäcklund transformation [4], the Darboux transformation [5], the Hirota bilinear method [6], the simplest equation method [7], the (G'/G)–expansion method [8], the Jacobi elliptic function expansion method [9], the Kudryashov method [10], the Lie symmetry method [11–14].

Furthermore conservation laws are essential in determining the extent of integrability of differential equations, development of numerical schemes, reduction and solutions of partial differential equations. See, for example [14,15] and references therein.

The third and fifth-order Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equations are derived in [16] as models for long-crested water waves which travel mostly unidirectionally. According to [16], such models tend to be more accurate on a much larger time scale. The authors of [16] used the results published in [17,18], where a variant of the higher order Boussinesq coupled systems were derived. For more details of the physical description of the equations the reader is referred to [19].

In this paper the third-order KdV-BBM equation

$$u_t + u_x + \frac{3}{2} u u_x + \nu u_{xxx} - \left(\frac{1}{6} - \nu\right) u_{txx} = 0, \quad \nu \neq \frac{1}{6}, \tag{1.1}$$

is the subject of our study. This equation is a generalization of the celebrated KdV and BBM equations. When $\nu = 1/6$, Eq. (1.1) reduces to the KdV equation and when $\nu = 0$, Eq. (1.1) becomes the BBM equation. Eq. (1.1) was investigated in [19] where the researchers presented conditions for the existence of hyperbolic, unbounded periodic and soliton solutions in terms of Weierstrass functions. Furthermore, an analysis for the initial value problem was developed together with a local and global well-posedness theory for (1.1).

Exact travelling wave solutions of (1.1) are obtained here by making use of different approaches. Moreover, we derive the conservation laws of the KdV-BBM equation by employing the variational derivative approach.

Travelling wave solutions of (1.1)

In this section we construct travelling wave solutions of (1.1) using direct integration, Kudryashov and extended Jacobi elliptic function expansion methods.

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Exact solutions of (1.1) using its Lie point symmetries

The vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{2.2}$$

where $\xi^i, i = 1,2$ and η depend on t,x and u, is a Lie point symmetry of (1.1) if

$$X^{[3]}\Delta|_{\Delta=0} = 0, \tag{2.3}$$

where

$$\Delta \equiv u_t + u_x + \frac{3}{2}uu_x + \nu u_{xxx} - \left(\frac{1}{6} - \nu\right)u_{txx},$$

and $X^{[3]}$ is the third prolongation [12,13] of (2.2) defined by

$$X^{[3]} = X + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}} + \zeta_{xxx}\frac{\partial}{\partial u_{xxx}} + \cdots, \tag{2.4}$$

where $\zeta_t, \zeta_x, \zeta_{xx}, \zeta_{txx}$ and ζ_{xxx} are determined as follows:

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2),\\
\zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2),\\
\zeta_{txx} &= D_x(\zeta_{tx}) - u_{txx} D_x(\xi^1) - u_{xxx} D_x(\xi^2),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2)
\end{aligned} \tag{2.5}$$

with the total derivatives D_t and D_x given by

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \cdots\\
D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots.
\end{aligned} \tag{2.6}$$

Expanding (2.3) we obtain

$$\begin{aligned}
&\frac{1}{6}\xi^2_{tuu}u_x^3 - \nu\xi^2_{uuu}u_x^4 - \nu u_t\xi^1_{uuu}u_x^3 - \nu\xi^2_{tuu}u_x^3 - \nu u_t\xi^2_{uuu}u_x^3 + \frac{1}{6}u_t\xi^2_{uuu}u_x^3 - 3\nu\xi^2_{xuu}u_x^3 + \nu\eta_{uuu}u_x^3 - \frac{3}{2}u\xi^2_u u_x^2 - \xi^2_u u_x^2 - \nu u_{tt}\xi^1_{uu}u_x^2 + \frac{1}{6}u_{tt}\xi^1_{uu}u_x^2 - 3\nu u_{tx}\xi^1_{uu}u_x^2\\
&- \nu u_{tt}\xi^1_{xx} - 3\nu u_{tx}\xi^2_{uu}u_x^2 + \frac{1}{2}u_{tx}\xi^2_{uu}u_x^2 - 6\nu u_{xx}\xi^2_{uu}u_x^2 - \nu u_t\xi^1_{tuu}u_x^2 + \frac{1}{6}u_t\xi^1_{tuu}u_x^2 - \nu u_t^2\xi^1_{uuu}u_x^2 + \frac{1}{6}u_t^2\xi^1_{uuu}u_x^2 - 3\nu u_t\xi^1_{xuu}u_x^2 - 2\nu\xi^2_{txu}u_x^2 + \frac{1}{3}\xi^2_{txu}u_x^2\\
&- 2\nu u_t\xi^2_{xuu}u_x^2 + \frac{1}{3}u_t\xi^2_{xuu}u_x^2 - 3\nu\xi^2_{xxu}u_x^2 + \nu\eta^1_{tuu}u_x^2 - \frac{1}{6}\eta_{tuu}u_x^2 + \nu u_t\eta_{uuu}u_x^2 - \frac{1}{6}u_t\eta_{uuu}u_x^2 + 3\nu\eta_{xuu}u_x^2 + \frac{3\eta u_x}{2} - \frac{3}{2}uu_t\xi^1_u u_x - u_t\xi^1_u u_x - \xi^2_t u_x - u_t\xi^2_u u_x\\
&- \frac{3}{2}u\xi^2_x u_x + \frac{3}{2}u\eta_u u_x - 2\nu u_t u_{xx}\xi^2_{xu} - 2\nu u_{tx}\xi^1_{tu}u_x + \frac{1}{3}u_{tx}\xi^1_{tu}u_x - 4\nu u_t u_{tx}\xi^1_{uu}u_x + \frac{2}{3}u_t u_{tx}\xi^1_{uu}u_x - 3\nu u_t u_{xx}\xi^1_{uu}u_x + \frac{1}{6}u_{tt}\xi^1_{xx} - 2\nu u_{tt}\xi^1_{xu}u_x + \frac{1}{3}u_{tt}\xi^1_{xu}u_x\\
&- 6\nu u_{tx}\xi^1_{xu}u_x - 3\nu u_{xx}\xi^2_{tu}u_x + \frac{1}{2}u_{xx}\xi^2_{tu}u_x - 3\nu u_t u_{xx}\xi^2_{uu}u_x + \frac{1}{2}u_t u_{xx}\xi^2_{uu}u_x - 4\nu u_{tx}\xi^2_{xu}u_x + \frac{2}{3}u_{tx}\xi^2_{xu}u_x - 9\nu u_{xx}\xi^2_{xu}u_x - \xi^2_x u_x + 2\nu u_{tx}\eta_{uu}u_x - \frac{1}{3}u_{tx}\eta_{uu}u_x\\
&+ 3\nu u_{xx}\eta_{uu}u_x - 2\nu\xi^1_u u_{ttx}u_x + \frac{1}{3}\xi^1_u u_{ttx}u_x - 3\nu\xi^1_u u_{txx}u_x - 3\nu\xi^2_u u_{txx}u_x + \frac{1}{2}\xi^2_u u_{txx}u_x - 4\nu\xi^2_u u_{xxx}u_x - 2\nu u_t\xi^1_{txu}u_x + \frac{1}{3}u_t\xi^1_{txu}u_x - 2\nu u_t^2\xi^1_{xuu}u_x\\
&+ \frac{1}{3}u_t^2\xi^1_{xuu}u_x - 3\nu u_t\xi^1_{xxu}u_x + \frac{1}{6}\xi^2_{txx}u_x - \nu u_t\xi^2_{xxu}u_x - \nu\xi^2_{txx}u_x + \eta_t + \frac{1}{6}u_t\xi^2_{xxu}u_x + 2\nu\eta_{txu}u_x - \frac{1}{3}\eta_{txu}u_x + 2\nu u_t\eta_{xuu}u_x - \frac{1}{3}u_t\eta_{xuu}u_x + 3\nu\eta_{xxu}u_x\\
&- 2\nu\xi^1_u u_{tx}^2 + \frac{1}{3}\xi^1_u u_{tx}^2 - 3\nu\xi^2_u u_{xx}^2 - u_t^2\xi^1_u - \frac{3}{2}uu_t\xi^1_x - u_t\xi^1_x + u_t\eta_u + \frac{3u\eta_x}{2} + \eta_x - \nu\xi^1_u u_{tt}u_{xx} - \nu\xi^2_{xxx}u_x + \frac{1}{6}\xi^1_u u_{tt}u_{xx} - 3\nu\xi^1_u u_{tx}u_{xx} - 3\nu\xi^2_u u_{tx}u_{xx}\\
&+ \frac{1}{2}\xi^2_u u_{tx}u_{xx} - \nu u_t u_{xx}\xi^1_{tu} + \frac{1}{3}u_{tx}\xi^1_{tx} + \frac{1}{6}u_t u_{xx}\xi^1_{tu} - 2\nu u_{tx}\xi^1_{tx} - \nu u_t^2 u_{xx}\xi^1_{uu} + \frac{1}{6}u_t^2 u_{xx}\xi^1_{uu} - 4\nu u_t u_{tx}\xi^1_{xu} + \frac{2}{3}u_t u_{tx}\xi^1_{xu} - 3\nu u_t u_{xx}\xi^1_{xu} - 3\nu u_{tx}\xi^1_{xx} - 2\nu u_{xx}\xi^2_{tx}\\
&+ \frac{1}{3}u_{xx}\xi^2_{tx} + \frac{1}{3}u_t u_{xx}\xi^2_{xu} - \nu u_{tx}\xi^2_{xx} + \frac{1}{6}u_{tx}\xi^2_{xx} + \eta_u u_x + \nu u_{xx}\eta^1_{tu} - \frac{1}{6}u_{xx}\eta_{tu} + \nu u_t u_{xx}\eta_{uu} - \frac{1}{6}u_t u_{xx}\eta_{uu} + 2\nu u_{tx}\eta_{xu} - \frac{1}{3}u_{tx}\eta_{xu} + 3\nu u_{xx}\eta_{xu} - 2\nu\xi^1_x u_{ttx}\\
&+ \frac{1}{3}\xi^1_x u_{ttx} - \nu\xi^1_t u_{txx} + \frac{1}{6}\xi^1_t u_{txx} - 2\nu u_t\xi^1_u u_{txx} + \frac{1}{3}u_t\xi^1_u u_{txx} - 2\nu\xi^2_x u_{txx} + \frac{1}{3}\xi^2_x u_{txx} + \nu\eta_u u_{txx} - \frac{1}{6}\eta_u u_{txx} - \nu u_t\xi^1_u u_{xxx} - \nu\xi^2_t u_{xxx} + \frac{1}{6}\xi^2_t u_{xxx} - \nu u_t\xi^2_u u_{xxx}\\
&+ \frac{1}{6}u_t\xi^2_u u_{xxx} - 3\nu\xi^2_x u_{xxx} + \nu\eta_u u_{xxx} - \nu u_t\xi^1_{txx} + \frac{1}{6}u_t\xi^1_{txx} - \nu u_t^2\xi^1_{xxu} + \frac{1}{6}u_t^2\xi^1_{xxu} - \nu u_t\xi^1_{xxx} + \nu\eta^1_{txx} - \frac{1}{6}\eta_{txx} + \nu u_t\eta_{xxu} - \frac{1}{6}u_t\eta^1_{xxu} - 3\nu u_{xx}\xi^2_{xx} - 3\nu\xi^1_x u_{txx}\\
&+ \nu\eta_{xxx} - u_t\xi^1_t = 0.
\end{aligned}$$

Now splitting the above equation with respect to the derivatives of u, yields the following system of overdetermined linear partial differential equations:

$$\xi^1_u = 0, \quad \xi^2_u = 0, \quad \eta_{uu} = 0, \quad \xi^1_x = 0,$$

$$2\eta_{xu} - \xi^2_{xx} = 0,$$

$$\eta_{xxu} + 2\xi^2_x = 0,$$

$$3\nu\eta_{xu} - 3\nu\xi^2_{xx} + \left(\nu - \frac{1}{6}\right)\eta_{tu} - 2\left(\nu - \frac{1}{6}\right)\xi^2_{tx} = 0,$$

$$\nu\eta_{xxx} + \eta_t + \left(\nu - \frac{1}{6}\right)\eta_{txx} + \left(\frac{3u}{2} + 1\right)\eta_x = 0,$$

$$\nu\xi^1_t - 3\nu\xi^2_x - \left(\nu - \frac{1}{6}\right)\xi^2_t - \left(\nu - \frac{1}{6}\right)\nu\eta_{xxu} = 0,$$

$$\frac{3}{2}\eta - \nu\xi^2_{xxx} - \xi^2_t + 2\left(\nu - \frac{1}{6}\right)\eta_{txu} + \left(\frac{3u}{2} + 1\right)\xi^1_t - \left(\nu - \frac{1}{6}\right)\xi^2_{txx} + \left(2\nu + \frac{1}{6} - \frac{3}{2}u\left(\nu - \frac{1}{6}\right)\right)\eta_{xxu} - \left(\frac{3u}{2} + 1\right)\xi^2_x = 0.$$

Solving the above system we obtain

$$\xi^1(t,x,u) = A_1 t + A_2, \tag{2.7}$$

$$\xi^2(t,x,u) = \frac{6A_1 vt}{6v-1} + A_3, \tag{2.8}$$

$$\eta(t,x,u) = \frac{A_1(2-3(6v-1)u)}{18v-3}, \tag{2.9}$$

where A_1, A_2 and A_3 are arbitrary constants. Thus the infinitesimal generators corresponding to (2.7)–(2.9) are

$$X_1 = \frac{\partial}{\partial x}, \tag{2.10}$$

$$X_2 = \frac{\partial}{\partial t}, \tag{2.11}$$

$$X_3 = 3t(6v-1)\frac{\partial}{\partial t} + 18vt\frac{\partial}{\partial x} + (2-3(6v-1)u)\frac{\partial}{\partial u}. \tag{2.12}$$

A linear combination [20] of the translation symmetries (2.10) and (2.11), namely $X = cX_1 + X_2$, yields the two invariants

$$z = x - ct \quad \text{and} \quad U = u, \tag{2.13}$$

which gives the group-invariant solution $U = U(z)$. Using this result, and with z as our new independent variable, (1.1) is transformed into the nonlinear ordinary differential equation (ODE)

$$\left(\frac{c}{6} - cv + v\right)U'''(z) + (1-c)U'(z) + \frac{3}{2}U(z)U'(z) = 0. \tag{2.14}$$

Integrating (2.14) with respect to z, we obtain

$$\left(\frac{c}{6} - cv + v\right)U''(z) + (1-c)U(z) + \frac{3}{4}U^2(z) + C_0 = 0, \tag{2.15}$$

where C_0 is an arbitrary constant of integration. The linear ODE (2.15) can be rewritten as

$$U''(z) + \frac{9}{2(c-6cv+6v)}U^2(z) + \frac{6(1-c)}{c-6cv+6v}U(z) + \frac{6C_0}{c-6cv+6v} = 0. \tag{2.16}$$

By introducing a new variable $y = y(z)$ in (2.16) defined by

$$U(z) = \frac{2}{3}(c-6cv+6v)y(z), \quad c \neq \frac{6v}{6v-1}, \tag{2.17}$$

Eq. (2.16) can be re-written in terms of $y(z)$ as

$$y'' + 3y^2 - \omega y + C_1 = 0, \tag{2.18}$$

where C_1 and ω are given by

$$C_1 = \frac{6C_0}{c-6cv+6v}, \quad \omega = -\frac{6(1-c)}{c-6cv+6v}. \tag{2.19}$$

The solutions of (2.16) can be expressed via those of Eq. (2.2) in [21]. We now turn our attention to (2.18). Multiplying (2.18) with y' and integrating with respect to z yields the first order linear ODE

$$y'^2 + 2y^3 - \omega y^2 + 2C_1 y + 2C_2 = 0, \tag{2.20}$$

where C_2 is an arbitrary constant of integration. Since the expression

$$2y^3 - \omega y^2 + 2C_1 y + 2C_2 \tag{2.21}$$

in (2.20) is a cubic function in $y(z)$, it is reasonable to assume that (2.20) can be written as

$$y'^2 = -2(y-\lambda_1)(y-\lambda_2)(y-\lambda_3). \tag{2.22}$$

where λ_1, λ_2 and λ_3 are roots of (2.21) ordered so that $\lambda_3 \geqslant \lambda_2 \geqslant \lambda_1$. The general solution of (2.18) can thus be expressed in terms of the Jacobi elliptic cosine amplitude function [21,22]

$$y(z) = \lambda_2 + (\lambda_3 - \lambda_2)cn^2\left(\sqrt{\frac{\lambda_3-\lambda_1}{2}}z, k^2\right), \quad k^2 = \frac{\lambda_3-\lambda_2}{\lambda_3-\lambda_1}, \tag{2.23}$$

where $0 \leqslant k^2 \leqslant 1$ and $(\lambda_3 - \lambda_1)/2 \geqslant 0$.

Reverting to the original variables, the solution of (1.1) is

$$u(t,x) = \frac{2}{3}(c-6cv+6v)\left\{\lambda_2 + (\lambda_3-\lambda_2)cn^2\left(\sqrt{\frac{\lambda_3-\lambda_1}{2}}z, k^2\right)\right\}. \tag{2.24}$$

By letting $c = 1, v = 0.083, \lambda_1 = 0.1, \lambda_2 = 0.2$ and $\lambda_3 = 0.4$, the graphical representation of (2.24) is rendered in Fig. 1.

Exact solutions of (1.1) using the Kudryashov method

In this section we invoke the Kudryashov method [10,11] to determine exact solutions of (1.1). We begin by assuming that the solutions to (2.14) can be written in the form

$$U(z) = \sum_{i=0}^{M} A_i Q^i(z), \tag{2.25}$$

where Q satisfies the Riccati equation

$$Q'(z) = Q^2(z) - Q(z). \tag{2.26}$$

The number M can be determined by using the balancing procedure as in [10] and A_i, $i = 0, 1, 2\ldots$ are constants to be determined. The Riccati equation has a solution in terms of elementary functions given by

$$Q(z) = \frac{1}{1+e^z}. \tag{2.27}$$

From (2.14), it can be seen that $M = 2$. Thus the solution (2.25) can be written as

$$U(z) = A_0 + A_1 Q(z) + A_2 Q^2(z). \tag{2.28}$$

Substituting (2.28) into (2.14) we obtain the following polynomial in $Q(z)$:

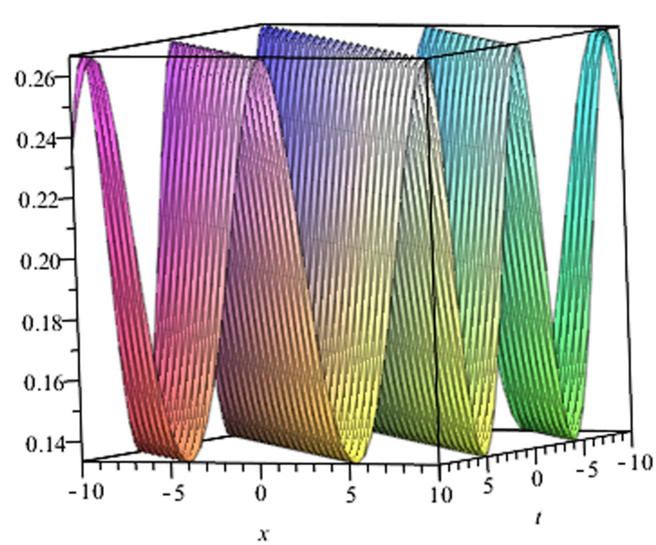

Fig. 1. Graphical representation of solution (2.24).

$$
\begin{aligned}
&12cvA_1Q^3(z) - 6cvA_1Q^4(z) - 7cvA_1Q^2(z) + cvA_1Q(z)\\
&\quad - 24cvA_2Q^5(z) + 54cvA_2Q^4(z) - 38cvA_2Q^3(z)\\
&\quad + 8cvA_2Q^2(z) - 2A_2Q^2(z) - 3A_2^2Q^4(z) + 3A_2^2Q^5(z)\\
&\quad - A_1Q(z) + \frac{3}{2}A_1^2Q^3(z) - \frac{3}{2}A_1^2Q^2(z) + A_1Q^2(z) + 2A_2Q^3(z)\\
&\quad - 9cA_2Q^4(z) + 4cA_2Q^5(z) - 8vA_2Q^2(z) + cA_1Q^4(z)\\
&\quad + \frac{5}{6}cA_1Q(z) + \frac{13}{3}cA_2Q^3(z) - 2cA_1Q^3(z) + \frac{2}{3}cA_2Q^2(z)\\
&\quad + 7vA_1Q^2(z) - vA_1Q(z) + 24vA_2Q^5(z) + \frac{1}{6}cA_1Q^2(z)\\
&\quad - 54vA_2Q^4(z) + 38vA_2Q^3(z) - 3A_0A_2Q^2(z)\\
&\quad + \frac{9}{2}A_1Q^4(z)A_2 + 3A_0A_2Q^3(z) - 12vA_1Q^3(z) - \frac{3}{2}A_0A_1Q(z)\\
&\quad - \frac{9}{2}A_1Q^3(z)A_2 + \frac{3}{2}A_0A_1Q^2(z) + 6vA_1Q^4(z) = 0. \qquad (2.29)
\end{aligned}
$$

Splitting Eq. (2.29) on powers of $Q(z)$ we obtain algebraic equations, viz.,

$$
\begin{aligned}
&\frac{9}{2}A_1A_2 + 6vA_1 - 54vA_2 + cA_1 - 9cA_2 - 6cvA_1 + 54cvA_2 - 3A_2^2 = 0,\\
&3A_0A_2 - \frac{9}{2}A_1A_2 - 12vA_1 + 38vA_2 + \frac{13}{3}cA_2 - 2cA_1 + 12cvA_1\\
&\quad - 38cvA_2 + 2A_2 + \frac{3}{2}A_1^2 = 0,\\
&4cA_2 + 24vA_2 + 3A_2^2 - 24cvA_2 = 0,\\
&\frac{3}{2}A_0A_1 - 3A_0A_2 + 7vA_1 - 8vA_2 + \frac{1}{6}cA_1 + \frac{2}{3}cA_2 - 7cvA_1\\
&\quad + 8cvA_2 + A_1 - \frac{3}{2}A_1^2 - 2A_2 = 0,\\
&\frac{3}{2}A_0A_1 + vA_1 + A_1 - \frac{5}{6}cA_1 - cvA_1 = 0. \qquad (2.30)
\end{aligned}
$$

Below we give one solution of interest obtained from solving (2.30).

$$
\begin{aligned}
A_0 &= \frac{1}{9}(6cv + 5c - 6v - 6),\\
A_1 &= \frac{1}{3}(24v - 24cv + 4c),\\
A_2 &= -\frac{1}{3}(24v - 24cv + 4c). \qquad (2.31)
\end{aligned}
$$

Thus, from (2.13), (2.28) and (2.31) we can write the solution of (1.1) and (2.14) as

$$
u(t,x) = \frac{1}{9}(6cv + 5c - 6v - 6) + \frac{1}{3}(24v - 24cv + 4c)\frac{e^z}{(1+e^z)^2}. \qquad (2.32)
$$

One possible graphical representation of (2.32) is given in Fig. 2. As expected [22], the result (2.32) is contained in (2.24). This can be seen if we let $\lambda_1 = \lambda_2$ in (2.24), which yields a soliton whose outline is akin to Fig. 2. Thus, we conclude that by using Kudryashov method we obtain a special case of solution (2.24).

Exact solutions of (1.1) using the extended Jacobi elliptic function expansion method

We now turn our attention to another interesting method of obtaining exact solutions, that is, the extended Jacobi elliptic function expansion method [23–26].

Cnoidal wave solutions of (1.1) using the extended Jacobi elliptic function expansion method

In this subsection we employ the Jacobi elliptic cosine function to obtain cnoidal wave solutions of (1.1).

We assume that our solutions can be expressed in the form

$$
U(z) = \sum_{i=-M}^{M} A_i H(z)^i, \qquad (2.33)
$$

where H is a solution to the first-order ODE [24,25]

$$
H'(z) = -\sqrt{(1 - H^2(z)(1 - \omega + \omega H^2(z))} \qquad (2.34)
$$

given by

$$
H(z) = cn(z, \omega). \qquad (2.35)
$$

We recall from the previous section that $M = 2$ and thus (2.33) is expanded to the form

$$
U(z) = A_{-2}H^{-2}(z) + A_{-1}H^{-1}(z) + A_0 + A_1H(z) + A_2H^2(z), \qquad (2.36)
$$

where, $A_i, i = -2, \ldots, 2$, are constants to be determined. In (2.35) and (2.34), the parameter $0 \leqslant \omega \leqslant 1$ is the modulus of the function. We now proceed to substitute (2.36) into the third-order ODE (2.14). Making use of (2.34) and splitting with respect to powers of $H(z)$, gives an overdetermined system of eight algebraic equations, namely

$$
\begin{aligned}
&12A_1cv\omega - 2A_1c\omega - 12A_1v\omega + 9A_2A_1 = 0,\\
&24A_2cv\omega - 4A_2c\omega - 24A_2v\omega + 3A_2^2 = 0,\\
&24A_{-2}cv\omega - 24A_{-2}cv - 4A_{-2}c\omega + 4A_{-2}c - 24A_{-2}v\omega\\
&\quad + 24A_{-2}v + 3A_{-2}^2 = 0,\\
&12A_{-1}cv\omega - 12A_{-1}cv - 2A_{-1}c\omega + 2A_{-1}c - 12A_{-1}v\omega\\
&\quad + 12A_{-1}v + 9A_{-2}A_{-1} = 0,
\end{aligned}
$$

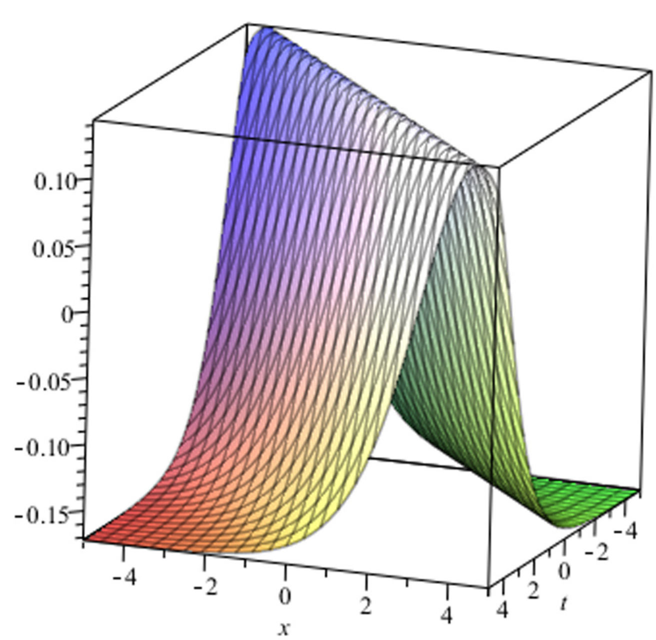

Fig. 2. Graphical representation of solution (2.32) for $c = 0.9$ and $v = 0.083$.

$$48A_{-2}cv - 96A_{-2}cv\omega + 16A_{-2}c\omega - 20A_{-2}c + 96A_{-2}v\omega - 48A_{-2}v + 9A_{-1}^2 + 12A_{-2} + 18A_{-2}A_0 = 0,$$

$$12A_{-1}cv\omega - 9A_{-2}A_1 - 6A_{-1}cv - 2A_{-1}c\omega - 12A_{-1}v\omega + 6A_{-1}v - 9A_0A_{-1} + 7A_{-1}c - 6A_{-1} = 0,$$

$$12A_1cv\omega - 6A_1cv - 2A_1c\omega - 12A_1v\omega + 6A_1v - 9A_0A_1 - 6A_1 - 9A_{-1}A_2 + 7A_1c = 0, \; +48A_2cv + 16A_2c\omega - 20A_2c - 96A_2cv\omega - 48A_2v + 9A_1^2 + 18A_0A_2 + 12A_2 + 96A_2v\omega = 0.$$

One possible set of values for the parameters of interest obtained from solving the above system is

$$\begin{aligned} A_{-2} &= \frac{1}{3}(24v\omega + 24cv + 4c\omega - 4c - 24cv\omega - 24v),\\ A_{-1} &= A_1 = 0,\\ A_0 &= \frac{1}{9}(48cv\omega - 24cv - 8c\omega + 10c - 48v\omega + 24v - 6)\\ A_2 &= \frac{1}{3}(24v\omega - 24cv\omega + 4c\omega). \end{aligned} \tag{2.37}$$

Consequently, the solution for (1.1) is

$$u(t,x) = A_{-2}nc^2(z,\omega) + A_0 + A_2cn^2(z,\omega), \tag{2.38}$$

where $nc = 1/cn$ [24,26] and $z = x - ct$. We now give a solution profile of (2.38) for $c = 0.5, v = 0.1, \omega = 0.01$ in Fig. 3.

Snoidal wave solutions of (1.1) using the extended Jacobi elliptic function expansion method

In a similar manner we can obtain the snoidal wave solutions of (1.1). However, in this case we use

$$H(z) = sn(z,w), \tag{2.39}$$

where $sn(z,w)$ is the Jacobi elliptic sine function, as a solution to the first-order ODE

$$H'(z) = \sqrt{(1 - H^2(z)(1 - \omega H^2(z))}. \tag{2.40}$$

Proceeding in the similar way, as before we obtain a set of algebraic equations. We now give one set of solutions obtained after solving this set of algebraic equations.

$$\begin{aligned} A_{-2} &= \frac{1}{3}(24cv - 4c - 24v),\\ A_{-1} &= A_1 = 0,\\ A_0 &= \frac{1}{9}(24v\omega + 10c + 24v - 24cv\omega - 24cv + 4c\omega - 6),\\ A_2 &= \frac{1}{3}\omega(24cv - 4c - 24v). \end{aligned} \tag{2.41}$$

Thus, in light of (2.41) the solution of (1.1) is

$$u(t,x) = A_0 + \frac{1}{\omega}A_2ns^2(z,\omega)\{1 + \omega sn^4(z,\omega)\}, \tag{2.42}$$

where A_0 and A_2 are given in (2.41). The corresponding graphical representation is given for $c = 0.5, v = 0.1$, and $\omega = 0.01$ in Fig. 4.

One-dimensional optimal system of subalgebras

We now calculate the optimal system of one-dimensional subalgebras for Eq. (1.1) and use it to find the optimal system of group-invariant solutions for Eq. (1.1). We follow the method given in [12]. Recall that the adjoint transformations are given by

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i,X_j] + \frac{1}{2}\epsilon^2[X_i,[X_i,X_j]] - \cdots, \tag{2.43}$$

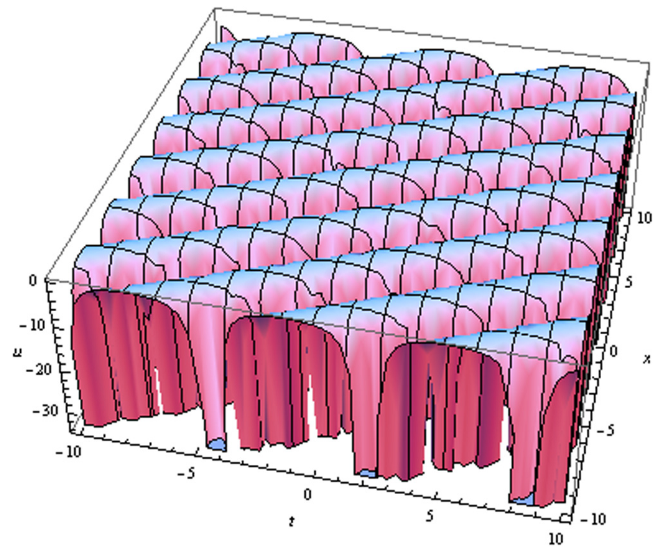

Fig. 3. Graphical representation of solution (2.38).

where $[X_i,X_j]$ is the commutator defined by

$$[X_i,X_j] = X_iX_j - X_jX_i. \tag{2.44}$$

We present the commutator table of the Lie symmetries and the adjoint representations of the symmetry group of (1.1) on its Lie algebra in Table 1 and Table 2, respectively. These two tables are then used to construct the optimal system of one-dimensional subalgebras for Eq. (1.1). As a result, after some calculations, one can obtain an optimal system of one-dimensional subalgebras given by $\{X_1, aX_1 + X_2, bX_1 + X_3\}$, where $a, b \in \mathbb{R}$.

Symmetry reductions and exact solutions of (1.1)

In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions and exact solutions of the KdV-BBM equation.

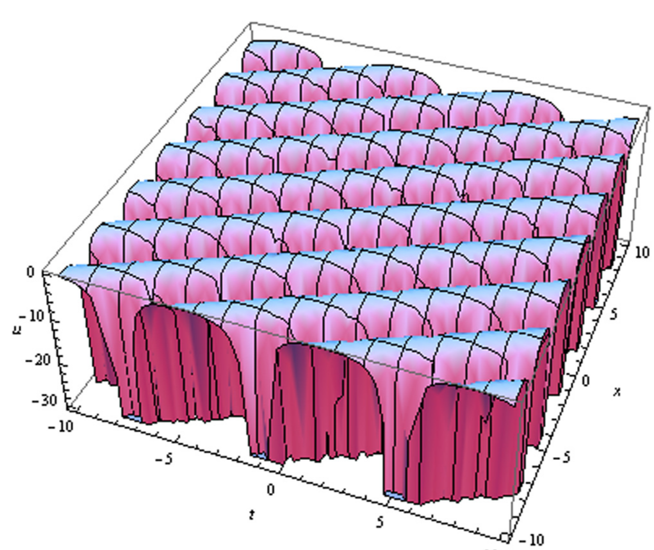

Fig. 4. Graphical representation of solution (2.42).

Case 1. X_1.

The symmetry X_1 gives rise to the following two invariants:

$$z = t, \quad f = u. \tag{2.45}$$

Now treating f as the new dependent variable and z as new independent variable, the KdV-BBM Eq. (1.1) transforms to

$$f'(z) = 0, \tag{2.46}$$

which integrates to $f(z) = K$, where K is a constant of integration. Hence the group-invariant solution of under X_1 is given by $u(t, x) = K$.

Case 2. $aX_1 + X_2$.

The symmetry $aX_1 + X_2$ gives rise to the following two invariants:

$$z = x - at, \quad f = u. \tag{2.47}$$

We note that these two invariants are identical to the invariants given in Eq. (2.13) with $a = c$ and hence the group-invariant solution, in this case, is given by (2.24).

Case 3. $bX_1 + X_3$.

The symmetry $bX_1 + X_3$ gives rise to the two invariants

$$z = \frac{b\ln(3t) + 3(6vt - 6vx + x)}{3 - 18v}, \quad f = t\left(\frac{2}{3 - 18v} + u\right). \tag{2.48}$$

By treating f as the new dependent variable and z as new independent variable, the KdV-BBM Eq. (1.1) transforms to

$$(6v - 1)\{bf''' + 3(6v - 1)f'\} - 9(6v - 1)(3f' - 2)f(z) + 6bf' = 0. \tag{2.49}$$

Conservation laws of (1.1)

In this work we compute second-order multipliers $\Lambda = \Lambda(t, x, u, u_x, u_{xx})$ [12,14,27,28]. Expanding the expression

$$\frac{\delta}{\delta u}\left[\Lambda(t, x, u, u_x, u_{xx})\left(u_t + u_x + \frac{3}{2}uu_x + vu_{xxx} - \left(\frac{1}{6} - v\right)u_{txx}\right)\right] = 0. \tag{3.50}$$

and splitting the resultant equation on derivatives of (u) yields the following fifteen determining equations:

$$3v\Lambda_{u_x u_{xx}} - \frac{1}{2}\Lambda_{u_x u_{xx}} = 0,$$

$$v\Lambda_{u_x u_{xx} u_{xx}} - \frac{1}{6}\Lambda_{u_x u_{xx} u_{xx}} = 0,$$

$$\frac{1}{3}\Lambda_{u_x} - 2v\Lambda_{u_x} = 0,$$

$$\frac{1}{6}\Lambda_{u_x u_{xx}} - v\Lambda_{u_x u_{xx}} = 0,$$

$$\Lambda_{u_{xx} u_{xx}} + \frac{1}{6}\Lambda_{u u_{xx}} - v\Lambda_{u u_{xx}} = 0,$$

$$\Lambda_{u_{xx} u_{xx} u_{xx}} + \frac{1}{6}\Lambda_{u u_{xx} u_{xx}} - v\Lambda_{u u_{xx} u_{xx}} = 0,$$

Table 1
Commutator table of the Lie algebra of Eq. (1.1).

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	$18vX_1 + 3(6v - 1)X_2$
X_3	0	$-18vX_1 - 3(6v - 1)X_2$	0

Table 2
Adjoint table of the Lie algebra of Eq. (1.1).

Ad	X_1	X_2	X_3
X_1	X_1	X_2	X_3
X_2	X_1	X_2	$-18\varepsilon vX_1 - 3\varepsilon(6v - 1)X_2 + X_3$
X_3	X_1	$\frac{6v}{6v-1}(e^{3\varepsilon(6v-1)} - 1)X_1 + e^{3\varepsilon(6v-1)}X_2$	X_3

$$\frac{1}{2}\Lambda_{u_x u_x} + \frac{1}{2}u_x\Lambda_{u u_x} + \frac{1}{2}\Lambda_{x u_x} - 3v\Lambda_{x u_x} - 3v\Lambda_{u u_x}u_x - 3vu_{xx}\Lambda_{u_x u_x} = 0,$$

$$\frac{3}{2}uu_x\Lambda_{u_{xx} u_{xx}} - v\Lambda_{x u_{xx}} + u_x\Lambda_{u_{xx} u_{xx}} - v\Lambda_{t u_{xx}} - 2v\Lambda_{u_x u_{xx}} - vu_x\Lambda_{u u_{xx}} + \frac{1}{6}\Lambda_{t u_{xx}} = 0,$$

$$\frac{3}{2}u_x\Lambda_{u_{xx} u_{xx} u_{xx}} + \frac{1}{6}\Lambda_{t u_{xx} u_{xx}} - vu_{xx}\Lambda_{u_x u_{xx} u_{xx}} - v\Lambda_{u u_{xx} u_{xx}}u_x + u_x\Lambda_{u_{xx} u_{xx} u_{xx}} - v\Lambda_{t u_{xx} u_{xx}} - 3v\Lambda_{u_x u_{xx}} - v\Lambda_{x u_{xx} u_{xx}} = 0,$$

$$\frac{1}{3}u_x\Lambda_{u u u_{xx}} + \frac{1}{3}\Lambda_{u u_x u_{xx}} - 2v\Lambda_{x u u_{xx}} - v\Lambda_{u u_x} + 2u_x\Lambda_{u u_{xx} u_{xx}} + 2u_{xx}\Lambda u_x u_{xx} u_{xx} + 2\Lambda_{x u_{xx} u_{xx}} + \frac{1}{3}\Lambda_{x u u_{xx}} + \frac{1}{6}\Lambda_{u u_x} - 2u_x v\Lambda_{u u u_{xx}} - 2u_{xx}v\Lambda_{u u_x u_{xx}} = 0,$$

$$\frac{1}{3}u_x\Lambda_{u_{xx} u_x u} - 2v\Lambda_{u u_{xx}} + \frac{1}{3}u_{xx}\Lambda_{u_x u_x u_{xx}} - 2v\Lambda_{x u_{xx} u_x} - v\Lambda_{u_x u_x} + 2\Lambda_{u_{xx} u_{xx}} + \frac{1}{3}\Lambda_{x u_{xx} u_x} + \frac{1}{6}\Lambda_{u_x u_x} + \frac{1}{3}\Lambda_{u u_{xx}} - 2vu_x\Lambda_{u u_x u_{xx}} - 2vu_{xx}\Lambda_{u_x u_x u_{xx}} = 0,$$

$$\frac{1}{3}u_x u_{xx}\Lambda_{u u_x u_x} - vu_x^2\Lambda_{u u_x u_x} - 2vu_x\Lambda_{x u u_x} - 2v\Lambda_{u u}u_x - 3vu_{xx}\Lambda_{u u_x} + \frac{1}{6}\Lambda_{x x u_x} - vu_{xx}^2\Lambda_{u_x u_x u_x} - 2vu_{xx}\Lambda_{x u_x u_x} + \frac{1}{3}\Lambda_{x u} + 2u_x\Lambda_{u u_{xx}} + 2\Lambda_{u_x u_{xx}}u_{xx} + \frac{1}{3}u_{xx}\Lambda_{x u_x u_x} - v\Lambda_{x x u_x} + \frac{1}{6}u_{xx}^2\Lambda_{u_x u_x u_x} + \frac{1}{3}u_x\Lambda_{x u u_x} + \frac{1}{3}\Lambda_{u u} + \frac{1}{6}u_x^2\Lambda_{u u u_x} - 2v\Lambda_{x u} + \frac{1}{2}\Lambda_{u u_x} + 2\Lambda_{x u_{xx}} - 2vu_x u_{xx}\Lambda_{u u_x u_x} - 2\Lambda_{u_x} = 0,$$

$$\frac{1}{3}\Lambda_{u u u_x} - vu_x^2\Lambda_{u u u} - 2v\Lambda_{x u u}u_x - vu_{xx}^2\Lambda_{u u_x u_x} - vu_{xx}\Lambda_{u u} - 2vu_{xx}\Lambda_{x u u_x} + 2\Lambda_{u u_x u_{xx}}u_x + \frac{1}{6}\Lambda_{x x u} - \Lambda_{u u_x}u_x - u_{xx}\Lambda_{u_x u_x} + u_x^2\Lambda_{u u u_{xx}} + 2u_x\Lambda_{x u u_{xx}} + u_{xx}^2\Lambda_{u_x u_x u_{xx}} + u_{xx}\Lambda_{u u_{xx}} + 2u_{xx}\Lambda_{x u_x u_{xx}} - v\Lambda_{x x u} + \frac{1}{3}u_{xx}\Lambda_{x u u_x} + \frac{1}{6}u_{xx}\Lambda_{u u} + \frac{1}{6}u_{xx}^2\Lambda_{u u_x u_x} + \frac{1}{3}u_x\Lambda_{x u u} + \frac{1}{6}u_x^2\Lambda_{u u u} - 2vu_x u_{xx}\Lambda_{u u u_x} - \Lambda_{x u_x} + \Lambda_{x x u_{xx}} = 0,$$

$$3uu_x u_{xx}\Lambda_{u_x u_{xx} u_{xx}} - 4vu_x u_{xx}\Lambda_{u u_x u_{xx}} - 2vu_x\Lambda_{t u u_{xx}} - 2vu_{xx}\Lambda_{t u_x u_{xx}} - 4vu_{xx}\Lambda_{x u_x u_{xx}} + 3uu_x^2\Lambda_{u u_{xx} u_{xx}} - 2u_{xx}^2 v\Lambda_{u_x u_x u_{xx}} - 2vu_x^2\Lambda_{u u u_{xx}} + 3uu_x\Lambda_{x u_{xx} u_{xx}} - 4vu_x\Lambda_{x u u_{xx}} + 2u_x u_{xx}\Lambda_{u_x u_{xx} u_{xx}} - 4vu_x\Lambda_{u u_x} - 4vu_{xx}\Lambda_{u_x u_x} - 2v\Lambda_{u u_{xx}} + 3uu_{xx}\Lambda_{u_{xx} u_{xx}} + \frac{1}{3}\Lambda_{x t u_{xx}} + \frac{1}{6}\Lambda_{t u_x} + 2u_{xx}\Lambda_{u_{xx} u_{xx}} + \frac{1}{3}\Lambda_{t u_x u_{xx}} + \frac{1}{3}u_x\Lambda_{t u u_{xx}} - v\Lambda_{t u_x} + 3u_x^2\Lambda_{u_{xx} u_{xx}} - 2v\Lambda_{x x u_{xx}} - 2v\Lambda_{t x u_{xx}} + 2\Lambda_{x u_{xx} u_{xx}}u_x - 4v\Lambda_{x u_x} + 2u_x^2\Lambda_{u u_{xx} u_{xx}} = 0,$$

$$\frac{9}{2}u_xu_{xx}\Lambda_{u_{xx}} - 3uu_{xx}\Lambda_{u_x} + \frac{1}{6}u_x^2\Lambda_{tuu} - v\Lambda_{xxx} + \frac{1}{3}u_x\Lambda_{txu} + 2u_{xx}^2\Lambda_{u_xu_{xx}}$$
$$+ \Lambda_{xxu_{xx}}u_x - \Lambda_{xu_x}u_x - v\Lambda_{xxt} + 2u_{xx}\Lambda_{xu_{xx}} - u_x^2\Lambda_{uu_x} + 2u_x^2\Lambda_{xuu_{xx}}$$
$$+ 3u_x^2\Lambda_{xu_{xx}} + \frac{1}{6}\Lambda_{tu_xu_x} + u_x^3\Lambda_{uuu_x} + \frac{1}{6}\Lambda_{tu} + 3u_x^3\Lambda_{uu_{xx}} + \frac{1}{3}\Lambda_{txu_x}$$
$$+ \frac{1}{6}\Lambda_{txx} + \frac{1}{3}\Lambda_{tuu_x} - vu_x^3\Lambda_{uuu} - vu_{xx}\Lambda_{tu} - 2vu_{xx}\Lambda_{txu_x} - vu_{xx}^3\Lambda_{u_xu_xu_x}$$
$$- 3vu_{xx}\Lambda_{xxu_x} + \frac{3}{2}uu_x\Lambda_{xxu_{xx}} - \frac{3}{2}uu_x\Lambda_{xu_x} - 3vu_x\Lambda_{xxu} - 3vu_{xx}^2\Lambda_{xu_xu_x}$$
$$- 3vu_x^2\Lambda_{xuu} + 3uu_x^2\Lambda_{xuu_{xx}} - 3vu_{xx}^2\Lambda_{uu_x} + \frac{3}{2}uu_x^3\Lambda_{uuu_{xx}} - \frac{3}{2}uu_x^2\Lambda_{uu_x}$$
$$- u_xu_{xx}\Lambda_{u_xu_x} + 3u_xu_{xx}\Lambda_{uu_{xx}} + 2u_x^2u_{xx}\Lambda_{uu_xu_{xx}} + 3u_x^2u_{xx}\Lambda_{u_xu_{xx}}$$
$$+ 3uu_{xx}^2\Lambda_{u_xu_{xx}} + u_xu_{xx}^2\Lambda_{u_xu_xu_{xx}} + 3uu_{xx}\Lambda_{xu_{xx}} + 2u_xu_{xx}\Lambda_{xu_xu_{xx}}$$
$$- vu_{xx}^2\Lambda_{tu_xu_x} - 3vu_{xx}\Lambda_{xu} - vu_x^2\Lambda_{tuu} - 2v\Lambda_{txu} + 3uu_x^2u_{xx}\Lambda_{uu_xu_{xx}}$$
$$- \frac{3}{2}u\Lambda_x - 2u_{xx}\Lambda_{u_x} - \Lambda_t - \Lambda_x - \frac{3}{2}u_x^2\Lambda_{u_x} - 6vu_x\Lambda_{xuu_x}u_{xx}$$
$$+ \frac{9}{2}u_xu_{xx}\Lambda_{uu_{xx}} - 3vu_xu_{xx}^2\Lambda_{uu_xu_x} - 3vu_x^2u_{xx}\Lambda_{uuu_x} - 3vu_xu_{xx}\Lambda_{uu}$$
$$- 2vuu_xu_{xx}\Lambda_{tuu_x} + \frac{3}{2}u_{xx}^2uu_x\Lambda_{u_xu_xu_{xx}} + 3uu_xu_{xx}\Lambda_{xu_xu_{xx}}$$
$$- \frac{3}{2}u_{xx}uu_x\Lambda_{u_xu_x} = 0.$$

Solving the above system of equations we obtain

$$\Lambda = C_1u + C_2, \tag{3.51}$$

where C_1 and C_2 are arbitrary constants. This yields the following two conservation laws:

$$T_1^t = \frac{1}{2}u^2 - \frac{1}{18}uu_{xx} + \frac{1}{3}vuu_{xx} - \frac{1}{6}vu_x^2 + \frac{1}{36}u_x^2,$$
$$T_1^x = \frac{1}{2}u^2 + \frac{1}{2}u^3 + vuu_{xx} - \frac{1}{9}uu_{tx} + \frac{2}{3}vuu_{tx} - \frac{1}{2}vu_x^2 + \frac{1}{18}u_xu_t - \frac{1}{3}vu_xu_t;$$

$$T_2^t = u - \frac{1}{18}u_{xx} + \frac{1}{3}vu_{xx},$$
$$T_2^x = u + \frac{3}{4}u^2 + vu_{xx} + \frac{2}{3}vu_{tx} - \frac{1}{9}u_{tx}.$$

Concluding remarks

In this paper we studied the Korteweg-de Vries-Benjamin-Bona-Mahony (KdV-BBM) equation. This equation describes the two-way propagation of waves. Lie symmetry analysis along with the Jacobi elliptic function expansion and Kudryashov methods was employed to construct its travelling wave solutions. Moreover conservation laws of the KdV-BBM equation were calculated using the multiplier approach. The usefulness of conservation laws was explained in the Introduction.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at https://doi.org/10.1016/j.rinp.2017.10.041.

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