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Coalition Robustness of Multiagent Systems

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COALITION ROBUSTNESS OF MULTIAGENT SYSTEMS

by

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of a thesis submitted by

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ABSTRACT

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Many multiagent systems are environments where distinct decision-makers compete, explicitly or implicitly, for scarce resources. In these competitive environments, it can be advantageous for agents to cooperate and form teams, or coalitions; this cooperation gives agents strategic advantage to compete for scarce resources. Multiagent systems thus can be characterized in terms of competition and cooperation. To evaluate the effectiveness of cooperation for particular coalitions, we derive measures based on comparing these different coalitions at their respective equilibria.

However, relying on equilibrium results leads to the interesting question of stability. Control theory and cooperative game theory have limitations that make it hard to apply them to study our questions about stabililty and evaluate cooperation in competitive environments. In this thesis we will lay a foundation towards a theory of coalition stability and robustness for multiagent systems. We then apply this condition to form a methodology to evaluate cooperation for market structure analysis.

ACKNOWLEDGMENTS

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Chapter 1

Introduction

1.1 Motivation

Many multiagent systems are environments where distinct decision-makers compete, explicitly or implicitly, for scarce resources. For example, different species in various ecosystems compete for food, water, sunlight, and basic survival. In political systems, people wage wars to control resources by seizing land, food, labor, or destroying infrastructure. Likewise, economic systems also exhibit competition, as firms struggle for market share by advertising, supplying better products and services, or even through unfair practices, such as creating barriers of entry or stealing trade secrets.

In these competitive environments, since individual agents have limited influence to accomplish their objectives, it can be advantageous for agents to cooperate and form teams, or coalitions; this cooperation gives agents strategic advantage to compete for scarce resources. In ecology such cooperation is called symbiosis, illustrated by the cooperative relationship between leaf-cutter ants, fungi in the Lepiotaceae family, and a particular bacterium which the ants use as a type of antibiotic to protect the fungi. The ants cut leaves to make food for the fungi and at the same time use a certain bacterium to kill molds that endanger the fungi. In exchange, the fungi, which now only grows in such ant colonies, produces food for the ant in specialized hyphal-tips known as gongylidia. Queen-ants also take the fungi seed with them to form other colonies. Politically, cooperation manifests in the form of strategic alliances. After West Germany joined the North Atlantic Treaty

Organization (NATO), the Warsaw Pact was formed in order to balance the competition. Economic systems also show cooperation at company and corporation levels, as merger and acquistions can be carried out to put the joining parties in an overall better economical position. Multiagent systems thus can be characterized in terms of competition and cooperation.

It is difficult, however, to evaluate the effectiveness of cooperation for particular coalitions. Is a coalition successful because of cooperation, or in spite of it? Addressing such issues can be useful in a variety of settings. For example, evaluating the cooperative influence between product lines controlled by a firm and a potential aquisition target is very useful to 1) the Department of Justice as to check whether a certain merger could result in a strong firm that can dominate the market, or 2) executives of firms as to structure their organizations into strongly cooperative units. In order to perform this kind of evaluation, one requires measures of cooperation relative to a particular competitive environment. In this thesis, we derive measures based on comparing these different coalitions at their respective equilibria. However, relying on equilibrium results leads to the interesting question of stability.

Control theory is rich with stability results, yet there is not much focus for coalition structures. Cooperative game theory, on the other hand, is focused on the study of stable coalitions known as the core. However, it does that without regard to how the coalitions are structured, or in other words the coalitional hierarchy. For examples, the classical formulation in cooperative game theory would assign a score to the strategic position of the Warsaw Pact without considering whether NATO was formed or disintegrated. These limitations make it hard to apply pure control theory or cooperative game theory to study our questions about stabililty and evaluate cooperation in competitive environments.

In this thesis we will lay a foundation towards a theory of coalition stability and robustness for multiagent systems. After showing some practical applications that motivate the theory, we will build up our results based on ideas from control theory and cooperative

game theory to lead to a stability robustness condition. We then apply this condition to form a methodology to evaluate cooperation for market structure analysis.

1.2   Organization of the Thesis

This thesis started out as a mentoring project sponsored by the Office of Research and Creative Activities at Brigham Young University. We explored the value of cooperation of different product lines within firms of a given market in [19]. Following [19] we then used the value of cooperation to cluster an organizational structure into meaningful divisions to maximize the value of cooperation within each unit [17]. Then in [18] we explored and solved the condition for stability of those systems, while also introducing the notion of coalition robustness in multiagent systems. Later works in [4] are also partially-based on our work in the said papers. Our work related to the problem of this thesis are also published in [16] and [20].

The following chapters correspond to [19], [17] and [18] respectively with some minor editing to suite a thesis format. We want to keep the chapters self-contained as much as possible, and in doing so we expect some redundancy as an acceptable trade off.

Chapter 2 formulates and discusses the concept of the value of cooperation, as well as giving some examples to emphasize the key concepts. Chapter 3 shows how the value of cooperation can be applied to analyze industrial organizational structures. It also builds and illustrates a simulation of such application. Chapter 4 generalizes from the formulation of Chapter 2 by introducing new concepts and definitions and deriving lemmas and theorems to solve the stability robustness problem.

Chapter 5 summarizes the ideas and adds one additional extension of the earlier results. Then we conclude the thesis by showing a plan of how our future work can proceed.

Chapter 2

The Value of Cooperation Within a Profit-Maximizing Organization [1]

This chapter derives a method for evaluating the cooperative influence among the product lines controlled by a firm. The idea is to let the profit maximizing dynamics of a given market structure define the value function for a particular coalition game. With this idea, we may aid anyone who needs to know about a product's place in the product network. For business managers, this means they can know the products their business offers which contribute to a greater whole, as opposed to those product lines which may be sold off with minimal impact. They may also discover which product lines would be most advantageous for their business. Given any set of products inside or outside the business, we may calculate the value of this set (with profit maximization as the objective).

Our method is also useful for the antitrust division of the Department of Justice. They are interested in maximizing total social welfare in a market by protecting market competition. To do this, they attempt to measure the control a particular company has on the market and take appropriate measures. Their preferred measure of the market power of the company is the Herfindahl-Hirshman index (I_{HH}) [15], the sum of the squares of each firm's market share, given by

$$I_{HH} = \sum_{i=1}^{N} s_i^2 \qquad (2.1)$$

This measure, however, relies on legal definitions of particular markets and focuses on a computation of market share. Market share, however, has been shown to be a weak indica-

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[1]This part appeared as [19] in the proceeding of the Joint Conferences on Information Sciences 2005, Salt Lake City, Utah . It also appeared partially as [20] in the proceedings of the IEEE International Conference on Control Applications 2005, Toronto, Canada.

tor of market power [6]. A more direct measure of market power that is insensitive to legal definitions of market boundaries but highly sensitive to the economics of the underlying product network would make a significant impact on antitrust efforts. The value of cooperation a firm is able to realize within a given economic environment is a step in the direction of computing market power directly. This work draws heavily from the theory of industrial organization and coalition games [15], [3], [10], [14], [8]. The most closely related work to our study is recent work on merger simulations. One paper [5] describes how the impact of a proposed merger can be computed by evaluating the post-merger equilibrium prices. The paper considers common functional forms of demand functions, and indicates how to conduct the merger simulation in each case. The value of cooperation proposed here is found through a kind of "reverse" merger simulation that explores the impact of splitting the firm into its constituent economic units to determine the value it is realizing by unifying the objectives of these basic units.

The next section introduces the dynamic framework motivating the profit gained at equilibrium as a viable value function. A coalition game is then formulated using this value function, and the Value of Cooperation and Relative Value of Cooperation are then introduced as measures on this game. A simple example is then provided to illustrate the ideas, and the conclusion and future work summarizing the work follow.

2.1 Profit Maximizing Dynamics

Consider a market, $\mathcal{M}$, of N products. Without loss of generality, give these products an arbitrary order and integer label so that $\mathcal{M} = \{1, 2, ..., N\}$. Let $p \in \mathbb{R}^N$ be the vector of (non-negative) prices for these N products, and let $q : \mathbb{R}^N \to \mathbb{R}^N$ be the (non-negative) demand for these products at prices p.

A firm, F is a subset of the N products in the market, $F \in 2^{\mathcal{M}}$. This implies that the firm controls the production and distribution of the products assigned to it. Most impor-

tantly for our analysis, since we consider a Bertrand market model [15], this implies that the firm may set the prices of the $n = |F|$ products assigned to it.

We suppose that the products of the market are partitioned between m firms. This implies that no two firms control the same product, $F_i \cap F_j = \emptyset \ \forall i \neq j$ and that the union of all products assigned to the m firms composes the entire market, $\bigcup_{i=1}^{m} F_i = \mathcal{M}$.

Let $c_j(q_j), j = 1, ..., N$ be the cost of production of q_j units of product j. The profit of the ith firm, is then given by

$$\pi_i = \sum_{j \in F_i} [q_j(p)p_j - c_j(q_j(p))] \tag{2.2}$$

A profit-maximizing firm under the Bertrand model of market behavior will tend to change its prices to maximize its short-term profit. We model this behavior by assuming that the firm will evolve the prices of its products in the direction of maximally improving its profits. That is, if product j belongs to firm i, then we expect the firm to evolve the price of product j as

$$\frac{dp_j(t)}{dt} = \left. \frac{\partial \pi_i(p)}{\partial p_j} \right|_{p(t)} \tag{2.3}$$

where $p(t)$ is the pricing vector for the entire market at time t.

Notice that these dynamics suggest that if the partial derivative of profits is negative with respect to the price of product j, that the firm should *decrease* the price of product j. This is in the direction of improving profits. Likewise, if the partial derivative were positive, the firm would increase the price of product j to improve profits. When the partial derivative is zero, the motivation is to hold the price at this locally profit-maximizing position.

Reordering the N market products so that each firm's products are grouped together, and letting n_i be the number of products controlled by firm i, we then can partition the pricing vector into components associated with each firm. If every firm in the market is

assumed to be profit maximizing, this yields the following market dynamics:

$$
\begin{bmatrix}
\dot{p}_1(t) \\
\vdots \\
\dot{p}_{n_1}(t) \\
\hline
\dot{p}_{n_1+1}(t) \\
\vdots \\
\dot{p}_{n_1+n_2}(t) \\
\hline
\vdots \\
\dot{p}_{1+\sum_{i=1}^{m-1} n_i}(t) \\
\vdots \\
\dot{p}_N(t)
\end{bmatrix}
=
\begin{bmatrix}
(\partial \pi_1 / \partial p_1)(p(t)) \\
\vdots \\
(\partial \pi_1 / \partial p_{n_1})(p(t)) \\
\hline
(\partial \pi_2 / \partial p_{n_1+1})(p(t)) \\
\vdots \\
(\partial \pi_2 / \partial p_{n_1+n_2})(p(t)) \\
\hline
\vdots \\
(\partial \pi_m / \partial p_{1+\sum_{i=1}^{m-1} n_i})(p(t)) \\
\vdots \\
(\partial \pi_m / \partial p_N)(p(t))
\end{bmatrix}
\tag{2.4}
$$

where the dot notation $\dot{p}(t)$ is used to represent $dp(t)/dt$. Notice that if the market system (2.4) has an equilibrium, such a pricing vector p_{eq} would represent prices from which no firm can improve its profits by unilaterally changing the prices over which it has control. Under certain technical conditions such an equilibrium can be shown to exist. Moreover, this equilibrium can often be shown to be asymptotically stable, in the sense that any pricing vector $p(0)$ will converge to the equilibrium p_{eq} as $t \to \infty$.

2.2 The Firm as a Coalition in a Multi-Coalition Environment

Under the assumption that the market dynamics are stabilizing, we expect price perturbations to re-equilibriate[2]. In this context, it is convenient to simplify the problem by only considering the profits of the firms at equilibrium. These profits define a payoff function reminiscent of those used to define coalition games.

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[2]This expectation is addressed in chapter 4.

Let $v(Fi) = \pi_i|_{p=p_{eq}}$ be the payoff or profit of firm i at the market equilibrium prices p_{eq}. In this way the firm may be thought of as a coalition of n_i players in an N-player cooperative game ($N = \sum n_i$). Each player is a one-product company that completely manages the production, distribution, and pricing decisions for its product. The firm, then, is a confederacy of these one-product companies that works together to maximize their combined profits or payoffs.

The theory of coalition games studies the behavior of such coalitions once the payoff function is defined for every possible coalition. The idea is that any given coalition F_i yields a well-defined payoff $v(F_i)$, and then a number of questions can be explored regarding how to distribute the payoff among the members of the coalition, etc.

Our situation is different because the payoff to a given firm doesn't just depend on the products it controls, but also on the market structure of the products outside the firm. For example, consider a 10-product market and a three product firm in the market. The payoff to the firm does not just depend on the prices of the three products it controls, but also on the prices of the other seven products. The profit-maximizing equilibrium prices of these other seven products, however, may be set differently depending on whether they belong to a single firm or whether they are controlled by seven different companies. Thus, the payoff to the three-product firm depends on the entire market structure.

Coalition game theory addresses such situations by considering partition systems and restricted games. For our purposes, it is sufficient to partition the N products of $\mathcal{M}$ into m firms and then assume that this structure is fixed outside of the particular firm that we are studying. This enables us to work with a well defined payoff function induced by the profit-maximizing dynamics of firms within the market without eliminating the multiple-coalition (i.e. multiple firm) cases of interest.

2.3   Value of Cooperation

To quantify the value of organizing a group of one-product companies into a single firm, we need to compare the profits the firm receives if it sets its prices as if each of its products were independent companies with those it realizes by fully capitalizing on cooperation between the products.

More precisely, let p_{eq} be the profit-maximizing equilibrium prices for the given market structure. In contrast, consider the new profit maximizing equilibrium prices achieved without cooperation if F_i were divided into its constituent one product companies and each independently optimized their prices. Let this second set of equilibrium prices serve as a basis for comparison, or reference, and be denoted p_{ref}. We can then define the following measure.

Definition 2.1. *The Value of Cooperation (VC) of a firm F_i in market M with structure $S = F_1, F_2, ..., F_m$ is given by:*

$$\text{VC}_{ref}(F_i, S) = \pi_i\big|_{p_{eq}} - \pi_i\big|_{p_{ref}} \tag{2.5}$$

This VC measure captures precisely the value realized by the firm due to cooperation within its organization. Note that the VC measure is always non-negative since the cooperating firm can always recover at least the non-cooperating, or reference, profits by simply setting the prices it controls in p_{eq} to those of p_{ref}.

As defined, the VC measure carries units of dollars and reflects a kind of absolute dollar-value of cooperation within the firm, thus making comparisons difficult. We, therefore, define a "relative" Value of Cooperation by normalizing VC_{ref} by the equilibrium profits as follows.

Definition 2.2. *The Relative Value of Cooperation (RVC) of a firm F_i in market M with structure $S = F_1, F_2, ..., F_m$ is given by:*

$$\text{RVC}_{ref}(F_i, S) = \frac{\pi_i\big|_{p_{eq}} - \pi_i\big|_{p_{ref}}}{\pi_i\big|_{p_{eq}}} \tag{2.6}$$

This RVC measure is naturally interpreted as the percentage of profits due to co-operation within the organization. It is bounded between zero and one, and enables direct comparison among firms of different sizes. By simply replacing the equilibrium and reference prices in the above definitions with the equilibriated profit-maximizing prices of the market structures being compared, one can easily use the thus modified VC and RVC to analyze the relative values of different organizational structures within a single firm. This is a natural application of the above framework, where the market is a firm and the firms are its organizational divisions.

Sometimes we may be interested in measuring the value of cooperation between structures other than the current market structure and the reference structure. This could be the case when considering mergers between firms, or when management is considering selling off a piece of the firm. In such cases it is easy to extend the definitions of VC and RVC by simply replacing the equilibrium and reference prices with the equilibriated profit-maximizing prices of the two market structures being compared.

It is instructive to contrast the VC and RVC with other measures used to characterize cooperative games. Hart and Mas-Colell defined a measure, called the potential, P, that computes the expected normalized worth of the game i.e. the per-capita potential, P/N, equals the average per-capita worth $(1/m)\sum_i(\pi_i)/(|Fi|)$. Given a market structure, this measure characterizes the expected profit of an average-sized firm (where size is measured with respect to the number of products the firm controls) in the market, even if such a firm does not actually exist.

Moreover, the potential has been connected to another measure, called the Shapley value, Φ_j, which yields the marginal contribution of each product in the market. This measure characterizes how the payoff of a coalition should be divided between members of the team. In both cases, Shapley value and the potential do not suggest anything about the intrinsic benefit of forming coalitions in the first place.

The Value of Cooperation, VC, and Relative Value of Cooperation, RVC, on the other hand, capture the natural significance for organizing production into multi-product firms. Nevertheless, these measures do not yield any information about how the profit of a firm should be efficiently invested into each of the firm's constituent production lines. Thus, the measures are inherently different from the potential or Shapley value of the coalition game that focus more on the value of a member of a coalition to the group rather than the value of the coalition as a whole.

Example 2.1. *To illustrate the point, consider a two product economy with linear demand given by*

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} -3.5 & -1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 100 \\ 100 \end{bmatrix} \tag{2.7}$$

Suppose that the unit production cost of each product is $c_1 = 10, c_2 = 10$. If we consider a market structure where each product is produced by an independent company, the profit function for each company becomes

$$\pi_1(t) = q_1(t)(p_1(t) - c_1)$$
$$= -3.5p_1^2 - p_1 p_2 + 135p_1 + 10p_2 - 1000 \tag{2.8}$$
$$\pi_2(t) = q_2(t)(p_2(t) - c_2)$$
$$= -2p_2^2 - 3p_1 p_2 + 30p_1 + 120p_2 - 1000 \tag{2.9}$$
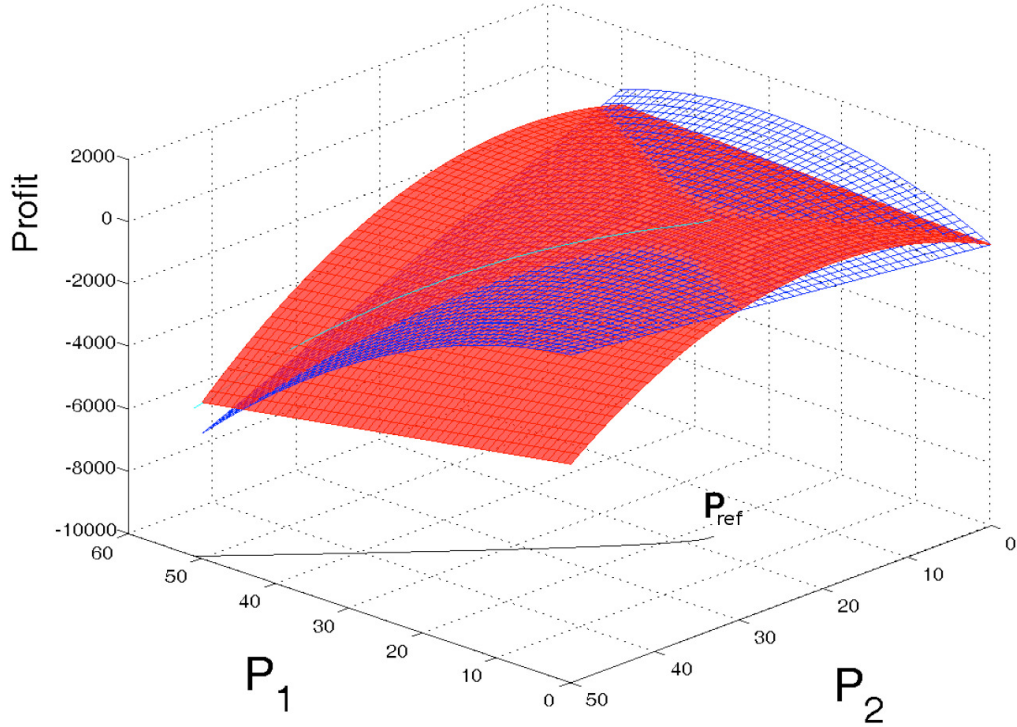
Figure 2.1: Two firm price trajectory and profit function.

Taking the partial derivatives of each profit function with respect to the appropriate pricing variable, we find the profit-maximizing market dynamics to be:

$$\begin{bmatrix} \frac{dp_1(t)}{dt} \\ \frac{dp_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -7 & -1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 135 \\ 120 \end{bmatrix} \qquad (2.10)$$

Figure 2.1 shows how the two-firm dynamics drive an initial pricing vector to a profit-maximizing equilibrium. This equilibrium price is

$$p_{ref} = \begin{bmatrix} 16.8 \\ 17.4 \end{bmatrix}$$

and the associated equilibrated profits are $\pi_1 = 161.84$, and $\pi_2 = 109.52$.

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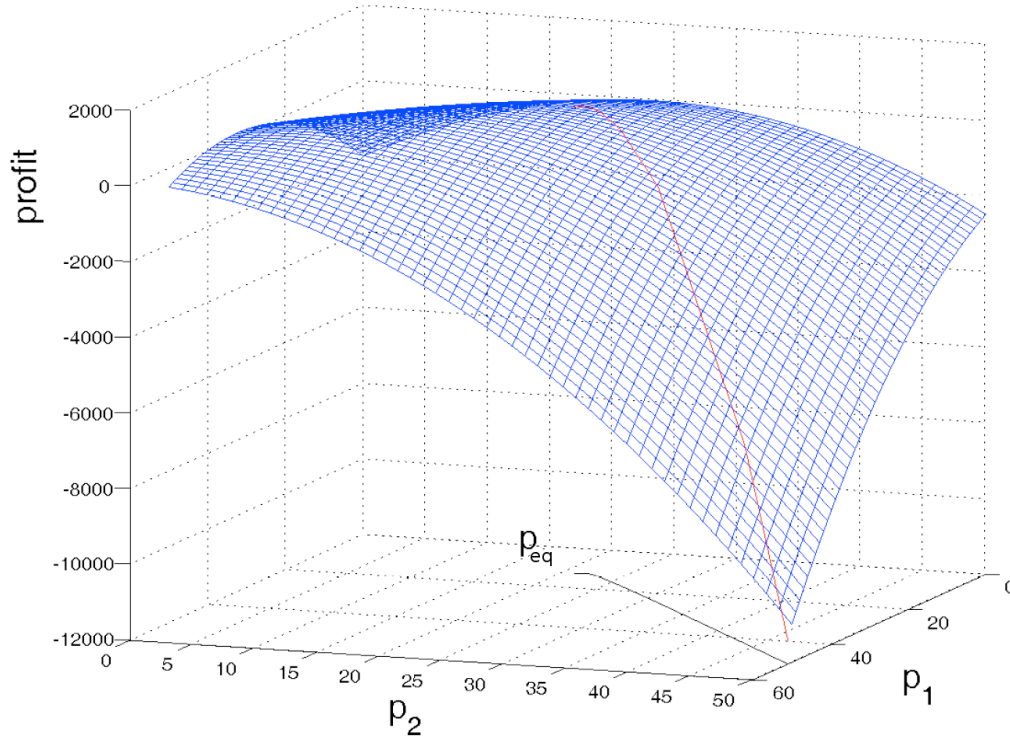
Figure 2.2: One firm price trajectory and profit function.

Now, consider a market structure where both products are controlled by the a same firm. In this case, the firm's profit function becomes

$$\pi(t) = q_1(t)(p_1(t) - c_1) + q_2(t)(p_2(t) - c_2)$$
$$= -3.5p_1^2 + 165p_1 - 4p_1p_2 + 130p_2 - 2p_2^2 - 2000 \qquad (2.11)$$

With this market structure, the firm adjusts the prices of both products to optimize the same objective. These new dynamics become:

$$\begin{bmatrix} \frac{dp_1(t)}{dt} \\ \frac{dp_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} -7 & -4 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} + \begin{bmatrix} 165 \\ 130 \end{bmatrix}. \qquad (2.12)$$

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Figure 2.2 shows how the single-firm dynamics drive an initial pricing vector to a profit-maximizing equilibrium. The new equilibrium price is given by

$$p_{eq} = \begin{bmatrix} 11.67 \\ 20.83 \end{bmatrix} \tag{2.13}$$

and the associated equilibriated profits are $\pi_{eq} = 316.667$. The value of cooperation in this example thus becomes

$$VC = \pi_{eq} - (\pi_1 + \pi_2) = 45.3067$$

$$RVC = 0.1431 \tag{2.14}$$

This suggests that in this market, just under 15% of the profits of the two-product firm are the direct result of it's interfirm cooperation.

2.4 Conclusion

This chapter explored quantitative measures to calibrate the value of cooperation within a specific firm in a given market. The idea is to assume profit-maximizing dynamics among the firms within the market and compare equilibrium profits in two different scenarios. The first scenario considers the firm as it is, as a single economic entity with a unified objective and exhibiting full cooperation between its various economic units. The second scenario considers splitting the firm into its constituent economic units and computing market equilibrium prices if these units were to fail to cooperate and acted completely independently out of self interest. The difference between the cooperative profits of the first scenario and the aggregate profits of the independent units of the second scenario define a measure we call the Value of Cooperation, VC, of the firm in its current market environment. A second related measure is the Relative Value of Cooperation, RVC, which normalizes the VC

measure by the cooperative profits to yield a unitless metric that reveals the percentage of profits derived from cooperation within the firm.

Quantifying the value of cooperation is a first step in understanding how firms exert market power in their respective environments. This information is important for both managers, who hope to leverage the information to better lead their organizations, and regulators, who want to monitor the impact of corporate decisions on social welfare. The following chapters, especially chapter 4, will concretely establish the relationship between the VC and RVC measures and market power and indicate how to compute approximations to these metrics from readily available market.

Chapter 3

Cooperation-based Clustering for Profit-maximizing

Organizational Design [1]

This chapter shows how the notion of relative value of cooperation (RVC) that we discussed in Section 2.3 can be used to analyze the relative economic advantage afforded by various organizational structures of a firm. The RVC measure does not consider human factors, but simply reflecting the value of cooperation of a firm's product lines. The value of cooperation is computed from transactions data by solving a regression problem to fit the parameters of the consumer demand function, and then simulating the resulting profit-maximizing dynamic system under various organizational structures. A hierarchical agglomerative clustering algorithm can be applied to reveal the optimal organizational substructure.

3.1 Introduction

Analyzing the impact of organizational structure on the performance of profit-maximizing organizations is a difficult task for business managers. Yet, informed design decisions are essential to long-term profitability. While these decisions are often ad hoc, today's large volumes of data make more systematic analyses possible. The key concept in such analyses is that of the relative value of cooperation (RVC) experienced by a firm, which captures the

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[1]This part appeared in the proceedings of the International Joint Conferences on Neural Networks, Vancouver BC as [17].

percentage of a firm's profits due strictly to the cooperative effects among the goods it sells [19].

Such a measure provides the scientific backing for sound organizational design decisions. For example, if a firm can identify which of its products have strong synergies with others, it can organize to ensure that decision makers of related products work closely together. This may include physically co-locating entities where interaction adds strong value to the organization, or it may result in decentralization when cooperation adds little value. Similarly, if a firm identifies pieces of its business that add little cooperative benefit to the organization as a whole, it may consider selling off these subunits. A subunit with a healthy balance sheet may sell for a high price without adversely affecting the firm's value of cooperation. On the other hand, the firm may pursue a different strategy of retaining its decoupled subunits but use the value of cooperation to identify an acquisition that strongly couples their mutual benefit. Thus, divisions of a firm that are quite independent may be cooperatively coupled through the acquisition of another business with the right cooperative effects. For example, a firm with two distinct independent divisions that have no value of cooperation may acquire another business unit that not only adds value of cooperation with each division, but does so in a way that the total business becomes tightly integrated. Moreover, the firm may identify an acquisition candidate that is struggling on its own, and thus is inexpensive, but brings the right cooperative effects to the organization to offset the risk of acquiring a struggling business. The value of cooperation thus becomes the lens through which a firm can better identify valuable opportunities in the market environment, or costly "baggage" in its own organizational structure.

In this chapter, we discuss the use of the Value of Cooperation inspired by the theory of industrial organization and coalition games [15], [3], [10], [14], [8]. Essentially, the profit maximizing dynamics of a given organizational structure define the value function for a particular coalition game. Researchers in [5] describe how the impact of a proposed merger can be computed by evaluating the post-merger equilibrium prices. They consider

common functional forms of demand functions and indicate how to conduct the merger simulation in each case. Our value of cooperation is computed through a kind of "reverse" merger simulation that explores the impact of splitting the firm into its constituent economic units to determine the value it is realizing by unifying the objectives of these basic units. We are then able to use this measure to drive a hierarchical agglomerative clustering algorithm in order to reveal the organizational substructure defined naturally from the cooperative effects of the global product network.

3.2   VC-based Clustering

Equipped with the value of cooperation as the measure of productive interaction between products, a firm may consider its internal organization and cluster its product to maximize the value of cooperation, indicating which decision makers in the organization should work together in setting prices, and which can work more independently. Knowing where cooperative value resides within an organization is the critical first step in exploring it.

We begin by describing how one may compute a firm's value of cooperation from data. The algorithmic process involves the following three steps:

1. Fit demand model from transactions data. This can often be accomplished through standard regression techniques. To obtain good estimates of the model parameters, however, it is important that the data be sufficiently "exciting". This can often be a challenge if prices have remained relatively constant or have only been changed in a very structured way (e.g. 20% off everything sales.)

2. Build corresponding profit-maximizing dynamic system. This step involves constructing the firm's profit function for each product in its offering. Importantly, this assumes knowledge of the total handling costs associated with selling each product, which information may often have to be estimated at best. Moreover, our initial results assume that this cost structure does not change with organization structure, i.e.,

although the economies of scale associated with selling more of a product can be easily accommodated, no cost benefit across product lines is built into the existing model[2].

3. Compute equilibrium for given organizational structures. With different assumptions about which products group together, the profit maximizing dynamics are altered yielding a new equilibrium point for the system.

A natural extension of the above algorithm would be to automate step 3 in such way that the organizational structure resulting in the largest value of cooperation is found. In other words, one is interested in finding the k clusters of n products for which the total value of cooperation is maximized for all k from 1 to n. We propose to do this through the use of the value of cooperation within a hierarchical agglomerative clustering framework.

VC-based hierarchical agglomerative clustering starts from the reference structure, where all products are assumed to act independently. Then, the two products which exhibit the strongest value of cooperation are merged, and the process is repeated, decreasing the total number of clusters by one until all of the firms products are finally merged into a single organizational structure. The hierarchical nesting adds a natural constraint to the problem, which yields a product hierarchy with a clear organizational interpretation.

The real novelty of our approach comes from its use of the profit-maximizing dynamics to define the resulting clusters. A typical approach to economic-driven clustering may compute the same demand function from data, expand it in a Taylor series around the market equilibrium, and then consider the first order terms as defining a graph over products, i.e., the product network. Various approaches to clustering this graph might then be considered. The approach discussed here, however, is a radical departure from such approaches by using the demand function to characterize a dynamic system, and then allowing this dynamic system to define clusters over the product network.

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[2]This limitation is an important focus of future work since synergistic costs can play as important a role in a total cooperation as synergistic sales. Nevertheless, the current work showing effects of synergistic sales demonstrates the key idea underlying cooperation-based analysis.

The use of VC-based hierarchical agglomerative clustering is best highlighted through an example. The following example will formulate the problem using simulated data.

Example 3.1. *In this simple example, a firm managing 15 products is considered. The demand function is taken to have the form: $q = Ap + B$ where $q \in \mathbb{R}^n, p \in \mathbb{R}^n$, and $q_i, p_i \geq 0 \ \forall \ i \in \{1,...,n\}$.*

This linear structure may have been fit directly from data, or it may be the result of linearizing another demand function around a nominal set of prices. Considering the reference structure where every product sets its price independently to maximize its own profit, each constituent product system has a profit function given by:

$$\pi_i = q_i(p)(p_i - c_i) \tag{3.1}$$

where $c_i \in \mathbb{R}^+$ is the marginal cost of the ith product. Note that a fixed cost could be added to the expression without affecting the results.

The vector B in the demand function is given by $B =$

$$\begin{bmatrix} 1130 & 330 & 330 & 1130 & 1030 & 1030 & 1930 & \ldots \\ \ldots & 330 & 2130 & 1125 & 2130 & 1930 & -150 & 300 & 330 \end{bmatrix}^T$$

and the cost vector is given by $C =$

$$\begin{bmatrix} 110 & 130 & 130 & 110 & 110 & 110 & 110 & \ldots \\ \ldots & 130 & 110 & 120 & 110 & 110 & 120 & 120 & 130 \end{bmatrix}^T.$$

Note that the relative strength of the own-price elasticities of various products is visible in the strongly diagonal structure of matrix A (Figure 3.1). In spite of this feature, Figure 3.2 demonstrates that over 50% of the firms profits result strictly from the cooperative effects between products.

The relative value of cooperation increases sharply as the first few products are grouped into their respective clusters. Once a critical clustering is achieved, however, no improvement in the value of cooperation is observed through subsequent centralization. This indicates that these sets of products are fairly independent, decoupled with respect to

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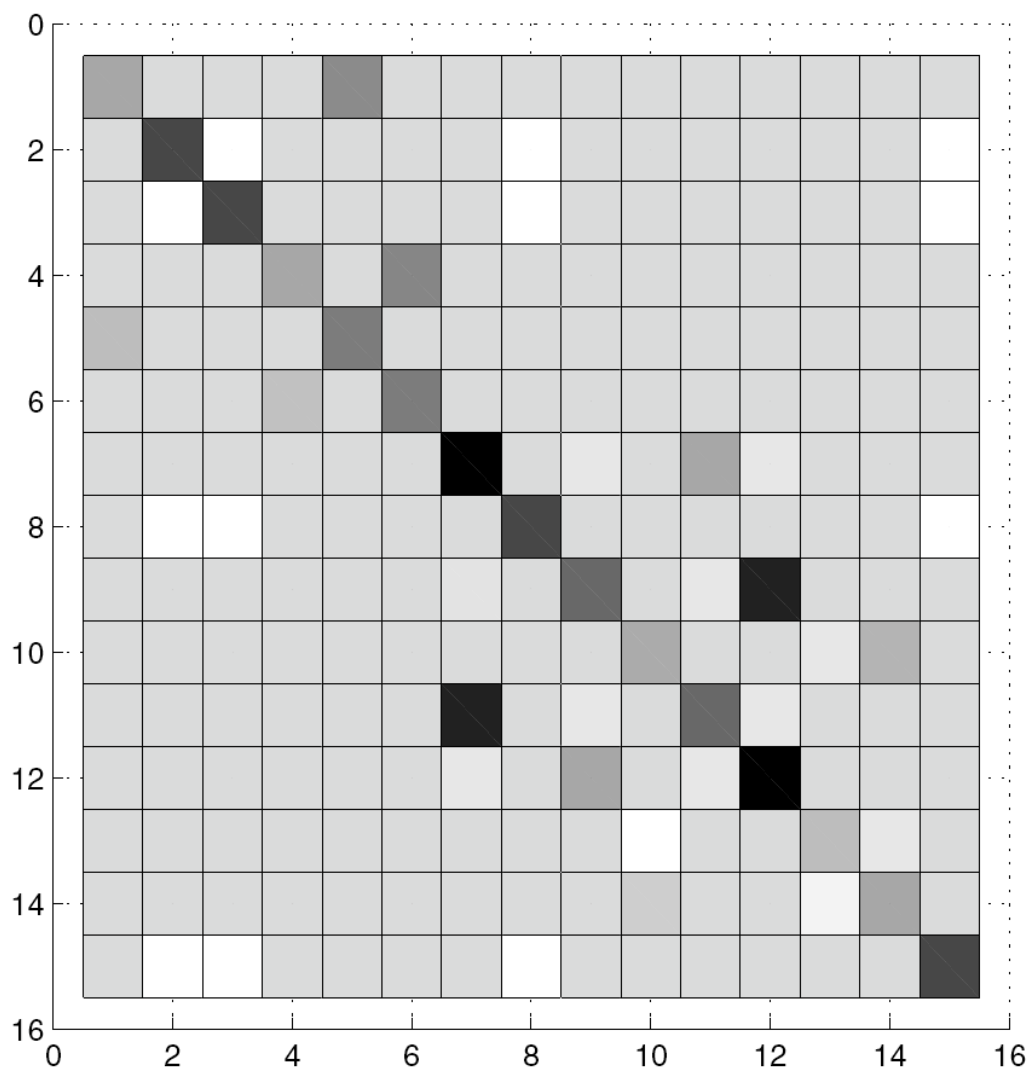
Figure 3.1: The A matrix of the Demand Function.

Figure 3.2: RVC vs. Product Groupings

Figure 3.3: VC-based Hieararchical Agglomerative Clustering of a 15 Product Firm. The highlighted row shows how the clusters are formed when there are exactly 5 clusters. This formation is interesting because as shown in Figure 3.2, no additional gain in the Value of Cooperation can be achieved by reducing the clusters. This formation consists of two 4-product firms, one 3-product firm, and two 2-product firms.
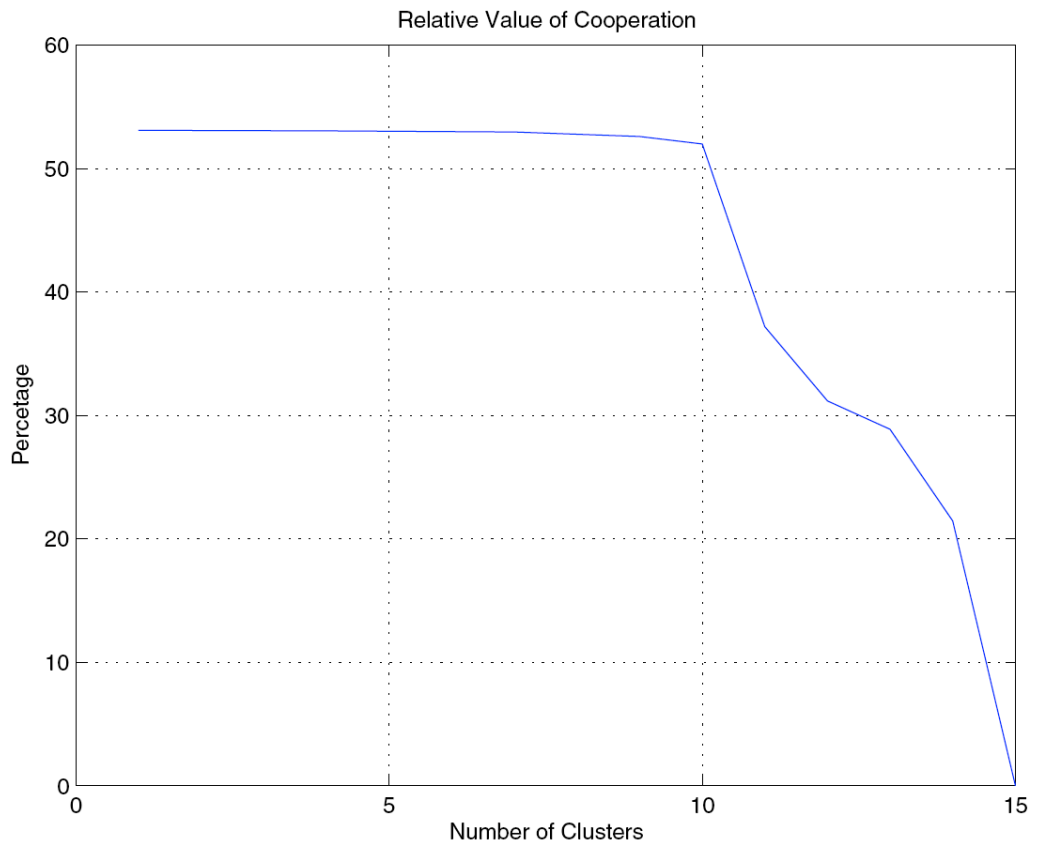
the market demand function. Cooperation-based clustering identifies these groups, even when they are not apparent from the market demand function directly.

The result of the cooperation-based clustering is shown in Figure 3.3. Each row indicates a set of clusters, beginning with 15 single-product clusters and ending with a single cluster of all the products. An interesting aspect of this clustering is apparent in the analysis of this figure with the RVC plot. We note that once the products have been grouped into five clusters, no more value of cooperation is derived through further clustering. This indicates that the firm is operating at the intersection of five rather independent markets, a fact that is not readily apparent from inspection of the demand function (see Figure 3.1).

3.3 Conclusion

We have illustrated the use of the value of cooperation for product clustering in the context of optimal organizational design through the use of the value of cooperation within a hierarchical agglomerative clustering framework. VC-based hierarchical agglomerative clustering starts from the reference structure, where all products are assumed to act independently. Then, the two products which exhibit the strongest value of cooperation are merged, and the process is repeated, decreasing the total number of clusters by one until all of the firms products are finally merged into a single organizational structure. The hierarchical nesting adds a natural constraint to the problem, which yields a product hierarchy with a clear organizational interpretation.

A real-world example would be very helpful in highlighting and validating our cooperation-based clustering idea. We would like to direct our future research to implementing the cooperation-based clustering framework to business. Such implementation would also highlight some of the challenges that may have been overlooked in our formulation.

Chapter 4

Coalition Robustness of Multiagent Systems[1]

Business networks provide one of the most compelling environments to study the conflicting effects of competition and cooperation on multi-agent dynamical systems. While firms engage various merger and divestiture strategies to create the desired cooperative environment that enhances their market power, governmental regulatory agencies enforce antitrust measures that protect competition as a means to limit the market power of these organizations. Merger simulation has subsequently evolved in recent years as a mechanism to study the impact of different organizational structures on the market. Nevertheless, typical economic models can often lead to competition dynamics that arbitrarily lose stability when considering different organizational structures. This work provides stability robustness conditions with respect to coalition structure for profit-maximizing dynamical systems with network demand, and partially convex utility. In particular, we show that stability of the coalition of all agents is sufficient to guarantee stability of all other coalition structures. These conditions are then leveraged to provide a systematic methodology for estimating a rich variety of demand systems that guarantee sensible stability results regardless of the structure of cooperation in the marketplace.

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[1]This part appeared in the proceedings of the American Control Conference 2008, Seattle WA as [18].

4.1 Firms, Market Power, and Merger Simulation

One of the most well-studied multi-agent systems is the marketplace. Market dynamics are governed by competition, nevertheless one of the most interesting features of the market is the spontaneous emergence of cooperation structures we call firms. Firms represent coalitions of agents that offset the computational limitations of individual agents to better compete for scarce resources. They orchestrate policies that attempt to drive profit-generating dynamics in the face of considerable uncertainty, both from the consumer market and from the competitive forces of other firms.

One way firms cope with market uncertainty is through growth. As firms deploy successful policies, they acquire capital that enable them to attract the cooperation of more agents in the marketplace. This can happen organically through the hiring of employees and the natural expansion of the firm's existing operations, or it can happen suddenly through mergers and acquisitions. Either way, such growth attempts to mitigate uncertainty by either entrenching the firm in the market niche known to have been previously successful, or by offsetting risk by diversifying the types of products or services the firm uses to compete for profits.

As firms generate wealth, they distribute a portion of it to their stake holders, who then engage the marketplace as consumers or investors of one kind or another. The ability of consumers to translate this wealth into an improved quality of life, however, depends significantly on the balance of power between firms in the marketplace. When firms are too strong, they do not have the incentive to innovate, and they can restrict the flow of existing goods and services to consumers unless premium prices are paid. When firms are too weak, they do not have the ability to innovate, nor do they generate the wealth their stake holders might otherwise have had to participate more fully as consumers or further investors in the marketplace. As a result, governments control the growth and strength of firms, either by stopping proposed mergers or by forcing firms to divide. This maintains

competition as an effective force to limit the market power of firms, and it ideally creates resonance between the welfare of consumers and the welfare of investors that fuel growth.

At the heart of both the firm's growth strategy and the government's regulation strategy, then, lies the ability to measure a firm's market power. In 1997 the US Department of Justice and the Federal Trade Commission's released guidelines governing the regulation of mergers within the United States [1]. This, in turn, precipitated growing interest in the use of "merger simulations" to estimate the effects of proposed mergers or acquisitions [12], [13], [22], [2] and [5].

Merger simulations predict post-merger prices based on a demand model of the relationship between prices charged and quantities sold by the firms under investigation in the relevant market. Assumptions or models about supply issues are also incorporated into the simulation. Under a Bertrand model of pricing, every firm sets the prices of its brands to maximize its profits. Equilibrium results when no firm can unilaterally change its prices to improve its profits. Simulations compare pre-merger prices and profits with post-merger prices and profits to analyze the impact of the merger. "Reverse" simulations compare prices and profits of an existing firm with those resulting from the division of the firm into constitutive components, thereby measuring the "Value of Cooperation" achieved by the strategic positioning of the firm as the coalition of those particular components within the context of the larger market [19], [17], and [20].

In this way, Value of Cooperation can be viewed as a quantification of market power, and merger simulation can be thought of as a Value of Cooperation measurement on the post-merger firm. The presence of market power alone, however, is not necessarily illegal, nor is it sufficient to give the firm monopolistic power, as the firm would also need to create barriers of entry to prevent new firms from competing. Likewise, there may be other measures used to quantify the impact of market structure or industrial organization on market conditions. Nevertheless, such measures typically compare a property of an

equilibrium of one market structure with that resulting from a different market structure, and are thus comparative static analyses that typically ignore dynamic issues.

Often, however, the demand models used in such simulations can lead to unstable equilibria, or even conditions where no equilibria exist at all for some market structures [5]. Such results are generally not the foreshadows of pending market doom should the right conspiracy be formed, but rather are simply dynamic limitations resulting from mathematical technicalities of the these models. None of the demand models typically used in economics, i.e. linear, log-linear (constant elasticity), logit, AIDS, and PCAIDS, guarantee the existence and stability of equilibria for all possible market structures.

Viewing the marketplace as a profit-maximizing multi-agent dynamical system (Section 4.2), this work resolves these issues by providing stability robustness conditions with respect to coalition structure for such systems when these systems have a particular network demand structure (Section 4.3). These conditions are then leveraged to provide a systematic methodology for empirically estimating a rich variety of AIDS-like demand systems from market data, using standard convex-optimization tools, that guarantee sensible stability results regardless of the structure of cooperation in the marketplace (Section 4.4).

4.2 Markets as Multi-Agent Systems

Consider a market consisting of n products, each produced and controlled by a single product division. These product divisions are the constitutive agents in our multi-agent system, $\mathcal{N}$, and they are arbitrarily ordered and numbered 1 to n. Following a Bertrand model of pricing, each agent has complete authority and control to price its product as it sees fit. The prices for all the products are public knowledge, known at any given time by all the agents, and denoted by the vector $x \in \mathbb{R}^n$. For convenience, we will assume that the prices are in units relative to the unit cost of production for each product. That is, x_i is the markup for product i.

We suppose that the aggregate effect of consumers in the market is given by a demand function, $q(x) : \mathbb{R}^n \to \mathbb{R}^n$, which characterizes how the quantity sold for each product varies with prices. Note that the demand, $q_i(x) : \mathbb{R}^n \to \mathbb{R}$, for product i depends, in general, not only on its own price, but on the prices of all the other products as well.

Each agent is equipped with a utility function that scores its reward as a function of the decisions of all the agents in the system. This utility function is a component of the market utility and is given by each product division's profits:

$$U_i(x) = x_i q_i(x). \tag{4.1}$$

A firm, F, is a coalition of agents, represented as a subset of $\mathcal{N}$. We allow the market to coalesce into $m \leq n$ firms, where every agent belongs to one and only one firm. Thus, the market structure, or industrial organization, $\mathscr{F} = \{F_1, F_2, ..., F_m\}$, is a partition of $\mathcal{N}$. We will write $\mathscr{F}^{-1}(i)$ for the firm to which agent i belongs.

We associate with each firm an objective or profit function given by the sum of the utility functions of the agents belonging to the firm,

$$U_F(x) = \sum_{i \in F} U_i(x) = \sum_{i \in F} x_i q_i(x). \tag{4.2}$$

By associating with a firm, an agent agrees to adjust the prices of its product to maximize the total profits or objective of the firm, rather than simply maximize its own utility. Thus, all agents belonging to the same firm adopt a common objective and effectively surrender their pricing authority to the firm, allowing the firm to lose money by underpricing in one division in order to induce a greater demand and profit in another division.

Each agent therefore changes its price in the direction of the gradient of the objective of the firm to which it belongs;

$$\dot{x}_i = \frac{\partial U_F}{\partial x_i}(x) = \frac{\partial \left[\sum_{i \in F} U_i \right]}{\partial x_i} = \sum_{i \in F} \frac{\partial U_i}{\partial x_i}(x). \tag{4.3}$$

Substituting from (4.1) for the profit structure of an agent's utility and writing them in vector notation, these dynamics become

$$\dot{x} = V_{\mathscr{F}}(x) = \left[D_{\mathscr{F}} \left(J_q^T(x) \right) \right] x + q(x), \tag{4.4}$$

where $J_q(x)$ is the Jacobian of the function $q(x)$, A^T denotes transpose of a matrix A, and $D_{\mathscr{F}}(A)$ is defined as: a) $d_{ij} = a_{ij}$ if $j \in \mathscr{F}^{-1}(i)$, and b) $d_{ij} = 0$ otherwise. Thus, if $\mathscr{F} = \{(1,2),3\}$ and A were given by

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad \text{then} \quad D_{\mathscr{F}}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Given a market structure and a demand function, Equation (4.4) thus represents the profit-maximizing dynamics of the multi-agent system and becomes the central focus of our analysis.

Our stability robustness problem, then, is to find conditions under which we can guarantee existence, uniqueness and stability of the equilibrium of Equation (4.4) for all market structures $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of $\mathscr{N}$.

Example 4.1. *Consider a market with three products with consumer demand given by:*

$$\begin{bmatrix} q_1(x) \\ q_2(x) \\ q_3(x) \end{bmatrix} = \begin{bmatrix} -3 & -5 & 4 \\ -4 & -4 & 3 \\ 1 & 2 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \tag{4.5}$$

Note that the demand is linear, and based on the signs of coefficients in the demand function, we can see that Products (1 and 2) are complements, while (1 and 3) and (2 and 3) are substitutes. That is to say, an increase in the price of Product 1 results in decreased

sales of both Products 1 (as you would expect) and 2 (i.e it is a complement to Product 1),

but an increase of sales of Product 3 (i.e. it is a substitute for Product 1).

The utility functions of the constitutive agents, meaning the three product divisions

that each control a single product, are thus given by

$$
\begin{aligned}
U_1 &= (-3x_1 - 5x_2 + 4x_3 + 80)\,x_1 \\
U_2 &= (-4x_1 - 4x_2 + 3x_3 + 90)\,x_2 \\
U_3 &= (x_1 + 2x_2 - 15x_3 + 80)\,x_3
\end{aligned}
\tag{4.6}
$$

Moreover, given any market structure $\mathcal{F}$, the profit-maximizing dynamics of this multi-agent system then become

$$
\dot{x} = D_{\mathcal{F}}\left(\begin{bmatrix} -3 & -4 & 1 \\ -5 & -4 & 2 \\ 4 & 3 & -15 \end{bmatrix}\right) x + q(x).
\tag{4.7}
$$

Now, let us compare the market dynamics for two different industrial organizations. First, we will consider the organization where every product division is its own firm, $\mathcal{F} = \{1,2,3\}$. In this case, the dynamics become:

$$
\dot{x} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)
$$

$$
\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 4 \\ -4 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix}
\tag{4.8}
$$

It is easy to verify that this system has a stable equilibrium at $x = (8.91, 8.11, 3.50)$ dollars. The demand at this point becomes $q = (26.72, 32.42, 52.63)$ units sold per unit time, and the profits for each firm are $U = (238.07, 262.93, 184.21)$ dollars per unit time.

Now let's consider the organization where Divisions 1 and 2 merge to form a single firm. This market structure is given by $\mathscr{F} = \{(1,2),3\}$, and the corresponding dynamics become:

$$\dot{x} = \begin{bmatrix} -3 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -15 \end{bmatrix} x + q(x)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -9 & 4 \\ -8 & -8 & 3 \\ 1 & 2 & -30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 80 \\ 90 \\ 80 \end{bmatrix} \tag{4.9}$$

From these dynamics, the system equilibrium is at $x = (6.40, 6.08, 3.29)$ dollars, corresponding to the demand of $q = (43.56, 49.95, 49.21)$ units sold per unit time and profits for the two firms of $U = (582.48, 161.90)$ dollars per unit time. Nevertheless, since the system has a positive eigenvalue 1.5297, this equilibrium point is unstable. As a result, these equilibrium values are never really attainable, the profits of \$582.48 for the merged firm can not actually be realized, because even small changes in prices will lead, according to this model, to a never ending price war that never converges. Note that there is no way to detect a priori that this particular market structure would be unstable with this particular demand system. The merger of Divisions 1 and 3, for example, corresponding to market structure $\mathscr{F} = \{(1,3),2\}$, is stable.

■

4.3 Stability Robustness Conditions

Example 4.1 demonstrates how otherwise reasonable models of market dynamics can fail when considering industrial organization issues. The demand model, which is of sufficient fidelity to address questions such as the complementary/substitutive relationship between products, drives the prediction that one possible merger will result in prices going to infinity. In reality, such a merger would not result in continually increasing prices; this result is simply an artifact of the model we have chosen. As a result, we see that this model is simply inadequate to describe market dynamics under changes in market structure, at least for some structures.

Nevertheless, if a model breaks down for some market structures by predicting unstable equilibria (or the lack of any equilibria, as happens for constant-elasticity models), can it be trusted to yield accurate results for any market structure? Whatever simplifications in the model cause it to drastically fail for some market structures might degrade its representation of the true dynamics under other market structures. The only safe course is to identify models that have sufficient fidelity to yield sensible results for every possible market structure.

Note that verifying the fidelity of a proposed model by checking the stability properties for all possible market structures is intractable; the number of possible market structures grows worse than exponential with n, the number of products, and real markets can involve thousands of products. As a result, we need tractable robustness conditions that can guarantee existence, uniqueness and stability of equilibria regardless of market structure.

To generate such conditions, we begin by defining the quantities we will use to check stability robustness of the system (4.4). For notational convenience let $F(i) = \mathscr{F}^{-1}(i)$ denote the firm to which the ith agent belongs. When introducing the lemmas, we will write $M_{m,n}(\mathbb{F})$ for the set of all $m \times n$ matrices whose entries are elements of the field $\mathbb{F}$, and we will abbreviate to $M_n(\mathbb{F})$ in the case of square matrices. For any square matrix $A \in M_n(\mathbb{C})$, we will denote its numerical range as $W(A) = \{x^*Ax \mid \|x\|_2 = 1\}$, and its

spectrum as $\sigma(A)$. For a subset S of a vector space, we will write $\mathrm{co}(S)$ to denote its convex hull. For two subsets A and B of a group $(G, +)$, we write $A + B = \{x + y \mid x \in A, y \in B\}$.

Lemma 4.1. *Given the system (4.4), the Jacobian of the system dynamics, $V_{\mathscr{F}}$, decomposes as:*

$$J_{V_{\mathscr{F}}}(x) = \left[A(x) + D_{\mathscr{F}}(A^T(x)) \right] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x), \tag{4.10}$$

where $A(x)$, $B_{\mathscr{F}}(x)$, and $C_{\mathscr{F}}(x)$ are given as follows:

$$A(x): \quad A_{ii}(x) = \tfrac{1}{2} \sum_{j=1}^{n} \frac{\partial^2 U_j}{\partial x_i^2}(x), \quad A_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) \tag{4.11}$$

$$B_{\mathscr{F}}(x): \quad B_{ii}(x) = 0, \quad B_{ij}(x) = \sum_{k \in F(i) \setminus \{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \tag{4.12}$$

$$C_{\mathscr{F}}(x): \quad C_{ii}(x) = -\sum_{j \notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2} \quad C_{ij}(x) = 0 \tag{4.13}$$

Proof. The diagonal entries of $J_{V_{\mathscr{F}}}(x)$ are given by,

$$
\begin{aligned}
J_{ii}(x) = \frac{\partial V_i}{\partial x_i}(x) &= \sum_{j \in F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\
&= \sum_{j=1}^{n} \frac{\partial^2 U_j}{\partial x_i^2}(x) - \sum_{j \notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \\
&= 2A_{ii}(x) + C_{ii}(x) = 2A_{ii}(x) + B_{ii}(x) + C_{ii}(x). \tag{4.14}
\end{aligned}
$$

For $j \neq i$, the off-diagonal $J_{ij}(x)$ is given by,

$$
\begin{aligned}
J_{ij}(x) = \frac{\partial V_i}{\partial x_j}(x) &= \sum_{k \in F(i)} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \\
&= \sum_{k \in F(i) \cap \{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) + \sum_{k \in F(i) \setminus \{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) \\
&= \sum_{k \in F(i) \cap \{i,j\}} \frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) + B_{ij}(x). \tag{4.15}
\end{aligned}
$$

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When $j \in F(i)$, we then have

$$J_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) + \frac{\partial^2 U_j}{\partial x_i \partial x_j}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + A_{ji}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + A_{ji}(x) + B_{ij}(x) + C_{ij}(x). \tag{4.16}$$

Otherwise, when $j \notin F(i)$, we then have

$$J_{ij}(x) = \frac{\partial^2 U_i}{\partial x_i \partial x_j}(x) + B_{ij}(x)$$

$$= A_{ij}(x) + B_{ij}(x) + C_{ij}(x) \tag{4.17}$$

Therefore,

$$J_{V_{\mathscr{F}}}(x) = \left[A(x) + D_{\mathscr{F}}(A^T(x)) \right] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x). \tag{4.18}$$

□

Definition 4.1. *The market structure consisting of a single firm, $\mathscr{F} = \{(1,2,...,n)\}$, that is, where all agents belong to the same coalition, is called the* Grand Structure, *denoted $\mathscr{G}$, and the associated firm is called the* Grand Coalition, *denoted G.*

Lemma 4.2. *Given by (4.13) and Definition 4.1, $C_{\mathscr{G}}(x) = 0$.*

Proof. This follows directly from the definition of $C_{\mathscr{F}}$ in (4.13), where the only nonzero elements are on the diagonal, and the diagonal elements become zero for the Grand Structure since all agents belong to the same firm. □

Definition 4.2. *A function* $h(x) : \mathbb{R}^n \to \mathbb{R}^m$ *is said to have* network structure *if there exist functions* $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ *such that*

$$h_i(x) = \sum_{j=1}^{n} f_{ij}(x_i, x_j), \quad i = 1, ..., n. \tag{4.19}$$

Lemma 4.3. *A demand function,* $q(x) : \mathbb{R}^n \to \mathbb{R}^n$, *with network structure induces network structure on the market utility function given by (4.1).*

Proof.

$$U_i(x) = x_i q_i(x) = x_i \sum_{j=1}^{n} f_{ij}(x_i, x_j) = \sum_{j=1}^{n} x_i f_{ij}(x_i, x_j) = \sum_{j=1}^{n} \hat{f}_{ij}(x_i, x_j). \tag{4.20}$$

$\square$

Lemma 4.4. *If the utility function,* $U(x)$, *associated with system (4.4) has network structure, then* $B_{\mathscr{F}}(x) = 0$ *for all market structures* $\mathscr{F}$.

Proof. Network structure of $U(x)$ implies there exist functions $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$ such that $U_i(x) = \sum_{j=1}^{n} f_{ij}(x_i, x_j), \quad i = 1, ..., n$. Hence, for $k \notin \{i, j\}$,

$$\begin{aligned}
\frac{\partial^2 U_k}{\partial x_i \partial x_j}(x) &= \sum_{l=1}^{n} \frac{\partial^2 f_{kl}(x_k, x_l)}{\partial x_i \partial x_j} \\
&= \frac{\partial^2 f_{ki}(x_k, x_i)}{\partial x_i \partial x_j} + \frac{\partial^2 f_{kj}(x_k, x_j)}{\partial x_i \partial x_j} \\
&= \frac{\partial}{\partial x_i} \frac{\partial f_{ki}(x_k, x_i)}{\partial x_j} + \frac{\partial}{\partial x_j} \frac{\partial f_{kj}(x_k, x_j)}{\partial x_i} = 0.
\end{aligned}$$

Therefore, following from (4.12), $B_{\mathscr{F}}(x) = 0$ for all market structures $\mathscr{F}$. $\square$

Definition 4.3. *A utility function* $U(x) : \mathbb{R}^n \to \mathbb{R}^n$ *is said to be* partially convex *if,*

$$\frac{\partial^2 U_j}{\partial x_i^2}(x) \geq 0 \quad \forall j \neq i, \ \forall x \in \mathbb{R}^n. \tag{4.21}$$

Lemma 4.5. *When utility functions of the system (4.4) are partially convex, $C_{\mathscr{F}}(x)$ is a negative semidefinite diagonal matrix.*

Proof. From the definition of parital convexity,

$$\frac{\partial^2 U_j}{\partial x_i^2}(x) \geq 0 \quad \forall j \neq i, \ \forall x \in \mathbb{R}^n.$$

From the definition of $C_{\mathscr{F}}$, its diagonal elements are $C_{ii}(x) = -\displaystyle\sum_{j \notin F(i)} \frac{\partial^2 U_j}{\partial x_i^2}(x) \leq 0$, and the off-diagonal elements are zero. Hence $C_{\mathscr{F}}$ is a diagonal matrix with non-positive diagonal elements. Recall that eigenvalues of a diagonal matrix are also its diagonal elements, the matrix $C_{\mathscr{F}}$ has only non-positive eigenvalues. Therefore, $C_{\mathscr{F}}$ is negative semidefinite. $\qquad\square$

Definition 4.4. *An n-product market with profit-maximizing dynamics given by (4.4), with demand function $q(x) : \mathbb{R}^n \to \mathbb{R}^n$ that has network structure, and with partially convex utility is said to be an* industrial organization network *for any market structure $\mathscr{F}$.*

Definitions 1 through 4 equip the models we will use to represent market dynamics with the technical structure we will need to guarantee stability robustness for all industrial organizations. In particular, industrial organization networks provide a model class with sufficient fidelity to explore questions involving changes in market structure. The following lemma comes from various parts in [7].

Lemma 4.6. *Let $A, B \in M_n(\mathbb{C})$.*

  (i) *$W(A)$ is compact and convex.*

 (ii) *$\mathrm{co}\big(\sigma(A)\big) \subseteq W(A)$.*

(iii) *$W(A+B) \subseteq W(A) + W(B)$.*

(iv) *A is normal $\Rightarrow \mathrm{co}\big(\sigma(A)\big) = W(A)$.*

Lemma 4.7. *For $A \in M_n(\mathbb{R})$,*

$$\max W(A + A^T) = \max \mathfrak{Re}W(A) + \max \mathfrak{Re}W(A^T).$$

Proof. Essentially follows from the definition of numerical range,

$$
\begin{aligned}
\max W(A + A^T) &= \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} x^*(A + A^T)x \\
&= \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} \left(x^*Ax + x^*A^Tx \right) = \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} \left(x^*Ax + \overline{x^*Ax} \right) \\
&= 2 \max_{\|x\|_2 = 1, x \in \mathbb{C}^n} \mathfrak{Re}(x^*Ax) = 2 \max \mathfrak{Re}W(A).
\end{aligned}
$$

Following the same reasoning, $\max W(A + A^T) = 2 \max \mathfrak{Re}W(A^T)$, hence $\max W(A + A^T) = \max \mathfrak{Re}W(A) + \max \mathfrak{Re}W(A^T)$. $\square$

The following lemma is from [11].

Lemma 4.8. *Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exist positive $\varepsilon, R \in \mathbb{R}$ such that for all $x \in \mathbb{R}^n$, $\|x\|_2 > R$,*

$$z^T \frac{\partial f}{\partial x}(x)z \leq -\varepsilon \|z\|_2^2 \quad \forall z \in \mathbb{R}^n.$$

Corollary 4.1. *Given $f : \mathbb{R}^n \to \mathbb{R}^n$, the equation $f(x) = y$ will have exactly one root for each y if there exists positive $\varepsilon \in \mathbb{R}$ such that,*

$$\max \mathfrak{Re}W \left(\frac{\partial f}{\partial x}(x) \right) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n.$$

Proof. For all $z \in \mathbb{R}^n$, $\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x) \right)$, and also, $\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in \mathbb{R}$, hence

$$\frac{z^T}{\|z\|_2} \frac{\partial f}{\partial x}(x) \frac{z}{\|z\|_2} \in W\left(\frac{\partial f}{\partial x}(x) \right) \cap \mathbb{R} \subseteq \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x) \right).$$

Thus, if $\max \mathfrak{Re} W\left(\frac{\partial f}{\partial x}(x)\right) \leq -\varepsilon$ then,

$$\frac{z^T}{\|z\|_2}\frac{\partial f}{\partial x}(x)\frac{z}{\|z\|_2} \leq \max \mathfrak{Re} W\left(\frac{\partial f}{\partial x}(x)\right) \leq -\varepsilon \Rightarrow z^T\frac{\partial f}{\partial x}(x)z \leq -\varepsilon\|z\|_2^2,$$

which satisfies the condition of Lemma 4.8. $\qquad\square$

Lemma 4.9. *For matrix $A \in M_n(\mathbb{R})$, $W\left(D_{\mathscr{F}}(A)\right) \subseteq W(A)$.*

Proof. For $F \subseteq \mathscr{F} = \{F_1, F_2, \ldots F_m\}$, let $I_F = \mathrm{diag}_{i=1}^n\left(\chi_F(i)\right)$, with $\chi_F(\cdot)$ being the membership function of F. Note that

$$D_{\mathscr{F}}(A) = \sum_{k=1}^m I_{F_k}AI_{F_k}.$$

Let $w \in W\left(D_{\mathscr{F}}(A)\right)$ and let $x \in \mathbb{C}^n$ such $\|x\|_2 = 1$ and $w = x^*D_{\mathscr{F}}(A)x$. Since $\sum_{k=1}^m I_{F_k} = I$, $\sum_{k=1}^m I_{F_k}x = x$, hence

$$1 = \|x\|_2^2 = x^*x = \left(\sum_{k=1}^m I_{F_k}x\right)^* \sum_{l=1}^m I_{F_l}x = \sum_{k=1,l=1}^m x^*I_{F_k}I_{F_l}x = \sum_{k=1}^m x^*I_{F_k}^2x = \sum_{k=1}^m \|I_{F_k}x\|_2^2.$$

Let $\mathscr{F}^+ = \{F \in \mathscr{F} : I_Fx \neq 0\}$. For $F \in \mathscr{F}^+$, let $y_F = \frac{I_Fx}{\|I_Fx\|_2}$. Therefore $\|y_F\|_2 = 1$ and $I_Fx = \|I_Fx\|_2 y_F$.

$$\begin{aligned} w = x^*D_{\mathscr{F}}(A)x &= x^*\left(\sum_{F \in \mathscr{F}} I_FAI_F\right)x = \sum_{F \in \mathscr{F}} x^*I_FAI_Fx \\ &= \sum_{F \in \mathscr{F}} (I_Fx)^*A(I_Fx) = \sum_{F \in \mathscr{F}^+} (I_Fx)^*A(I_Fx) \\ &= \sum_{F \in \mathscr{F}^+} (\|I_Fx\|_2 y_F)^*A(\|I_Fx\|_2 y_F) \\ &= \sum_{F \in \mathscr{F}^+} \|I_Fx\|_2^2(y_F^*Ay_F), \end{aligned} \qquad (4.22)$$

while $\sum_{F \in \mathscr{F}^+} \|I_Fx\|_2^2 = \sum_{F \in \mathscr{F}} \|I_Fx\|_2^2 = 1$. Therefore, w is a convex combination of $y_F^*Ay_F$, which are in $W(A)$ because $\|y_F\|_2 = 1$. $W(A)$ is convex (Lemma 4.6) $\Rightarrow w \in W(A)$. $\qquad\square$

These lemmas demonstrate intermediate results that will enable us to provide stability robustness conditions for profit-maximizing dynamics under any coalition structure. In particular, Lemma 8 and Corollary 1 provide the machinery used to guarantee existence and uniqueness of an equilibrium for every market structure. To demonstrate stability of these equilibria using Lyapunov's indirect method, Lemma 1 provides a decomposition of the Jacobian of the system dynamics that simplify under certain technical assumptions. Lemmas 2-5 then invoke these technical assumptions to characterize an industrial organization network and simplify the expression for the Jacobian of its dynamics. Finally, Lemmas 6, 7, and 9 then yield the machinery to demonstrate how a simple check on the stability of the Grand Structure dynamics will guarantee stability for all other market structures. We now state and prove the stability robustness theorem.

Theorem 4.1. *Consider an n-product market with agent set $N = \{1, 2, ..., n\}$ and an industrial organization network characterized by (4.4). Let the Grand Coalition, G, of this network be given as in Definition 1, with objective function, U_G, as specified in (4.2). Under these conditions, then (4.4) will have a unique and stable equilibrium for all $\mathscr{F} \in \Delta$, where Δ is the set of all partitions of $\mathscr{N}$, if there exists positive $\varepsilon \in \mathbb{R}$ such that*

$$\max \sigma\big(H(x)\big) \leq -\varepsilon \quad \forall x \in \mathbb{R}^n, \tag{4.23}$$

where $H(x)$ is the Hessian matrix of the objective function $U_G(x)$.

Proof. Let $\mathscr{F}$ be an arbitrary market structure in Δ. Let $J_{V_{\mathscr{F}}}(x)$ be the Jacobian matrix of $V_{\mathscr{F}}(x)$ given by (4.4). Following from Lemma 4.1,

$$J_{V_{\mathscr{F}}}(x) = \big[A(x) + D_{\mathscr{F}}\big(A^T(x)\big)\big] + B_{\mathscr{F}}(x) + C_{\mathscr{F}}(x).$$

The network structure of demand, $q(x)$, and thus also of utility, $U(x)$, then imply that $B_{\mathscr{F}}(x) = 0$ as shown in Lemma 4.4. In the case that $\mathscr{F}$ is the Grand Structure, we know

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from Lemma 4.2 that $C_{\mathcal{G}}(x) = 0$. Thus, $J_{V_{\mathcal{G}}}(x) = A(x) + A^T(x) = H(x)$. In general, however, we have $J_{V_{\mathcal{F}}} = A(x) + D_{\mathcal{F}}(A^T(x)) + C_{\mathcal{F}}(x)$. From Lemma 4.6 this yields,

$$W\left(J_{V_{\mathcal{F}}}(x)\right) = W\left(A(x) + D_{\mathcal{F}}\left(A^T(x)\right) + C_{\mathcal{F}}(x)\right)$$
$$\subseteq W\left(A(x)\right) + W\left(D_{\mathcal{F}}\left(A^T(x)\right)\right) + W(C_{\mathcal{F}}(x)).$$

From Lemma 4.9, $W\left(D_{\mathcal{F}}\left(A^T(x)\right)\right) \subseteq W\left(A^T(x)\right)$, hence

$$W\left(J_{V_{\mathcal{F}}}(x)\right) \subseteq W\left(A(x)\right) + W\left(A^T(x)\right) + W\left(C_{\mathcal{F}}(x)\right).$$

As a result,

$$\max \mathfrak{Re} W\left(J_{V_{\mathcal{F}}}(x)\right) \leq \max \mathfrak{Re} W\left(A(x)\right) + \mathfrak{Re} W\left(A^T(x)\right) + \max \mathfrak{Re} W\left(C_{\mathcal{F}}(x)\right).$$

Due to Lemma 4.5, $W\left(C_{\mathcal{F}}(x)\right) \leq 0$. Also, from Lemma 4.7,

$$\max \mathfrak{Re} W\left(A(x)\right) + \mathfrak{Re} W\left(A^T(x)\right) = \max W\left(A(x) + A^T(x)\right) = \max W\left(H(x)\right) = \max \sigma(H(x)).$$

Following that, $\max \mathfrak{Re} W\left(J_{V_{\mathcal{F}}}(x)\right) \leq \max \sigma(H(x)) \leq -\varepsilon$. By Corollary 4.1, we can conclude that the equation $V_{\mathcal{F}}(x) = 0$ has exactly one solution x_e. Hence the market structure $\mathcal{F}$ yields exactly one equilibrium x_e. Moreover, since the Jacobian evaluated at the equilibrium point, $J_{V_{\mathcal{F}}}(x_e)$, satisfies,

$$\max \mathfrak{Re} \sigma(J_{V_{\mathcal{F}}}(x_e)) \leq \max \mathfrak{Re} W(J_{V_{\mathcal{F}}}(x_e)) \leq -\varepsilon < 0,$$

then the equilibrium x_e is locally stable due to Lyapunov's indirect method. $\qquad\square$

4.4 Demand Estimation for Industrial Organization Networks

This section shows how we apply the stability robustness condition in Theorem 4.1 to a class of AIDS-like demand models. We will begin to cover first our main tool, semidefinite programming [21] [9], used in finding the model parameters that best fit the data, while meeting the sufficient condition given in Theorem 4.1.

4.4.1 Semidefinite Programming

In semidefinite programming, one minimizes a convex function subject to the constraint that an affine combination of symmetric matrices is positive semidefinite. As the authors of [21] noted, such a constraint is nonlinear and nonsmooth, but convex. In fact, it is shown in [21] that although semidefinite programs are much more general than linear programs, they are not much harder to solve. Most interior-point methods for linear programming have been generalized to semidefinite programs. As in linear programming, these methods have polynomial worst-case complexity, and perform very well in practice.

Let us show the canonical form of a semidefinite program,

$$
\begin{aligned}
\text{minimize} \quad & f_0(x) \\
\text{subject to} \quad & \sigma\left(\Psi_0 + \sum_{i=1}^{n} \Psi_i x_i\right) \leq 0,
\end{aligned}
\tag{4.24}
$$

where $f_0(x)$ is convex and Ψ_i are symmetric for $i = 0, 1, \ldots, n$.

4.4.2 Demand Estimation with Stability Robustness Constraint

Now we will show our methodology applying to a class of demand models. Let us first do so by describing our model, after which we shall show that both the requirements given in Definition 4.2 and Definition 4.3 are met. This demand model is based on the concept of effective price: we recognize that changing prices from different price ranges will yield different effects on demand. Therefore, let $f_i(x_i)$ be a function representing the effective

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price of product i, our demand function will be,

$$q = Pf(x) + b, \text{where } f(x) = \big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big), \tag{4.25}$$

for some $n \times n$ matrix P and $n \times 1$ vector b.

Use demand functions given by Equation (4.25)

Let us show that this demand model given in (4.25) satisfies all the assumptions of an industrial organization network. First, it can be shown that the network assumption in Definition 4.2 is met, because

$$q_i(x) = \sum_{j=1}^{n} p_{ij} f_j(x_j). \tag{4.26}$$

Also the partially convex requirement, as defined in Definition 4.3, is met. For $j \neq i$,

$$\frac{\partial^2 U_j}{\partial x_i^2}(x) = \frac{\partial^2 [x_j q_j(x)]}{\partial x_i^2} = x_j \frac{\partial}{\partial x_i} \frac{\partial q_j(x)}{\partial x_i} = x_j \frac{\partial}{\partial x_i} \left(p_{ji} \frac{\partial f_j(x_j)}{\partial x_i} \right) = 0 \geq 0. \tag{4.27}$$

Use splines to design the effective price functions, f(x), in the demand model

These functions should be monotone and will serve as basis functions in a nonlinear regression when fitting P and b from data. The choice of f(x) can be guided by data or use professional expertise to characterize price sensitivity in the market.

Substitute the desired effective price functions to build a semidefinite program. Note that this program samples $H(x)$ to try to enforce that $\sigma(H(x)) \leq -\varepsilon$ everywhere

This is the most important step in the process. Let us be detailed in showing how it is carried out. Assuming that we are given K data points $(\mathbf{q_i}, \mathbf{x_i})$, $i = 1, 2, \ldots K$, where $\mathbf{q_i} \in \mathbb{R}^n$ are quantity demanded at a price setting $\mathbf{x_i} \in \mathbb{R}^n$, our objective is to minimize the regression error. For example, if the regression error is measured by the l_2 norm, then we have a least

square regression problem,

$$\text{find} \quad P \in M_n(\mathbb{R}), b \in \mathbb{R}^n \quad \text{to}$$

$$\text{minimize} \quad \sum_{i=1}^{K} \|P\mathbf{x_i} + b - \mathbf{q_i}\|_2^2. \tag{4.28}$$

In addition, we need to ensure that the condition (4.23) is met. This condition needs to be held for an infinite number of $x \in \mathbb{R}^n$. However, by looking carefully at,

$$H(x) = \frac{\partial^2 U_G}{\partial^2 x} = \left[P\text{diag}\left(\frac{df_i}{dx_i}(x)\right) + \text{diag}\left(\frac{df_i}{dx_i}(x)\right) P^T \right] + \text{diag}\left(\sum_j p_{ij} x_j \frac{d^2 f_i}{dx_i^2}(x)\right), \tag{4.29}$$

we recognize that if we require that the effective price functions are linear for $x \in \mathscr{C}_n(R) = \{\mathbb{R}^n, \|x\|_2 > R\}$ for some radius R, then, $H(x)$ is unchanged for $x \in \mathscr{C}_n(R)$. Therefore we only need to meet the constraint (4.23) for a compact ball $x \in \mathscr{B}_n(R) = \{x \in \mathbb{R}^n \mid \|x\|_2 \leq R\}$. In fact, we will make one step further by sampling the points in this ball, so that the number of points to check is finite. This is often done in practice. So, let $S = \{\mathbf{s_j}\}$ be a finite sample of $x \in \mathscr{B}_n(R)$, constraint (4.23) can be approximated by,

$$\max \sigma\left(H(\mathbf{s_j})\right) \leq -\varepsilon \quad \forall \mathbf{s_j} \in S, \tag{4.30}$$

If we let $\mathbf{y} = \begin{bmatrix} p_{11} & \dots p_{1n} & b_1 \dots p_{n1} & \dots & p_{nn} & b_n \end{bmatrix}^T$, $\Pi \in M_{Kn,n^2+n}(\mathbb{R})$, $\Pi = \text{diag}(\Sigma, \Sigma, \dots, \Sigma)$, where $\Sigma \in M_{K,n+1}$,

$$[\Sigma]_i = \begin{bmatrix} f_1(x_i) & \dots f_n(x_i) & 1 \end{bmatrix},$$

$$\mathbf{z} = \begin{bmatrix} q_{11} & \cdots & q_{K1} & \cdots q_{1n} & \cdots q_{Kn} \end{bmatrix}^{T},$$ and $l = n^2 + n$, then the regression objective be-comes,

$$\text{find} \quad \mathbf{y} \in \mathbb{R}^{l} \quad \text{to}$$

$$\text{minimize} \quad \|\Pi\mathbf{y} - \mathbf{z}\|_2^2. \tag{4.31}$$

Also, let $\Phi_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$, $\Phi_{ij}(\mathbf{s}) = \text{diag}_{i=1}^{n}(\sum s_j \frac{\partial^2 f_i}{\partial x_i}(s_i))$, $\Theta_{ij}(\mathbf{s}) \in M_n(\mathbb{R})$ hav-ing two non-zero $(i,j)^{th}$ and $(j,i)^{th}$ entries with value $\frac{\partial f_i}{\partial x_i}(s_i)$, $\Psi_{ij} \in M_{n\|S\|}(\mathbb{R})$, $\Psi_{ij} = \text{diag}_{\mathbf{s_k} \in \mathbf{S}}\left[\Phi_{ij}(\mathbf{s_k}) + \Theta_{ij}(\mathbf{s_k})\right]$, and $\Psi_0 \in M_{n\|S\|}(\mathbb{R}), \Psi_0 = \text{diag}(\varepsilon, \varepsilon, \ldots \varepsilon)$, then the regres-sion constraint becomes

$$\text{subject to} \quad \max \sigma \left[\Psi_0 + \sum_{i=1, j=1}^{n} \Psi_{ij} y_{(n+1)i+j}\right] \leq 0. \tag{4.32}$$

(4.31) and (4.32) together constitute a semidefinite program.

Solve for y - or equivalently - P **and** b

Solving the least square semidefinite program in (4.31) and (4.32) yields the network de-mand function, $q(x)$, that best fits the data, and guarantees that the conditions from Theorem 4.1 on $H(x)$ that guarantee stability robustness for all market structures are met.

4.4.3 Numerical Experiment

Consider 100 data points generated by the log-linear model,

$$\log q(x) = \begin{bmatrix} -0.57 & 0.10 & -0.12 \\ 0.20 & -1.00 & 0.11 \\ -0.02 & 0.06 & -0.68 \end{bmatrix} \log x + \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + w, \tag{4.33}$$
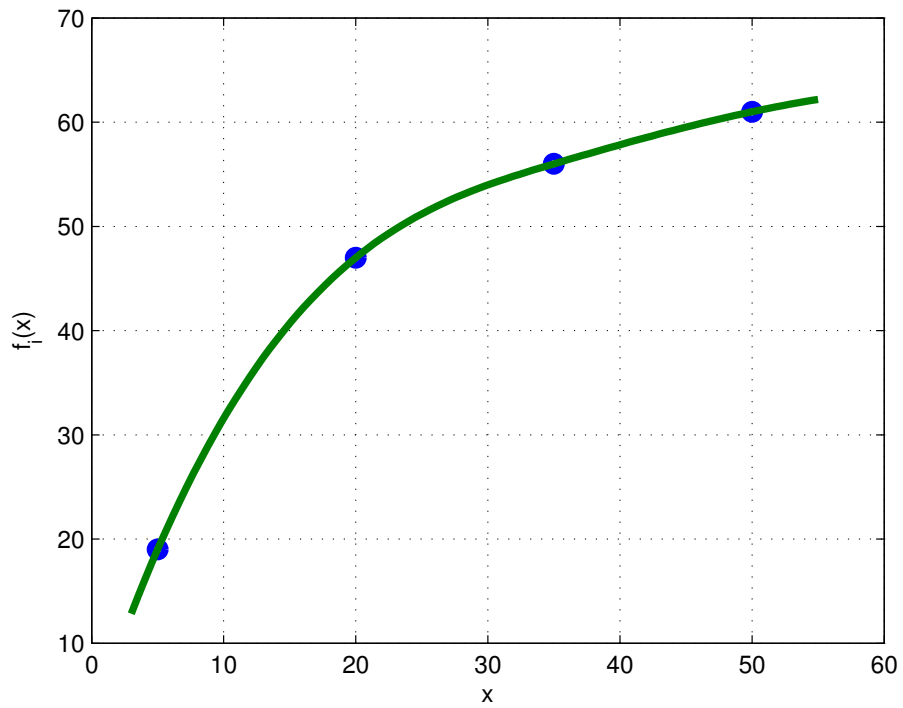
Figure 4.1: Plot showing price sensitivity: a spline going through points $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$.
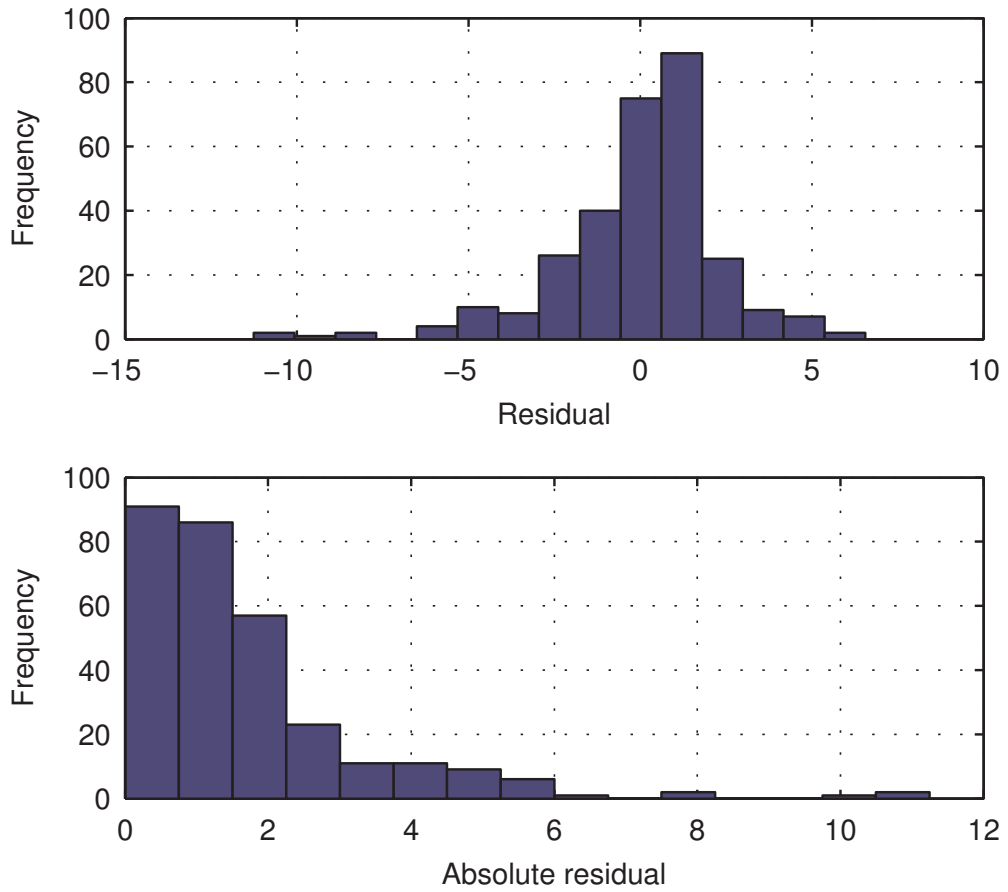
Figure 4.2: Plot showing the histogram of residuals. The top one is the histogram residual values, while the bottom shows a histogram of absolute residuals. These results indicate that the industrial organization network model fits the data extremely well.

where w is white noise with standard deviation 1. We choose $f_i(\cdot)$ to be the same function for each dimension: a spline going through $(5, 19)$, $(20, 47)$, $(35, 56)$, and $(50, 61)$ (we chose these points by looking at the generated data, and roughly estimating the effects of different price ranges on demand.) A plot showing this spline is shown in Figure 4.1.

Based on this spline, we perform a semidefinite regression to fit the demand function $q = Af(x) + b$ while meeting the robustness condition. The optimal parameters become:

$$q(x) = \begin{bmatrix} -5.70 & 0.96 & -1.23 \\ 1.96 & -10.00 & 1.17 \\ -0.24 & 0.59 & -6.82 \end{bmatrix} f(x) + \begin{bmatrix} 481.22 \\ 636.45 \\ 563.00 \end{bmatrix}. \tag{4.34}$$

These matrices do not look quite the same as the matrices in the original model because our regression model is not in logarithm scale. To see how our model fits the demand, we plot of percentage difference of demands between our regression model and the log-linear model in Figure 4.2. Since we have 100 data points, and each data point reflects the demand of three different products, we show in our plot the histogram of 300 differences, and the histogram of 300 absolute error. While the demanded quantities range between 150 and 400 units, the differences range between 0 and 12 units. For 90% of the data points, the difference is less than 1.5 percent. The maximal difference is about 3.5 percent. Our model fits the data quite well, but more importantly, it guarantees existence, uniqueness, and stability of equilibriums under all market structures.

Note also see that the complementary/substitutive relationships between different products are also preserved. In the log-linear model, we see that the pairs of products 1 and 2, and 2 and 3 are substitutes, while products 1 and 3 are complements. This is also reflected by the sign of elements of P.

Finally, we show how our demand model reflects own-price demand by plotting q_i with respect to x_i, while fixing both other two prices at 20. The shape looks quite realistic
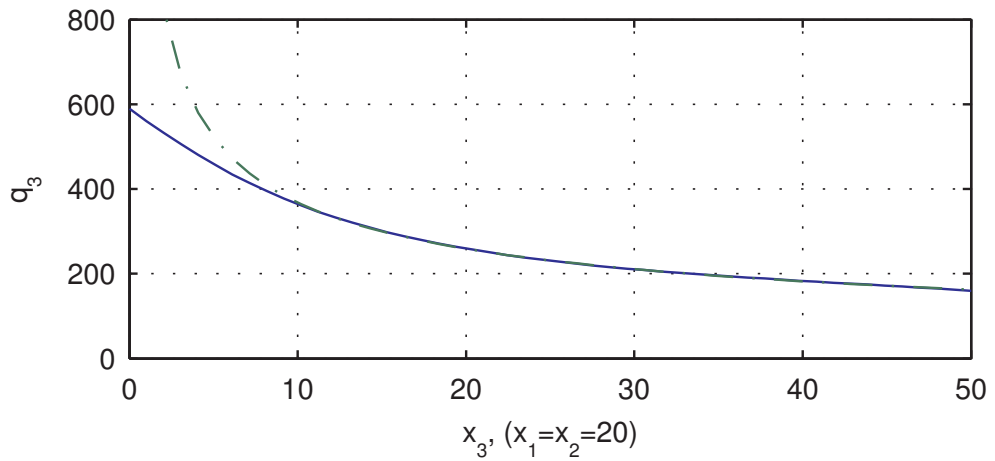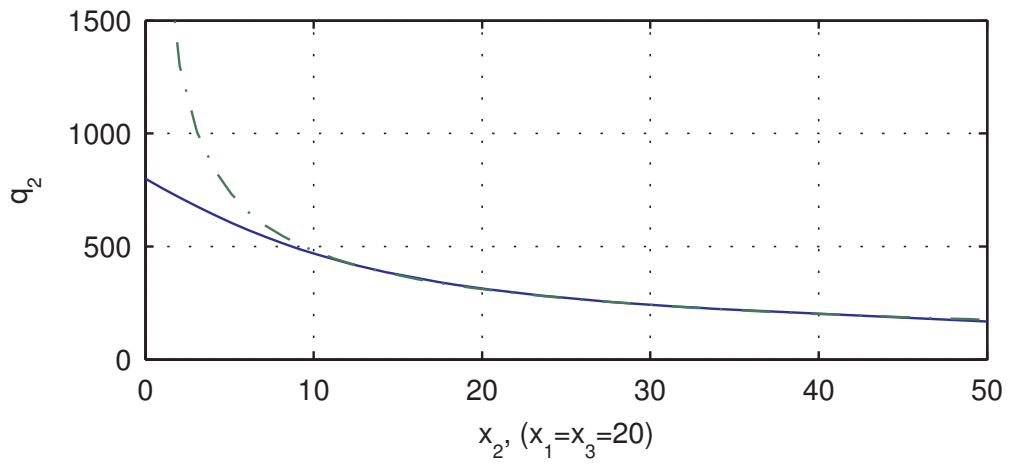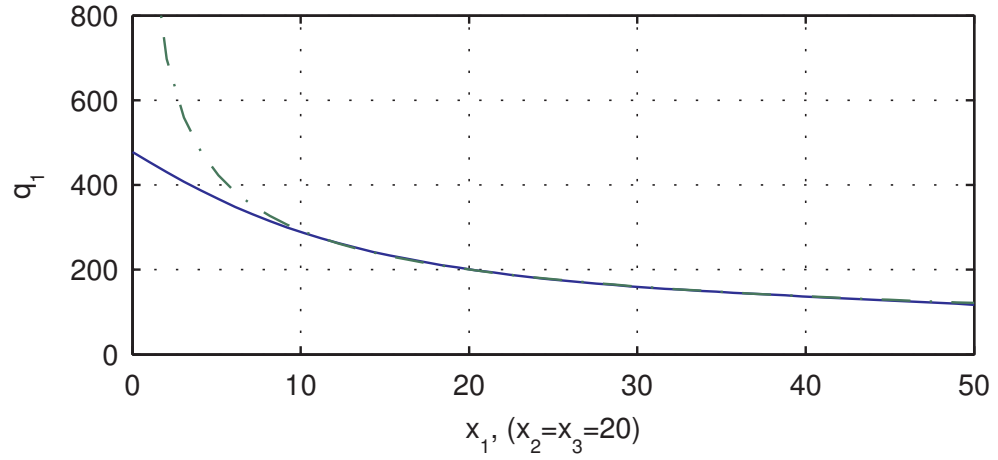
Figure 4.3: Demand plots of each product with respect to its own price, fixing the other two prices at 20. The solid lines plot our demand functions, and the dashed lines plot the loglinear demand functions.

(Figure 4.3), as it shows a decreasing function that gets flatter when price increases, reflecting the law of diminishing returns. These results suggest the method is quite practical.

We also show the log linear demand function in the same plot. The difference between our demand function and the log linear demand function is when price is close to 0, and due to nature of logarithm, log-linear demand increases exponentially fast.

4.5 Global Stability [2]

In this section, we will extend the result in Theorem 4.1 significantly by showing that the stability robustness condition will not only guarantee local stability, but in fact also yield global stability for all coalition structures.

Proposition 4.1. *A system given by*

$$\dot{x} = f(x) \tag{4.35}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally exponentially stable if

$$\exists \varepsilon > 0 \quad \forall x \in \mathbb{R}^n \quad \mathfrak{Re}W\left(\frac{\partial f}{\partial x}(x)\right) \leq -\varepsilon. \tag{4.36}$$

Proof. We will write $\|\cdot\|$ to denotes the L_2 norm. Notice that since we are working with real vectors as $x, f(x) \in \mathbb{R}^n$, for clarity we will use the transpose operator x^T in place of the conjugate transpose operator x^* as these are only different in a complex vector space. For convenience, we write x to mean $x(t)$, the system state vector at time t and x_0 to mean $x(0)$, the initial state vector of the system.

From Corollary 4.1, there exists x_e such that $f(x_e) = 0$. Let $V(x) = \|x - x_e\|^2 = (x - x_e)^T (x - x_e)$. We shall prove that $V(x)$ is exponentially decreasing to 0.

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[2]This result was not a part of [18].

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Taking the derivative

$$\dot{V}(x) = \frac{d}{dt}(x - x_e)^T(x - x_e) + (x - x_e)^T\frac{d}{dt}(x - x_e) = 2(x - x_e)^T f(x). \quad (4.37)$$

Let L be the line segment $y(s) = s(x - x_e) + x_e$ with $s \in [0, 1]$ connecting x_e and x. Thus, $dy = (x - x_e)ds$. Using the fundamental theorem of calculus for line integrals, we can compute $f(x)$ from $\frac{\partial f}{\partial x}(y(s))$ with $s \in [0, 1]$.

$$f(x) = f(x_e) + \int_L \frac{\partial f}{\partial x}(y)dy = \int_0^1 \frac{\partial f}{\partial x}(y(s))(x - x_e)ds. \quad (4.38)$$

Hence

$$\begin{aligned}
\dot{V}(x) &= 2\int_0^1 (x - x_e)^T\frac{\partial f}{\partial x}(y(s))(x - x_e)ds \\
&= 2\int_0^1 \frac{(x - x_e)^T}{\|x - x_e\|}\frac{\partial f}{\partial x}(y(s))\frac{x - x_e}{\|x - x_e\|}\|x - x_e\|^2 ds \\
&\leq 2\int_0^1 \max\Re\mathfrak{e}W\left(\frac{\partial f}{\partial x}(y(s))\right)\|x - x_e\|^2 ds \\
&\leq -2\int_0^1 \varepsilon\|x - x_e\|^2 ds = -2\left(\int_0^1 \varepsilon ds\right)V(x) = -2\varepsilon V(x). \quad (4.39)
\end{aligned}$$

Thus $\frac{\dot{V}(x)}{V(x)} \leq -2\varepsilon \Rightarrow \log\left(\frac{V(x)}{V(x_0)}\right) \leq -2\varepsilon t \Rightarrow V(x) \leq V(x_0)\exp(-2\varepsilon t)$. Therefore $x \to x_e$ when $t \to \infty$. $\qquad \square$

Applying the above proposition to the proof of Theorem 4.1, it is now proven that the coalition stability-robustness condition in (4.23) guarantees global stability for all coalition structures.

4.6 Conclusion

In this chapter we demonstrated stability robustness conditions with respect to coalition structure for a class of profit-maximizing nonlinear systems. These conditions were then

leveraged to provide a systematic methodology for estimating a rich variety of demand systems from data that guarantee sensible stability results regardless of the structure of cooperation within the marketplace.

The importance of these results emerges from the ability for regulators and managers alike to reliably conduct market power analyses using merger simulation and reverse merger simulation techniques. In such studies one can compute, for example, the value of cooperation of a firm as a measure of its market power.

Chapter 5

Conclusion and Future Work

5.1 Summary

In this thesis we laid a foundation towards a theory of coalition stability and robustness for multiagent systems. After showing some practical applications that motivated and stemmed the theory, we built up our results based on ideas from control theory and cooperative game theory to lead to a stability robustness condition. We then applied this condition to form a methodology to evaluate cooperation for market structure analysis.

Our first practical application was to quantify the value of cooperation. Quantifying the value of cooperation is a first step in understanding how firms exert market power in their respective environments. This information is important for both managers, who hope to leverage the information to better lead their organizations, and regulators, who want to monitor the impact of corporate decisions on social welfare.

We have illustrated the use of the value of cooperation for product clustering in the context of optimal organizational design as our second practical application. We proposed to do this through the use of the value of cooperation within a hierarchical agglomerative clustering framework.VC-based hierarchical agglomerative clustering starts from the reference structure, where all products were assumed to act independently. Then, the two products which exhibit the strongest value of cooperation were merged, and the process is repeated, decreasing the total number of clusters by one until all of the firms products are finally merged into a single organizational structure. The hierarchical nesting added a nat-

ural constraint to the problem, which yields a product hierarchy with a clear organizational interpretation.

In both of these applications, we relied on equilibrium results, which in turn required a theoretical understanding of stability of the systems under various coalitional structures. Using ideas from control theory and cooperative game theory, we demonstrated stability robustness conditions with respect to coalition structure for a class of profit-maximizing nonlinear systems. These conditions were then leveraged to provide a systematic methodology for estimating a rich variety of demand systems from data that guarantee sensible stability results regardless of the structure of cooperation within the marketplace.

Finally we added an extension to the theory, by strengthening the stability into global exponential stability, a much stronger type of stability without adding additional constraint to our coalition robustness condition. Global stability is important because it guarantees that the system will end up at its unique equilibrium state without regard to its initial condition, which gives the equilibrium state more meaning and authority as a reference state to represent the system itself.

5.2   Future Work

Discrete systems can be viewed as approximations of continuous system where the state is asummed constant in each time interval. On the other hand, continuous systems can be considered as the limits of discrete systems when the unit interval shrink to infinitesimal length. Many stability results, for example the Lyapunov indirect methods, apply both for discrete and continous systems. Due to this duality between discrete and continuous systems, we expect our theory to apply to discrete systems as well. In order to do this, we will need to formulate our framework in a discrete way: at each point in time each agent can choose the best actions given the current state of the system, and move accordingly. These types of system may be more realistic than continuous systems in applications where the outcomes are clear enough for decision makers to choose the optimal move each time.

We are also interested in leveraging the coalition robustness condition to design multiagents systems that benefits from having dynamic coalitions between agents. For example, we may be interested UAVs that may work in teams when they are within range of each others and change team structures when they are out of range. Another example would be distributed optimization systems that decompose the objective functions into parts and distribute the parts to coalitions of machines within the same local network to optimize locally. With a carefully design of the objective function we can guarantee stability regardless of what coalitions are formed. That would in turn result in robust and flexible multiagent systems.

Finally, we also want to check these conditions in real firms and markets. We would like to know whether our coalition robustness condition is realistic; whether real application meet the condition; and in case they do not meet the condition, what would the implication be? A practical implementation of these theoretical results will play an important role in our future study to answer these questions.

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