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Analytic solution for the space-time fractional Klein-Gordon and coupled conformable Boussinesq equations

Muhannad A. Shallal [a], Hawraz N. Jabbar [a], Khalid K. Ali [b,*]

[a] Department of Mathematics, College of Science, University of Kirkuk, Kirkuk, Iraq
[b] Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City, Cairo, Egypt

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ABSTRACT

In this paper, we constructed a travelling wave solution for space-time fractional nonlinear partial differential equations by using the modified extended Tanh method with Riccati equation. The method is used to obtain analytic solutions for the space-time fractional Klein-Gordon and coupled conformable space-time fractional Boussinesq equations. The fractional complex transforms and the properties of modified Riemann-Liouville derivative have been used to convert these equations into nonlinear ordinary differential equations.

Introduction

The study of solutions of nonlinear partial differential equations is very important in the understanding of many physical phenomena in many science and engineering applications. In light of this, several methods have been employed to treat varieties of such problems among which Kudryashov method [1,2], the modified extended tanh method [3] and some other analytical and numerical methods among others, see [4–24].

The Klein-Gordon equation appears in different real world applications, such as the quantum field theory, nonlinear optics and solid state physics. Fractional Klein-Gordon equation has been studied by many researchers for instance, homotopy perturbation method by Baleanu et al [25,26], and approximate analytical solution for linear and nonlinear time fractional order Klein-Gordon equations by Tamsir and Srivastava [27]. The coupled Boussinesq equations are modeling for two way propagation of surface waves in a uniform horizontal channel [28]. Number of studies have been introduced to solve coupled Boussinesq equations for example, the expansion method [29], and new transformation and new approach [30,31]. This paper is organized as follows: In Section "Description of the fractional calculus", the description of the fractional calculus is demonstrated. In Section "Analysis of the method", analysis of the method is given to illustrate how fractional differential equations are converted into integer-order differential equations. In Section "Application", the application modified extended Tanh method is used to obtain the analytic solutions for the space-time fractional Klein-Gordon and coupled conformable space-time fractional Boussinesq equations. Section "Co nclusion" conclude the paper.

Description of the fractional calculus

The Jumarie's modified Riemann-Liouville derivative of a continuous (not necessarily differentiable) function $u(t)$ of order α is defined as follows [32]:

$$D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^{\infty}(t-\xi)^{-\alpha}(u(\xi)-u(0))d\xi, \quad 0<\alpha<1, \tag{1}$$

where $\Gamma(.)$ is the well-known gamma function. Some other properties include:

i). $D_t^{\alpha}t^c = \frac{\Gamma(1+c)}{\Gamma(1+c-\alpha)}t^{c-\alpha}$,
ii). $D_t^{\alpha}(au(t)+bv(t)) = aD_t^{\alpha}u(t)+bD_t^{\alpha}v(t)$, where a and b are constants,
iii). $D_t^{\alpha}u(\xi) = \sigma\frac{du}{d\xi}D_t^{\alpha}(\xi)$,

where σ is fractional indice, see [33].

* Corresponding author.
E-mail address: khalidkaram2012@azhar.edu.eg (K.K. Ali).

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Analysis of the method

We present the modified extended tanh expansion method by considering the following nonlinear fractional differential equation of the form:

$$G\left(u, D_t^{\alpha}u, D_x^{\alpha}u, D_{tt}^{2\alpha}u, D_{xx}^{2\alpha}u, D_t^{\alpha}D_x^{\alpha}u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{2}$$

where α is order of the derivative of the function $u = u(x,t)$. Also, we use the wave transformation

$$u(x,t) = U(\xi), \quad \xi = k\frac{x^{\alpha}}{\Gamma(\alpha+1)} - c\frac{t^{\alpha}}{\Gamma(\alpha+1)} - x_0, \tag{3}$$

where k and c are nonzero constants. Substitution of wave transformation (3) into (2), we obtain an ordinary differential equation of the form

$$P(U, U', U'', \ldots) = 0, \tag{4}$$

where, $'$ is a derivative w.r.t ξ. Further, the solution is assumed to be of the finite series of the form:

$$U(\xi) = a_0 + \sum_{n=1}^{n=N}\left(a_n\Phi^n(\xi) + \frac{b_n}{\Phi^n(\xi)}\right), \tag{5}$$

where $a_0, a_n, b_n, n = 1, 2, \ldots N$ are nonzero constants to be computed; where N is a positive integer determined by balancing the highest order derivative with the highest nonlinear terms in the equation, and $\Phi(\xi)$ satisfies the Riccati differential equation:

$$\Phi'(\xi) = d + \Phi^2(\xi), \tag{6}$$

where d is a constant. Further, the Riccati differential equation in (6) has solutions of the form:

(i) if $d < 0$, then

$$\Phi(\xi) = -\sqrt{-d}\tanh(\sqrt{-d}\xi),$$
$$\Phi(\xi) = -\sqrt{-d}\coth(\sqrt{-d}\xi),$$

(ii) if $d = 0$, then

$$\Phi(\xi) = -\frac{1}{\xi},$$

(iii) if $d > 0$, then

$$\Phi(\xi) = \sqrt{d}\tanh(\sqrt{d}\xi),$$
$$\Phi(\xi) = -\sqrt{d}\coth(\sqrt{d}\xi).$$

Substituting Eq. (5) and its necessary derivatives into (4) gives a polynomial in $\Phi(\xi)$. Collecting coefficients of the obtained polynomials and subsequently setting each one to zero, we will get a set of over-determined algebraic equations for $a_0, a_n, b_n (n = 1, 2, \ldots)$, and b with the aid of symbolic computation using Mathematica. Finally, solving the algebraic equations and the above possible solutions of Raccati equation into (6), we obtain the solution of Eq. (2).

Application

To test the efficiency of the method, the analytic solutions of the space-time fractional Klein-Gordon and coupled conformable space-time fractional Boussinesq equations are organized.

The space-time fractional Klein-Gordon equation

Consider the Klein-Gordon equation with space-time fractional derivatives of the form:

$$u_{tt}^{2\alpha} - u_{xx}^{2\alpha} - au - \mu u^3 = 0. \tag{7}$$

On using the wave transformation

$$u(x,t) = U(\xi), \quad \xi = k\frac{x^{\alpha}}{\Gamma(\alpha+1)} - c\frac{t^{\alpha}}{\Gamma(\alpha+1)} - x_0, \tag{8}$$

we get a reduced ordinary differential equation as follows

$$(k^2 - c^2)U'' + aU + \mu U^3 = 0. \tag{9}$$

Balancing the highest order derivative with the highest nonlinear order $[U''; (U)^3]$ in Eq. (9), we get $N = 1$.

And it offers a truncated series from Eq. (5) as:

$$U(\xi) = a_0 + a_1\Phi(\xi) + b_1\Phi^{-1}(\xi). \tag{10}$$

Then, substituting Eq. (10) and its necessary derivatives together with Eq. (6) into Eq. (9); collecting the coefficients of same degree of $\Phi(\xi)$ and thereafter setting them to zero, we get the following algebraic equations:

$$a_0 + \mu a_0^3 + 6\mu a_0 a_1 b_1 = 0,$$

$$2(k^2 - c^2)a_1 d + aa_1 + 3\mu a_0^2 a_1 + 3\mu a_1^2 b_1 = 0, \quad 3\mu a_0 a_1^2 = 0,$$

$$2(k^2 - c^2)a_1 + \mu a_1^3 = 0, \quad 2db_1(k^2 - c^2) + ab_1 + 3\mu a_0^2 b_1 + 3\mu a_1 b_1^2 = 0,$$

$$3\mu a_0 b_1^2 = 0, \quad 2d^2 b_1(k^2 - c^2) + \mu b_1^3 = 0.$$

Solving the above system, we get the following:

Case 1.

$$a_0 = 0,$$
$$a_1 = \mp\frac{\sqrt{2}\sqrt{c^2 - k^2}}{\sqrt{\mu}},$$
$$b_1 = \mp\frac{a}{2\sqrt{2\mu(c^2 - k^2)}},$$
$$d = \frac{a}{8(c^2 - k^2)}.$$

Which produces

$$u_1(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2 - k^2}}{\sqrt{\mu}}\sqrt{d}\tan(\sqrt{d}\xi) \pm \frac{a}{2\sqrt{d}\sqrt{2\mu(c^2 - k^2)}}\cot(\sqrt{d}\xi), \; d > 0,$$

$$u_2(x,t) = \mp\frac{\sqrt{2}\sqrt{c^2 - k^2}}{\sqrt{\mu}}\sqrt{d}\cot(\sqrt{d}\xi) \mp \frac{a}{2\sqrt{d}\sqrt{2\mu(c^2 - k^2)}}\tan(\sqrt{d}\xi), \; d > 0,$$

$$u_3(x,t) = \mp\frac{\sqrt{2}\sqrt{c^2 - k^2}}{\sqrt{\mu}}\sqrt{-d}\tanh(\sqrt{-d}\xi) \mp \frac{a}{2\sqrt{-d}\sqrt{2\mu(c^2 - k^2)}}\coth(\sqrt{-d}\xi), \; d < 0,$$

$$u_4(x,t) = \mp\frac{\sqrt{2}\sqrt{c^2 - k^2}}{\sqrt{\mu}}\sqrt{-d}\coth(\sqrt{-d}\xi) \mp \frac{a}{2\sqrt{-d}\sqrt{2\mu(c^2 - k^2)}}\tanh(\sqrt{-d}\xi), \; d < 0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 2.

$$a_0 = 0,$$
$$a_1 = \mp\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}},$$
$$b_1 = \mp\frac{a}{2\sqrt{2\mu(c^2-k^2)}},$$
$$d = -\frac{a}{4(c^2-k^2)}.$$

Which produces

$$u_5(x,t) = \mp\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{d}\tan(\sqrt{d}\xi) \mp \frac{a}{2\sqrt{d}\sqrt{2\mu(c^2-k^2)}}\cot(\sqrt{d}\xi),\ d>0,$$
$$u_6(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{d}\cot(\sqrt{d}\xi) \pm \frac{a}{2\sqrt{d}\sqrt{2\mu(c^2-k^2)}}\tan(\sqrt{d}\xi),\ d>0,$$
$$u_7(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{-d}\tanh(\sqrt{-d}\xi) \pm \frac{a}{2\sqrt{-d}\sqrt{2\mu(c^2-k^2)}}\coth(\sqrt{-d}\xi),\ d<0,$$
$$u_8(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{-d}\coth(\sqrt{-d}\xi) \pm \frac{a}{2\sqrt{-d}\sqrt{2\mu(c^2-k^2)}}\tanh(\sqrt{-d}\xi),\ d<0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 3.

$$a_0 = a_1 = 0,$$
$$b_1 = \mp\frac{a}{\sqrt{2\mu(c^2-k^2)}},$$
$$d = \frac{a}{2(c^2-k^2)}.$$

Which produces

$$u_9(x,t) = \pm\frac{a}{\sqrt{d}\sqrt{2\mu(c^2-k^2)}}\cot(\sqrt{d}\xi),\ d>0,$$
$$u_{10}(x,t) = \mp\frac{a}{\sqrt{d}\sqrt{2\mu(c^2-k^2)}}\tan(\sqrt{d}\xi),\ d>0,$$
$$u_{11}(x,t) = \mp\frac{a}{\sqrt{-d}\sqrt{2\mu(c^2-k^2)}}\coth(\sqrt{-d}\xi),\ d<0,$$
$$u_{12}(x,t) = \mp\frac{a}{\sqrt{-d}\sqrt{2\mu(c^2-k^2)}}\tanh(\sqrt{-d}\xi),\ d<0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 4.

$$a_0 = b_1 = 0,$$
$$a_1 = \mp\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}},$$
$$d = \frac{a}{2(c^2-k^2)}.$$

Which produces

$$u_5(x,t) = \mp\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{d}\tan(\sqrt{d}\xi),\ d>0,$$
$$u_6(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{d}\cot(\sqrt{d}\xi),\ d>0,$$
$$u_7(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{-d}\tanh(\sqrt{-d}\xi),\ d<0,$$
$$u_8(x,t) = \pm\frac{\sqrt{2}\sqrt{c^2-k^2}}{\sqrt{\mu}}\sqrt{-d}\coth(\sqrt{-d}\xi),\ d<0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Thus, we have obtained several new wave solutions. Comparing our results with the results presented in [34,35] shows that our solutions are different and novel. Now, plotting these solutions at different time levels and different values of, shows the motion of solitary waves as shown in Fig. 1.

Coupled conformable space-time fractional Boussinesq equations

Consider the coupled conformable space-time fractional Boussinesq equations of the form:

$$u_t + v_x = 0. \tag{11}$$

$$v_t^\alpha + \lambda(u^2)_x - \mu u_{xxx}^{3\alpha} = 0. \tag{12}$$

On using the wave transformation

$$u(x,t) = U(\xi),\ v(x,t) = V(\xi), \quad \xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0, \tag{13}$$

we get a reduced ordinary differential equation as follows

$$-cU' + kV' = 0,$$
$$-cV' + \lambda(U^2)' - \mu k^3 U''' = 0.$$

Integrating the above equations once, and assuming the constant of integration zero, we get

$$-cU + kV = 0, \tag{14}$$

$$-cV + \lambda(U^2) - \mu k^3 U'' = 0. \tag{15}$$

Using Eq. (14) and (15) yields:

$$-c^2U + \lambda K(U^2) - \mu k^4 U'' = 0. \tag{16}$$

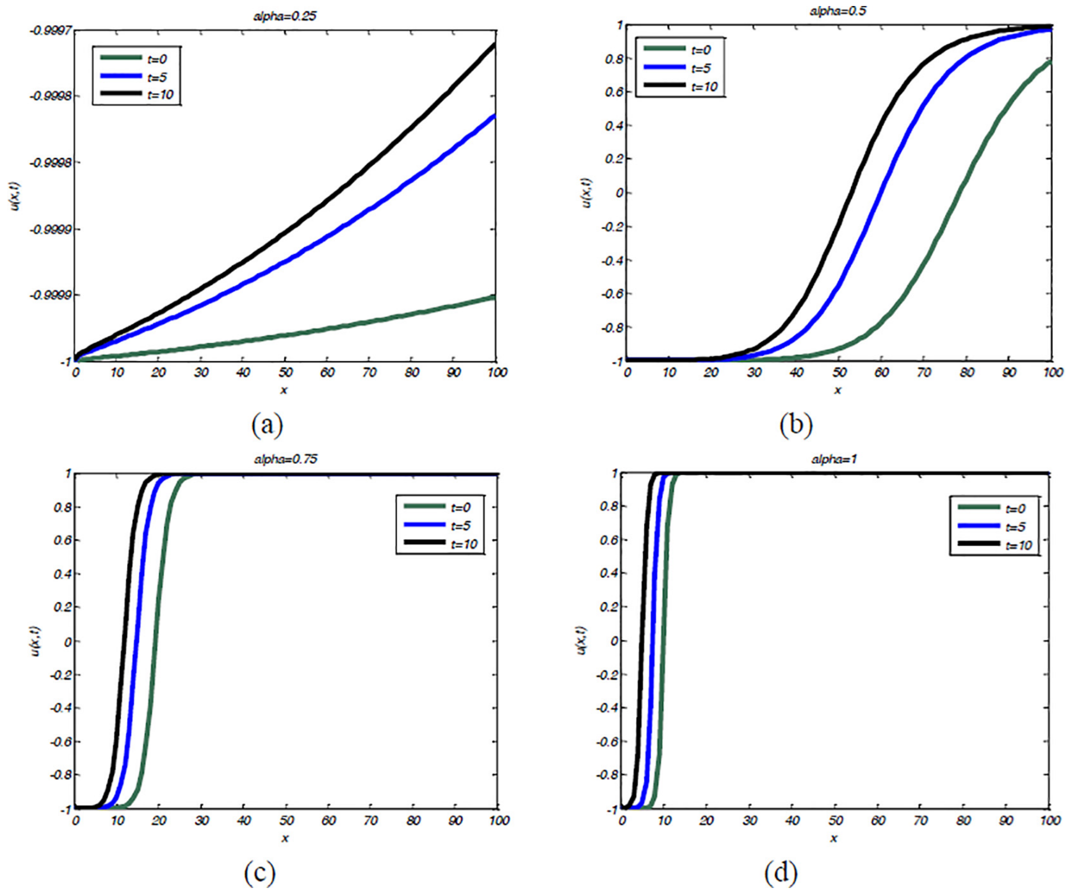

Fig. 1. The exact solutions for the space-time fractional Klein-Gordon equation with, subsituting the values $c = -1/2, k = 1, a = 1, \mu = -1, 0 \leqslant x \leq 100$, and $\alpha = 0.25, 0.5, 0.75, 1$ at different time levels.

Balancing the highest order derivative with the highest nonlinear order $[U''; (U)^2]$ in Eq. (16), we get $N = 2$.

And it offers a truncated series from Eq. (5) as:

$$U(\xi) = a_0 + a_1\Phi(\xi) + a_1\Phi^2(\xi) + b_1\Phi^{-1}(\xi) + b_1\Phi^{-2}(\xi). \tag{17}$$

Then, substituting Eq. (17) and its necessary derivatives together with Eq. (6) into Eq. (16); collecting the coefficients of same degree of $\Phi(\xi)$ and thereafter setting them to zero, we get the following algebraic equations:

$$c^2a_0 - 2a_1b_1k\lambda - 2a_2b_2k\lambda - a_0^2k\lambda + 2\mu a_2d^2k^4 + 2\mu b_2k^4 = 0,$$

$$c^2a_1 - 2a_0a_1k\lambda - 2a_2b_1k\lambda + 2\mu a_1dk^4 = 0,$$

$$c^2a_2 - a_1^2k\lambda - 2a_0a_2k\lambda + 8\mu a_2dk^4 = 0,$$

$$2a_1a_2k\lambda - 2\mu a_1k^4 = 0,$$

$$a_2^2k\lambda - 6\mu a_2k^4 = 0,$$

$$c^2b_1 - 2a_0b_1k\lambda - 2b_2a_1k\lambda + 2\mu b_1dk^4 = 0,$$

$$c^2b_2 - b_1^2k\lambda - 2a_0b_2k\lambda + 8\mu b_2dk^4 = 0,$$

$$2b_1b_2k\lambda - 2\mu b_1d^2k^4 = 0,$$

$$b_2^2k\lambda - 6\mu b_2d^2k^4 = 0,$$

Solving the above system, we get the following:

Case 1.

$$a_1 = b_1 = b_2 = 0,$$
$$a_0 = -\frac{c^2}{2k\lambda},$$
$$a_2 = \frac{6\mu k^3}{\lambda},$$
$$d = -\frac{c^2}{4k^4\mu}.$$

Which produces

$$u_1(x,t) = -\frac{c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda}d\tanh^2(\sqrt{-d}\xi),\ d < 0,$$

$$u_2(x,t) = -\frac{c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda}d\coth^2(\sqrt{-d}\xi),\ d < 0,$$

$$v_1(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda}d\tanh^2(\sqrt{-d}\xi)\right),\ d < 0,$$

$$v_2(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda}d\coth^2(\sqrt{-d}\xi)\right),\ d < 0,$$

$$u_3(x,t) = -\frac{c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda}d\tan^2(\sqrt{d}\xi),\ d > 0,$$

$$u_4(x,t) = -\frac{c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda}d\cot^2(\sqrt{d}\xi),\ d > 0,$$

$$v_3(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda}d\tan^2(\sqrt{d}\xi)\right),\ d > 0,$$

$$v_4(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda}d\cot^2(\sqrt{d}\xi)\right),\ d > 0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 2.

$$a_1 = b_1 = b_2 = 0,$$

$$a_0 = \frac{3c^2}{2k\lambda},$$

$$a_2 = \frac{6\mu k^3}{\lambda},$$

$$d = \frac{c^2}{4k^4\mu}.$$

Which produces

$$u_5(x,t) = \frac{3c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda} d\tanh^2(\sqrt{-d}\xi),\ d<0,$$

$$u_6(x,t) = \frac{3c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda} d\coth^2(\sqrt{-d}\xi),\ d<0,$$

$$v_5(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda} d\tanh^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$v_6(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} - \frac{6\mu k^3}{\lambda} d\coth^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$u_7(x,t) = \frac{3c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda} d\tan^2(\sqrt{d}\xi),\ d>0,$$

$$u_8(x,t) = \frac{3c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda} d\cot^2(\sqrt{d}\xi),\ d>0,$$

$$v_7(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda} d\tan^2(\sqrt{d}\xi)\right),\ d>0,$$

$$v_8(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} + \frac{6\mu k^3}{\lambda} d\cot^2(\sqrt{d}\xi)\right),\ d>0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 3.

$$a_1 = b_1 = a_2 = 0,$$

$$a_0 = -\frac{c^2}{2k\lambda},$$

$$b_2 = \frac{4c^4}{8\mu k^5\lambda},$$

$$d = -\frac{c^2}{4k^4\mu}.$$

Which produces

$$u_9(x,t) = -\frac{c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\tanh^2(\sqrt{-d}\xi),\ d<0,$$

$$u_{10}(x,t) = -\frac{c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\coth^2(\sqrt{-d}\xi),\ d<0,$$

$$v_9(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\tanh^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$v_{10}(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\coth^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$u_{11}(x,t) = -\frac{c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\tan^2(\sqrt{d}\xi),\ d>0,$$

$$u_{12}(x,t) = -\frac{c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\cot^2(\sqrt{d}\xi),\ d>0,$$

$$v_{11}(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\tan^2(\sqrt{d}\xi)\right),\ d>0,$$

$$v_{12}(x,t) = \frac{c}{k}\left(-\frac{c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\cot^2(\sqrt{d}\xi)\right),\ d>0,$$

where

$$\xi = k\frac{x^\alpha}{\Gamma(\alpha+1)} - c\frac{t^\alpha}{\Gamma(\alpha+1)} - x_0.$$

Case 4.

$$a_1 = b_1 = a_2 = 0,$$

$$a_0 = \frac{3c^2}{2k\lambda},$$

$$b_2 = \frac{4c^4}{8\mu k^5\lambda},$$

$$d = \frac{c^2}{4k^4\mu}.$$

Which produces

$$u_{13}(x,t) = \frac{3c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\tanh^2(\sqrt{-d}\xi),\ d<0,$$

$$u_{14}(x,t) = \frac{3c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\coth^2(\sqrt{-d}\xi),\ d<0,$$

$$v_{13}(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\tanh^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$v_{14}(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} - \frac{4c^4}{8\mu k^5\lambda} d\coth^2(\sqrt{-d}\xi)\right),\ d<0,$$

$$u_{15}(x,t) = \frac{3c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\tan^2(\sqrt{d}\xi),\ d>0,$$

$$u_{16}(x,t) = \frac{3c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\cot^2(\sqrt{d}\xi),\ d>0,$$

$$v_{15}(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\tan^2(\sqrt{d}\xi)\right),\ d>0,$$

$$v_{16}(x,t) = \frac{c}{k}\left(\frac{3c^2}{2k\lambda} + \frac{4c^4}{8\mu k^5\lambda} d\cot^2(\sqrt{d}\xi)\right),\ d>0,$$

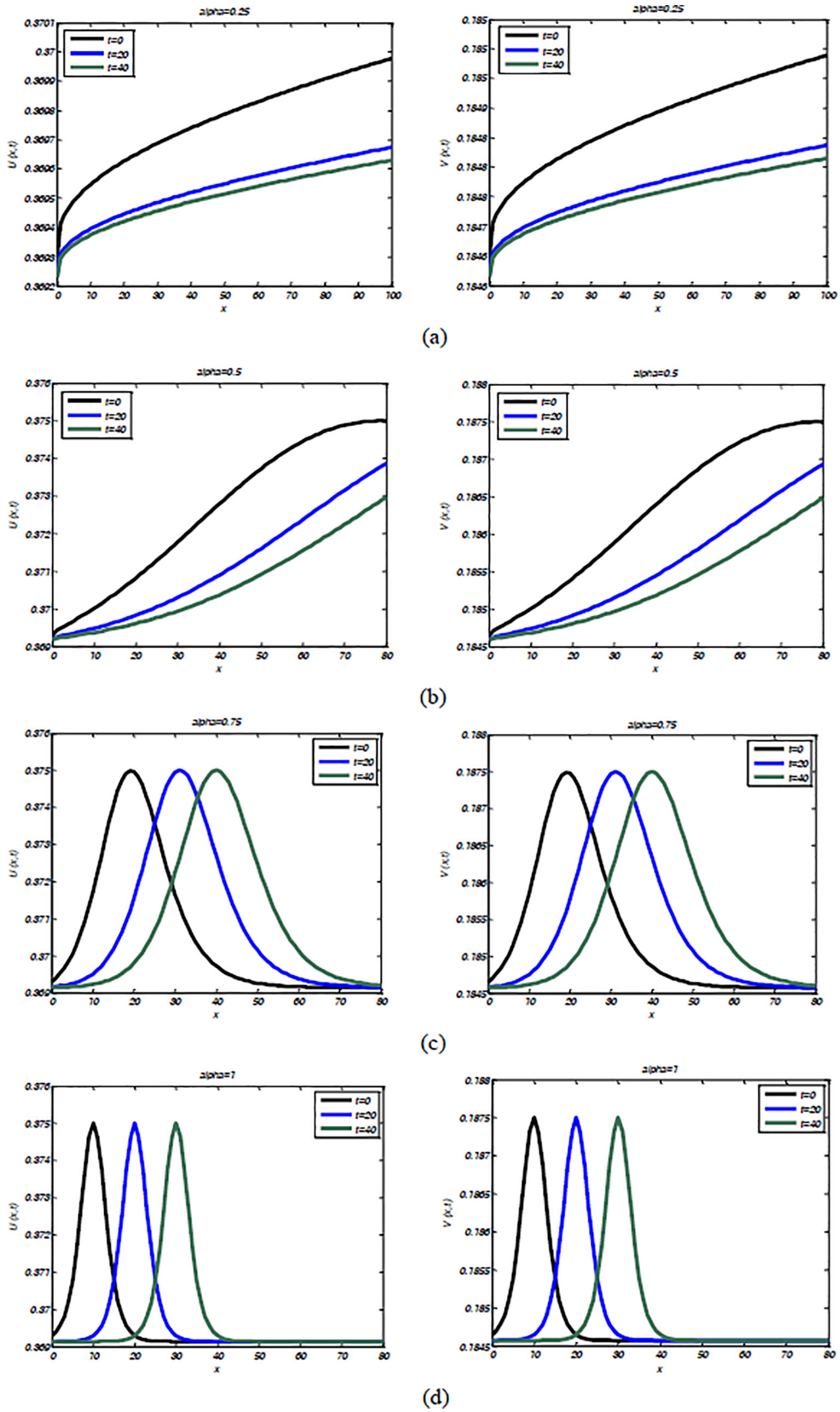

Fig. 2. The exact solutions $u(x,t), v(x,t)$ for coupled conformable space-time fractional Boussinesq equations with, subsituting the values $c = 1/2, k = 1, \lambda = 1, \mu = -1$, $x_0 = 10, 0 \leqslant x \leq 100$, and $\alpha = 0.25, 0.5, 0.75, 1$ at different time levels.

where

$$\xi = k\frac{x^{\alpha}}{\Gamma(\alpha+1)} - c\frac{t^{\alpha}}{\Gamma(\alpha+1)} - x_0.$$

Thus, we have obtained several new wave solutions. Comparing our results with the results presented in [29] shows that our solutions are different and novel. Now, plotting these solutions at different time levels and different values of, shows the motion of solitary waves as shown in Fig. 2.

Conclusion

In this paper, modified extended Tanh method with Riccati equation has been successfully applied to find analytic solutions of the space-time fractional Klein-Gordon and coupled conformable space-time fractional Boussinesq equations. The results showed that the proposed method is a powerful and an efficient method. The method is simple and concise. Therefore it's applicable to solve other linear and nonlinear fractional partial differential equations in engineering and mathematical physics.

Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttps://doi.org/10.1016/j.rinp.2017.12.051.

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