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Interaction of the magnetic quadrupole moment of a non-relativistic particle with an electric field in a rotating frame

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A B S T R A C T

We study the interaction of magnetic quadrupole moment of neutral particle systems (such as atoms or molecules) with a radial electric field for non-relativistic particles in a rotating frame that tends to a uniform effective magnetic field perpendicular to the plane of motion of the neutral particle. We solve the corresponding Schrödinger equation and obtain the eigenfunctions, in terms of Heun polynomials, and the energy levels of the field for that system.

1. Introduction

It is well known that in general relativity, topological defects, such as monopoles, domain walls or cosmic strings, produce a curved space–time which leads to a shift of the energy levels and a change in the wave function of the physical system. Some quantum field theories may comprise stable field configurations with linear defects. An example of such defect lines is the so-called 'cosmic strings' [1–3]. The cosmic strings are spatial lines with trapped energy density, analogous to vortex lines in superfluids and superconductors, as well as line defects in crystals [4]. In fact, it is an interesting outcome of theoretical physics that relativistic constructs of cosmology and general relativity, such as cosmic strings, can be applied in condensed matter physics, for instance, in the description of defects in solids or the application of relativistic equations to various aspects of graphene. Let us mention some examples. The authors of Ref. [5] discuss the Aharonov–Bohm effect

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which persists in the ray-optics limit, and the possibilities for producing different homogeneous effective fields by exploiting the conical shape. In Ref. [6] the geometry of topological defects is utilized to describe a disclination in a graphene layer and the massless Dirac equation in this curved background describes the electrons therein. A geometric approach helped the study of geometric phases in graphitic cones in Ref. [7]. In Ref. [8], the conical geometry of some topological insulators was seen to induce electric polarization determined by the cone's aperture angle. It was concluded in Ref. [9] that the geometry associated to topological defects has a role on the electronic properties of graphene which is central for future investigations of quantum computation in such systems. In Ref. [10], the Kaluza–Klein approach was used to describe topological defects in a graphene layer, and a geometrical model was proposed to discuss the quantum flux in K-spin subspace. The authors of Ref. [11] used a geometric theory to describe a rotating fullerene molecule as a two-dimensional spherical space in a rotating frame with topological defects submitted to a non-Abelian gauge field. The influence of defects was investigated from many perspectives, such as the scattering of particles [12–14], or the interaction of the harmonic oscillator with topological defects [15,16]. We note also that examples of calculation of Landau levels in the presence of topological defects are described in Refs. [17–19]. An example of the effects of a non-trivial topology of space–time, in the gravitation context, is that the energy levels of an atom placed in a gravitational field will be shifted as a result of the interaction of the atom with space–time curvature Ref. [20]. Therefore, we have to consider the topology of the space–time in order to completely describe the physics of system.

In this paper, we examine the interaction between the magnetic quadrupole moment of a neutral particle system (such as atoms or molecules) with a radial electric field for non-relativistic particles by solving the corresponding Schrödinger equation. Investigations of the interaction between quantum fields and curved space–time include, for instance, one-electron atoms [20–22], similarities between topological defects in space–time with defects in solid continua called distortions, which can be classified as dislocations and disclinations [23], the description of dislocations and disclinations of solids in the framework of three-dimensional gravity [24] and the bound states of electrons and holes to such disclinations [25], and studies in crystals and condensed matter physics [26–32]. The authors of Ref. [33] showed that in a chiral conical space–time, the wave functions, energy spectra, and scattering amplitudes associated with a quantum scalar particle depend on the global features of this space–time. Ref. [34] considers the Landau levels in the non-relativistic dynamics of a neutral particle with a permanent magnetic dipole moment which interacts with an external electric field in the curved space–time background with or without a torsion field. The quantum scattering of an electron by a topological defect called dispiration, with an externally applied magnetic field along its axis, was examined in Ref. [35]. For an investigation of the Landau levels within the relativistic dynamics of a neutral particle with a permanent magnetic dipole moment interacting with an external electric field in the curved space–time background with a torsion field, see Ref. [36].

In general, the interest of particle systems with multipole moments stems from quantum effects such as geometric quantum phases [37]. Of particular interest are recent investigations of atoms with magnetic quadrupole moment; for instance, many quantum effects such as the scalar Aharonov–Bohm effect, the dual of the Aharonov–Bohm effect, the Aharonov–Casher effect, or the He–McKellar–Wilkens effect are associated with systems of particles that possess multipole moments [38]. In Ref. [39], the effects of rotation on a neutral particles system where Landau-type quantization stems from the interaction of the magnetic quadrupole moment of a neutral atom or molecule with external fields, are such that rotating effects can modify the cyclotron frequency and breaks the degeneracy of the analogue of the Landau levels. The same authors also observed that the energy spectrum of an atom with magnetic quadrupole moment is modified, in contrast to the Landau-type levels, and there is a restriction on the possible values of the cyclotron frequency determined by the rotation and scalar potential proportional to the inverse of the radial distance [40]; that analysis of an atom with a magnetic quadrupole moment in the presence of a time-dependent magnetic field shows that the time-dependent magnetic field induces an electric field that interacts with the atom's magnetic quadrupole moment and gives rise to a Landau-type quantization and that a time-independent Schrödinger equation can be obtained (without the interaction between the magnetic quadrupole moment of the atom and the time-dependent magnetic field) and can be solved exactly [41]; and the interaction between the magnetic quadrupole moment and an electric field is similar to the Coulomb potential and, by confining this atom to harmonic and linear confining potentials, a quantum effect characterized by the dependence of the angular frequency on the quantum numbers of the system is obtained, and it is shown that the possible values of the angular frequency associated with the ground state of the system are determined by a third-degree algebraic equation [42].

In the present work, we consider a non-relativistic neutral particle with a magnetic quadrupole moment in a region that possesses a uniform effective magnetic field. We analyse the effects of rotation and a static scalar potential with the Schrödinger equation. Recent investigations of the Schrödinger equation in a curved space–time include Refs. [43–49]. We analyse the influence of the topological defect on the equation of motion, the energy spectrum and the wave-function. In Section 2, we analyse the effects of rotation and a scalar potential on the system and study the interaction of magnetic quadrupole moment with electric field. We present our conclusions in Section 3.

2. Schrödinger equation

Hereafter, we establish and solve the Schrödinger which describes a non-relativistic scalar field which interacts with a magnetic quadrupole. In the geometric approach, the medium with a disclination has the line element in cylindrical coordinates (in units such that $c = 1$) [1,50,51], given by

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\varphi^2 + dz^2, \tag{1}$$

where $-\infty < z < \infty$, $-\infty < t < \infty$, $\rho \geq 0$ and $0 \leq \varphi \leq 2\pi$. It is related to the Lorentz metric $ds^2 = -dT^2 + dX^2 + dY^2 + dZ^2$, by the change of coordinates $X = \rho\cos(\alpha\varphi)$, $Y = \rho\sin(\alpha\varphi)$, $Z = z$ and $T = t$ [52]. This metric is equivalent to the boundary condition with periodicity of $2\pi\alpha$ instead of 2π around the z-axis. In the Volterra process [53] of disclination creation, this corresponds to remove ($\alpha < 1$) or insert ($\alpha > 1$) a wedge of material of dihedral angle $\lambda = 2\pi(\alpha - 1)$. This metric corresponds to a locally flat medium with a conical singularity at the origin. The only nonzero components of the Riemann curvature tensor and the Ricci tensor [54] are given by $R_{12}^{12} = R_1^2 = R_2^2 = 2\pi\frac{1-\alpha}{\alpha}\delta_2(\rho)$, where $\delta_2(\rho)$ is the two-dimensional delta function in flat space. From expression above, it follows that if $0 < \alpha < 1$ ($-2\pi < \lambda < 0$) the defect carries positive curvature and if $1 < \alpha < \infty$ ($0 < \lambda < \infty$) the defect carries negative curvature. This fact is very important in the curved space theory of amorphous solids [55] where geometrical frustration, the incompatibility between a given local order and the geometry of Euclidean space, is relieved by propagation of the local order in a space of constant curvature. Disclinations carrying curvature of sign opposite that of the curvature of the background space must be introduced [56] in order to reduce the mean curvature of the model to zero, yielding a distorted(locally curved) structure with the desired local order, perforated by disclination lines: a sensible structural model for amorphous solids. For early papers that discussed the coupling in Eq. (1) that describes the rotating reference frame for mesoscopic systems, see Ref. [57] where the authors discuss a new quantum interference effect between the split paths of a coherent particle beam in a uniformly rotating frame and they demonstrated that the effect should be observable even in a slow-rotating frame, and Ref. [58] in which the authors studied the effects caused by the rotation of an electron (or hole) in the presence of a screw, and the influence of the dislocation and the rotation on both the persistent current and magnetization.

From the geometrical point of view, the metric in Eq. (1) describes a Minkowski space–time with a conical singularity. The covariant metric tensor which corresponds to the spatial part of Eq. (1) is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2\rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\alpha^2\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where we order the coordinates as ρ, φ, z, and where $g^{\mu\nu}$ are the contravariant components of the inverse metric tensor of the tensor $g_{\mu\nu}$.

In non-relativistic systems, the non-inertial effects of rotation on quantum systems with a magnetic quadrupole moment in a rotating frame are described by means of the Schrödinger equation,

$$\left(\frac{\vec{\pi}^2}{2m} - \vec{\omega}\cdot\vec{L} + V(\rho)\right)\Psi(\mathbf{r}) = i\hbar\partial_t\Psi(\mathbf{r}), \tag{3}$$

for which we consider the effect of rotation on the non-relativistic quantum particle with a constant angular velocity by $\vec{\omega} = \omega\hat{z}$ and $\vec{\pi} = \vec{p} - e\vec{A}$ [39,46,57–59]. In Eq. (3) we shall need the Laplacian

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\right), \tag{4}$$

so that from Eq. (3), we obtain $\vec{\pi}^2\psi(r)$ as follows:

$$\vec{\pi}^2\Psi(\mathbf{r}) = p^2\Psi(\mathbf{r}) - e\mathbf{p}\cdot\mathbf{A}\Psi(\mathbf{r}) - e\mathbf{A}\cdot\mathbf{p}\Psi(\mathbf{r}) + e^2A^2\Psi(\mathbf{r}). \tag{5}$$

Thus, by replacing $\vec{p} = -i\vec{\nabla}$ into Eq. (5), we obtain

$$\vec{\pi}^2\Psi(\mathbf{r}) = -\nabla^2\Psi(\mathbf{r}) + 2ie\mathbf{A}\cdot\nabla\Psi(\mathbf{r}) + e^2A^2\Psi(\mathbf{r}). \tag{6}$$

In terms of the spatial part of the metric in Eq. (2), and using $g = det\left(g_{\mu\nu}\right) = \alpha^2\rho^2$ we find that, in Eq. (6),

$$\vec{\nabla}\Psi(\mathbf{r}) = \frac{\partial\Psi(\mathbf{r})}{\partial\rho}\hat{\rho} + \frac{1}{\alpha\rho}\frac{\partial\Psi(\mathbf{r})}{\partial\varphi}\hat{\varphi} + \frac{\partial\Psi(\mathbf{r})}{\partial z}\hat{z}. \tag{7}$$

Hereafter, we consider $A_\rho = 0$ and $A_z = 0$, and we have $\nabla\cdot\mathbf{A} = 0$, Thereby, Eq. (6) turns into

$$\vec{\pi}^2\Psi(\mathbf{r}) = -\frac{\partial^2\Psi(\mathbf{r})}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\Psi(\mathbf{r})}{\partial\rho} - \frac{1}{\alpha^2\rho^2}\frac{\partial^2\Psi(\mathbf{r})}{\partial\phi^2} - \frac{\partial^2\Psi(\mathbf{r})}{\partial z^2} + \frac{2ieA_\varphi}{\alpha\rho}\frac{\partial\Psi(\mathbf{r})}{\partial\phi} + e^2A_\varphi^2\Psi(\mathbf{r}) \tag{8}$$

If we consider $\partial_\varphi = i\ell$, $\partial_z = ik$ and $A_\varphi = \frac{1}{2}\lambda\mathcal{M}\rho$ and we replace into Eq. (8), we get

$$\vec{\pi}^2\Psi(\mathbf{r}) = -\frac{\partial^2\Psi(\mathbf{r})}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial\Psi(\mathbf{r})}{\partial\rho} + \frac{\ell^2}{\alpha^2\rho^2}\Psi(\mathbf{r}) + k^2\Psi(\mathbf{r}) - \frac{e\ell\lambda\mathcal{M}}{\alpha}\Psi(\mathbf{r}) + \frac{e^2\lambda^2\mathcal{M}^2\rho^2}{4}\Psi(\mathbf{r}). \tag{9}$$

The angular momentum operator $\vec{L}$ in Eq. (3) is given by

$$\vec{L} = \vec{r}\times\vec{\pi}, \tag{10}$$

where $\vec{\pi}$ is the generalized momentum,

$$\vec{\pi} = -i\vec{\nabla}_\alpha - \vec{\mathcal{M}}\times\vec{E}, \tag{11}$$

where $\vec{E}$ is the electric field in the laboratory frame, and $\vec{\mathcal{M}}$ is a vector with components

$$\left(\vec{\mathcal{M}}\right)_i \equiv \sum_j \mathcal{M}_{ij}\partial_j, \tag{12}$$

(see, for instance, Ref. [60]) with $\mathcal{M}_{ij}$ a symmetric and traceless tensor known as the 'magnetic quadrupole moment tensor' (analogous to the vector $\mathcal{Q}_i = \sum_j \mathcal{Q}_{ij}\partial_j$ of Ref. [61] where $\mathcal{Q}_{ij}$ is the electric quadrupole moment tensor).

In Eq. (3), we will utilize a static scalar potential $V(\rho)$ [62],

$$V(\rho) = a_1\rho + a_2\rho^2 - \frac{a_3}{\rho} + \frac{a_4}{\rho^2}. \tag{13}$$

This potential has many applications in particle physics, quantum field theory, molecular and solid state physics. The potential is a model of central potential consisting of radial oscillator harmonics potential, linear and Kratzer potential [49]. The Kratzer potential has two terms the first is the

Coulomb potential and the other is the inverse square term potential that remove the degeneracy. Also the linear term and Coulomb terms are named as Cornell potential [63].

Hereafter, we shall consider magnetic quadrupole moment tensors with only two non-null components [64,65]:

$$\mathcal{M}_{\rho z} = \mathcal{M}_{z\rho} = \mathcal{M}, \tag{14}$$

where $\mathcal{M}$ is a constant ($\mathcal{M} > 0$). This magnetic quadrupole moment interacts with a radial electric field chosen such that an analytical solution is possible (similar to Eq. (2) of Ref. [66]):

$$\vec{E} = \frac{1}{2}\lambda\rho^2\hat{\rho}, \tag{15}$$

with λ being a constant associated with a non-uniform distribution of electric charges inside a non-conducting cylinder. The interaction between this magnetic quadrupole moment and this radial electric field gives rise to a uniform effective magnetic field defined as $\vec{B} = \vec{\nabla} \times \vec{\mathcal{M}} \times \vec{E} = \lambda\mathcal{M}\hat{z}$, where the unit vector $\hat{z}$ points in the $+z$ direction, so that $\vec{B}$ is perpendicular to the plane of motion of the particle. For a moving neutral particle with a magnetic quadrupole moment. This magnetic quadrupole moment interacts with a radial electric field $E_\rho = \frac{1}{2}\lambda\rho^2$ with λ being a constant associated with a non-uniform distribution of electric charges inside a non-conductor cylinder.

We can write the angular momentum operator as

$$\vec{L} = \vec{r} \times (-i\vec{\nabla}_\alpha - \vec{\mathcal{M}} \times \vec{E}). \tag{16}$$

We use the angular momentum operator in Eq. (10) with Eqs. (14) and (15) so that $\vec{\mathcal{M}} \times \vec{E} = \frac{1}{2}\lambda\mathcal{M}\rho\hat{\varphi}$.

The interaction is time-independent so that one can write $\Psi(t, r, \varphi, z) = e^{-i(\mathcal{E}t-\ell\varphi-kz)}\psi(\rho)$, where $\mathcal{E}$ is the energy of the scalar boson, $\ell = 0, 1, 2, \ldots$, and k is a real number. If we consider only the radial component, the non-minimal substitution leads to

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\left(\frac{\ell^2}{\alpha^2} + 2ma_4\right) + \frac{2ma_3}{\rho} - 2ma_1\rho - \rho^2\eta^2 + \kappa^2\right]\psi(\rho) = 0 \tag{17}$$

where

$$\kappa^2 = \frac{\mathcal{M}}{2}\lambda\,\frac{\ell}{\alpha} - k^2 + \frac{2m\omega\ell}{\alpha} + 2\mathcal{E}m, \tag{18a}$$

$$\eta^2 = \frac{\mathcal{M}^2}{4}\lambda^2 + m\mathcal{M}\omega\lambda + 2ma_2, \tag{18b}$$

with $\mathcal{M}$ as in Eq. (14).

Let us introduce a change of variables given by $r = \sqrt{\eta}\,\rho$ so that the radial equation (17) becomes

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}\left(\frac{l^2}{\alpha^2} + 2ma_4\right) + \frac{1}{r}\left(\frac{2ma_3\sqrt{\eta}}{\eta}\right) - \frac{2ma_1}{\eta\sqrt{\eta}} - r^2 + \frac{\kappa^2}{\eta}\right]\psi\,(r) = 0. \tag{19}$$

If we write the radial wave function as

$$\psi\,(r) = e^{-\frac{r^2}{2}-\frac{Cr}{2}}\,r^{\left(\frac{\ell^2}{\alpha^2}+D\right)}h\,(r)\,, \tag{20}$$

where the constants C and D are determined as

$$C = \frac{2ma_1}{\eta^{\frac{3}{2}}}, \qquad D = -\frac{\ell^2}{\alpha^2} \pm \alpha\sqrt{\ell^2 + 2ma_4\alpha^2}, \tag{21}$$

where the positive sign is physically acceptable. Therefore Eq. (19) can be written as (see Refs. [67–70])

$$h''(r)+\left(-C+\frac{1+2D+\frac{2\ell^2}{\alpha^2}}{r}-2r\right)h'(r)+\left\{\frac{C^2}{4}-2\left(1+D+\frac{\ell^2}{\alpha^2}\right)+\frac{\kappa^2}{\eta}\right.$$
$$\left.+\left[-\frac{C}{2}\left(1+2D+2\frac{\ell^2}{\alpha^2}\right)+\frac{2ma_3}{\sqrt{\eta}}\right]\frac{1}{r}\right\}h(r)=0. \tag{22}$$

Note that this has the form of the biconfluent Heun equation,

$$H''(s)+\left(-2s-b+\frac{1+a}{s}\right)H'(s)+\left[-2-a+c+\frac{-b|a+1|/2-d/2}{s}\right]H(s)=0. \tag{23}$$

By comparing this with Eq. (22), we see that the parameters in Eq. (23) are related to the physical parameters as

$$a=2\alpha\sqrt{\ell^2+2ma_4\alpha^2},\quad b=\frac{2ma_1}{\left(\frac{\mathcal{M}^2}{4}\lambda^2+m\mathcal{M}\omega\lambda+2ma_2\right)^{\frac{3}{4}}},$$
$$c=\frac{1}{\eta}\left(\frac{\mathcal{M}}{2}\lambda\,\frac{\ell}{\alpha}-k^2+\frac{2m\omega\ell}{\alpha}+2\mathcal{E}m\right)+\frac{m^2a_1^2}{\left(\frac{\mathcal{M}^2}{4}\lambda^2+m\mathcal{M}\omega\lambda+2ma_2\right)^{\frac{3}{2}}}, \tag{24}$$
$$d=\frac{-4ma_3}{\left(\frac{\mathcal{M}^2}{4}\lambda^2+m\mathcal{M}\omega\lambda+2ma_2\right)^{\frac{1}{4}}}.$$

In order to determine the power series solution of Eq. (23) and its energy eigenvalues, we utilize the Frobenius method; that is, we express $H(s)$ of Eq. (23) as

$$H(s)=\sum_{n=0}c_ns^{n+p}, \tag{25}$$

where p is to be determined, and we substitute Eq. (25) into Eq. (23), to obtain

$$\begin{aligned}&p(p+a)c_0s^{p-2}\\&+\left[-bpc_0-\frac{1}{2}\left(b\,|a+1|+d\right)+(p+1)(p+1+a)c_1\right]s^{p-1}\\&+\sum_{j=0}(j+p+2)\,(j+p+2+a)\,c_{j+2}s^{p+j}\\&-\frac{1}{2}\sum_{j=0}\left[b\,|a+1|+d+2b\,(j+p+1)\right]c_{j+1}s^{p+j}\\&+\sum_{j=0}\left(c-a-2\,(j+p+1)\right)c_js^{p+j}=0.\end{aligned} \tag{26}$$

From the coefficient of s^{p-2}, we see than

$$p=0\text{ or }p=-a, \tag{27}$$

the coefficient of s^{p-1} gives

$$c_1=\frac{bp+\frac{1}{2}(b|a+1|+d)}{(p+1)(p+1+a)}c_0, \tag{28}$$

and the coefficients of s^{p+j}, $j=0,1,2,\ldots$, lead to

$$c_{j+2}=\frac{b(p+j+1)+\frac{1}{2}(b|a+1|+d)}{(p+j+2)(p+j+2+a)}c_{j+1}+\frac{a-c+2(p+j+1)}{(p+j+2)(p+j+2+a)}c_j. \tag{29}$$

For the sake of this paper, we shall consider only the solution $p=0$ from Eq. (27). Then Eq. (29) is

$$c_{j+2}=\frac{b(j+1)+\frac{1}{2}(b|a+1|+d)}{(j+2)(j+2+a)}c_{j+1}+\frac{a-c+2j+2}{(j+2)(j+2+a)}c_j. \tag{30}$$

For the power series in Eq. (25) to terminate and reduce to a polynomial, there has to be a value of j, denoted n_0, such that the numerator of the last term in Eq. (29) is zero, which gives two conditions:

$$c-a-2=2n_0, \qquad n_0=1,2,3,\ldots \tag{31}$$

and

$$c_{n_0+1}=0. \tag{32}$$

From Eqs. (31), (24) and (18b), the energy eigenvalues are given by

$$\begin{aligned}\mathcal{E}_{n_0,\ell}=&\frac{k^2}{2m}-\frac{\ell\left(\lambda\mathcal{M}+4m\omega\right)}{4m\alpha}+\frac{\sqrt{8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)}}{2m}\\&\left[1+n_0-\frac{4m^2a_1^2}{\left[8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)\right]^{\frac{3}{2}}}+\alpha\sqrt{\ell^2+2m\alpha^2a_4}\right].\end{aligned} \tag{33}$$

From Eq. (28), we find c_1 in terms of c_0,

$$c_1=\frac{b\,|a+1|+d}{2(a+1)}c_0. \tag{34}$$

If we take $c_0=1$, we find, from Eqs. (30) and (34),

$$c_2=\frac{\left[b+\frac{1}{2}\left(b\,|a+1|+d\right)\right]\left[\frac{1}{2}\left(b\,|a+1|+d\right)\right]}{2\left(1+a\right)\left(2+a\right)}-\frac{n_0}{2+a}, \tag{35}$$

and

$$\begin{aligned}c_3=&\frac{1}{12\left(a+3\right)}\left\{\frac{4\left(1-n_0\right)\left(b\,|a+1|+d\right)}{a+1}+\frac{\left(4b+b\,|a+1|+d\right)}{2\left(a+2\right)}\left[-4n_0\right.\right.\\&\left.\left.+\frac{\left(b\,|a+1|+d\right)\left(2b+b\,|a+1|+d\right)}{2\left(a+1\right)}\right]\right\}\end{aligned} \tag{36}$$

For $n_0=1$, we have $c_2=0$, so that the Heun polynomial is linear and the corresponding energy eigenvalues are

$$\begin{aligned}\mathcal{E}_{1,\ell}=&\frac{k^2}{2m}-\frac{\ell\left(\lambda\mathcal{M}+4m\omega\right)}{4m\alpha}+\frac{\sqrt{8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)}}{2m}\\&\left\{2-\frac{4m^2a_1^2}{\left(8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)\right)^{\frac{3}{2}}}+\alpha\sqrt{\ell^2+2m\alpha^2a_4}\right\},\end{aligned} \tag{37}$$

with the corresponding wave function given by

$$h_{1,\ell}(\sqrt{\eta}\,\rho)=c_0+c_1\sqrt{\eta}\,\rho=1+\left(\frac{b\,|a+1|+d}{2(a+1)}\right)\sqrt{\eta}\,\rho, \tag{38}$$

so that

$$\psi_{1,\ell}(\sqrt{\eta}\,\rho)=N_{1,\ell}e^{-\frac{\eta\,\rho^2}{2}}e^{-\frac{C\sqrt{\eta}\,\rho}{2}}\left(\sqrt{\eta}\rho\right)^{\left[\frac{\ell^2}{\alpha^2}+D\right]}\left[1+\left(\frac{b\,|a+1|+d}{2(a+1)}\right)\sqrt{\eta}\,\rho\right], \tag{39}$$

where $N_{1,\ell}$ is a normalization constant. By setting Eq. (35) equal to zero, with $n_0=1$, we find

$$\frac{\left[b+\frac{1}{2}\left(b\,|a+1|+d\right)\right]\left[\frac{1}{2}\left(b\,|a+1|+d\right)\right]}{2\left(a+1\right)\left(a+2\right)}=\frac{1}{a+2}. \tag{40}$$

In the above equation, we have a constraint equation including free parameters a, b, c and quantum numbers n and ℓ. We may rewrite the constraint such that a free parameter, say a, is obtained as a function of n and ℓ and the other free parameters can vary independently. It means that we should consider a as $a_{n_0,\ell}$ or equivalent a_4 should be consider as $a_{4_{n_0,\ell}}$. In this way, we can write the above equation as

$$\frac{\left[b+\frac{1}{2}\left(b\left|a_{1,\ell}+1\right|+d\right)\right]\left[\frac{1}{2}\left(b\left|a_{1,\ell}+1\right|+d\right)\right]}{2\left(a_{1,\ell}+1\right)\left(a_{1,\ell}+2\right)}=\frac{1}{a_{1,\ell}+2}. \tag{41}$$

Next if we choose $n_0 = 2$, which means we interrupt the series with $c_3 = 0$, then we find

$$\begin{aligned} h_{2,\ell}\left(\sqrt{\eta}\,\rho\right) =& c_0+c_1\sqrt{\eta}\,\rho+c_2\eta\rho^2=1+\left(\frac{b\,|a+1|+d}{2(a+1)}\right)\sqrt{\eta}\,\rho \\ &+\left(\frac{\left(b+\frac{1}{2}\left(b\,|a+1|+d\right)\right)\left(\frac{1}{2}\left(b\,|a+1|+d\right)\right)}{2\,(1+a)\,(2+a)}-\frac{2}{2+a}\right)\eta\rho^2, \end{aligned} \tag{42}$$

so that

$$\begin{aligned} \psi_{2,\ell}(\sqrt{\eta}\,\rho) =& N_{2,\ell}e^{-\frac{\eta\rho^2}{2}}e^{-\frac{C\sqrt{\eta}\,\rho}{2}}(\eta\rho)^{\left[\frac{\ell^2}{\alpha^2}+D\right]}\left[1+\left(\frac{b\,|a+1|+d}{2(a+1)}\right)\sqrt{\eta}\,\rho\right. \\ &\left.+\left(\frac{\left(b+\frac{1}{2}\left(b\,|a+1|+d\right)\right)\left(\frac{1}{2}\left(b\,|a+1|+d\right)\right)}{2\,(1+a)\,(2+a)}-\frac{2}{2+a}\right)\eta\rho^2\right], \end{aligned} \tag{43}$$

and the energy eigenvalue is

$$\begin{aligned} \mathcal{E}_{2,\ell} =& \frac{k^2}{2m}-\frac{\ell\left(\lambda\mathcal{M}+4m\omega\right)}{4m\alpha}+\frac{\sqrt{8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)}}{2m} \\ &\left\{3-\frac{4m^2a_1^2}{\left(8ma_2+\lambda\mathcal{M}\left(\lambda\mathcal{M}+4m\omega\right)\right)^{\frac{3}{2}}}+\alpha\sqrt{\ell^2+2m\alpha^2a_4}\right\}. \end{aligned} \tag{44}$$

3. Conclusion

In this contribution, we have investigated the Schrödinger equation describing the interaction of neutral particle system (atom or molecule) with a magnetic quadrupole moment with a radial electric field for non-relativistic particles in a rotating frame. We considered a uniform effective magnetic field perpendicular to the plane of motion of the neutral particle. We obtained the eigenfunctions and the energy levels of the field in that background. For such non-relativistic systems, we solved the differential equation for the radial part of the wave function by using the Frobenius method; we observed that this part of the wave function corresponds to the biconfluent Heun equation. In order to obtain a normalized radial wave function, we imposed that the biconfluent Heun series terminate and we found explicit expression for lowest Heun polynomials, along with their respective energy eigenvalues in terms of the physical system's parameters.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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