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Self-stress on a dielectric ball and Casimir–Polder forces

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ABSTRACT

It has always been conventionally understood that, in the dilute limit, the Casimir energy of interaction between bodies or the Casimir self-energy of a dielectric body could be identified with the sum of the van der Waals or Casimir–Polder energies of the constituents of the bodies. Recently, this proposition for self-energies has been challenged by Avni and Leonhardt (2018), who find that the energy or self-stress of a homogeneous dielectric ball with permittivity ε begins with a term of order $\varepsilon-1$. Here we demonstrate that this cannot be correct. The only possible origin of a term linear in $\varepsilon-1$ lies in the bulk energy, that energy which would be present if either the material of the body, or of its surroundings, filled all space. Since Avni and Leonhardt correctly subtract the bulk terms, the linear term they find likely arises from their omission of an integral over the transverse stress tensor.

1. Introduction

From the beginning of the subject, it has been recognized that van der Waals or Casimir–Polder forces between neutral atoms [1] and Casimir (also known as quantum-vacuum or dispersion)

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forces between dielectric or conducting bodies [2] have the same origin. This was made manifest by Lifshitz [3] and by Dzyaloshinskii, Lifshitz, and Pitaevskii [4] who generalized Casimir's calculation of the force between perfectly conducting plates to that between dispersive dielectric slabs. In the dilute limit, the resulting energy is just the sum of the Casimir–Polder energies.

Later it was shown that the Casimir self-energy of a dielectric ball in the dilute limit [5] is identical to that obtained by summing the van der Waals energies [6]. Even though the Casimir self-energy of a dielectric ball of permittivity ε is divergent, in the dilute limit the first term, of order $(\varepsilon-1)^2$, is finite, and unambiguously coincides with the sum of Casimir–Polder energies. This quadratic dependence on the susceptibility reflects the nature of the van der Waals two-body interaction, which is quadratic in the polarizability.

Thus it was surprising when Avni and Leonhardt [7] re-examined the Casimir self-energy for a homogeneous dielectric ball and found, after discarding divergent terms, that the Casimir energy has a leading term of order $\varepsilon-1$ in the dilute limit. However, they did obtain the correct result for a perfectly conducting spherical shell [8], or more generally, for the finite energy for a diaphanous or isorefractive ball (where the speed of light is the same inside and outside the ball) [9]. They attribute the discrepancy to their different method of regularization: instead of point-splitting in time or transverse coordinates, they consider the difference in the radial–radial stress tensor at finite displacements inside and outside the spherical boundary. However, we show that their method is erroneous. The only possible source of a linear term is the bulk pressure, which they, like we, subtract. In this paper we explicitly compute that bulk pressure, which indeed begins with order $\varepsilon-1$, and show that it cannot give a finite remainder, and that on physical grounds it must be subtracted. What is left has to behave like $(\varepsilon-1)^2$ in the dilute limit. Because of the error in their method of computing the pressure on the spherical surface, the claim of Ref. [7] that the Casimir energy of the dielectric ball "shutters the picture of the equivalence between the macroscopic effect and pairwise summation" is incorrect. We can indeed ascribe the Casimir force to the sum of Casimir–Polder or van der Waals energies, where in dense media pairwise summation must be supplemented by multi-particle interactions.

2. Bulk pressure on a dielectric ball

We first examine the bulk pressure, as it is the only possible source of a linear $\varepsilon-1$ term. We consider a ball, of radius a, made of homogeneous, isotropic material, with permittivity ε and permeability μ inside, surrounded by an infinite background characterized by electrical properties ε', μ'. The pressure on the spherical surface of the ball was first worked out in the general case in Ref. [10]:

$$p=\frac{1}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}e^{iy\tilde\tau}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\delta)\left[x\frac{d}{dx}\ln D_l+f_l(x)-f_l(x')\right], \tag{2.1}$$

where $\tilde\tau=\frac{\tau}{a}\to 0$ is a time-splitting regulator, $\delta\to 0$ is a transverse spatial (angular) regulator,

$$D_l=(s_l(x)e_l'(x')-s_l'(x)e_l(x'))^2-\xi^2(s_l(x)e_l'(x')+s_l'(x)e_l(x'))^2, \tag{2.2}$$

with

$$\xi=\frac{\sqrt{\frac{\varepsilon}{\varepsilon'}\frac{\mu'}{\mu}}-1}{\sqrt{\frac{\varepsilon}{\varepsilon'}\frac{\mu'}{\mu}}+1}, \tag{2.3}$$

and $y=\zeta a$, $\zeta=-i\omega$ is the imaginary frequency, $x=|y|\sqrt{\varepsilon\mu}$, $x'=|y|\sqrt{\varepsilon'\mu'}$, while the modified Riccati–Bessel functions are

$$s_l(x)=\sqrt{\frac{\pi x}{2}}I_{l+1/2}(x),\quad e_l(x)=\sqrt{\frac{2x}{\pi}}K_{l+1/2}(x). \tag{2.4}$$

The terms subtracted in Eq. (2.1) are the terms present if either medium filled all of space:

$$f_l(x)=2x[s_l'(x)e_l'(x)-e_l(x)s_l''(x)]. \tag{2.5}$$

More precisely, this amounts to the removal of the pressure that the interior medium would exert at $r = a$ if it filled all space, less the pressure that the exterior medium would exert at $r = a$ if it filled all space. In other words, only the scattering part of the Green's function is included [11]. We refer to the removal of the difference in the non-scattering parts as the bulk subtraction.

The difference of bulk pressures vanishes if the speed of light both inside and outside the ball is the same, $\sqrt{\varepsilon\mu} = \sqrt{\varepsilon'\mu'}$, which is true for a perfectly conducting shell of negligible thickness in vacuum, or more generally, for a diaphanous or isorefractive ball. We will discuss this case in Section 4; we merely note here that Ref. [7] agrees with the usual results in these cases. This agreement is beside the point, because in these situations the pressure is an even function of ξ, so a linear term can never arise. But for a purely dielectric ball the bulk terms play a crucial role. We will here explicitly evaluate those terms; in some sense, the failure to exclude them properly is what gives rise to terms proportional to $\varepsilon - 1$ in Ref. [7], as these are the only possible source of such effects.

It is easy to evaluate these terms directly:

$$p^{(0)} = -\frac{1}{16\pi^2 a^4}\int_{-\infty}^{\infty} dy\, e^{iy\tilde{\tau}} \sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)[f_l(x) - f_l(x')]. \tag{2.6}$$

Let us suppose the ball is characterized by $\varepsilon \neq 1$, $\mu = 1$, and is surrounded by vacuum. The term linear in $\varepsilon - 1$ is

$$p_1^{(0)} = -\frac{\varepsilon - 1}{32\pi^2 a^4}\int_{-\infty}^{\infty} dy\, e^{iy\tilde{\tau}} \sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\, x\frac{d}{dx}f_l(x), \tag{2.7}$$

where we have now assumed that there is no dispersion, and now $x = |y|$. We use the summation formula

$$\sum_{l=0}^{\infty}(2l+1)P_l(\cos\delta)e_l(x)s_l(y) = \frac{xy}{\rho}e^{-\rho}, \quad \rho = \sqrt{x^2 + y^2 - 2xy\cos\delta}. \tag{2.8}$$

This gives rather immediately

$$p_1^{(0)} = -\frac{\varepsilon - 1}{16\pi^2 a^4}\int_0^{\infty} dx\, x\cos x\tilde{\tau}\left(2 + e^{-x\tilde{\delta}}\left[-2 + x\left(\tilde{\delta} - 4/\tilde{\delta}\right)\right]\right), \tag{2.9}$$

where $\tilde{\delta} = \sqrt{2}\sqrt{1 - \cos\delta} \approx \delta$. The term independent of $\tilde{\delta}$ is a Fresnel integral:

$$\int_0^{\infty} dx\, x\cos x\tilde{\tau} = -\frac{1}{\tilde{\tau}^2}, \tag{2.10}$$

and the remaining integrals are absolutely convergent:

$$p_1^{(0)} = -\frac{\varepsilon - 1}{8\pi^2 a^4}\left[-\frac{1}{\tilde{\tau}^2} - \frac{\tilde{\delta}^2 - \tilde{\tau}^2}{\left(\tilde{\delta}^2 + \tilde{\tau}^2\right)^2} + \frac{\tilde{\delta}^2 - 3\tilde{\tau}^2}{\left(\tilde{\delta}^2 + \tilde{\tau}^2\right)^3}\left(\tilde{\delta}^2 - 4\right)\right]. \tag{2.11}$$

Evidently, spatial point-splitting is insufficient to regulate all divergences. For simplicity, we now set $\delta = 0$; then this reduces to

$$p_1^{(0)} = -\frac{\varepsilon - 1}{2\pi^2 a^4}\frac{3}{\tilde{\tau}^4}, \tag{2.12}$$

which is precisely what would be obtained if δ were set equal to zero from the outset. Recalling that $\tilde{\tau} = \tau/a$, we see that this corresponds to an energy given by

$$4\pi a^2 p_1^{(0)} = -\frac{\partial E_1^{(0)}}{\partial a}, \quad \text{or} \quad E_1^{(0)} = \frac{\varepsilon - 1}{\pi a}\frac{2}{\tilde{\tau}^4}. \tag{2.13}$$

Thus the portion of the bulk term linear in $\varepsilon - 1$ is purely divergent, with no finite remainder, in contradistinction to the linear term found in Ref. [7]. To provide some kind of "physical" interpretation of the energy expression in Eq. (2.13), one might note that the fundamental parameter in the energy expression is the ratio between the geometric size a and the distance τ $(= c\tau)$ covered by light during the cutoff time τ. Characteristic for a volume energy is that this parameter has to be raised to a power of four.

We can easily evaluate $p^{(0)}$ exactly from Eq. (2.6) in just the same way:

$$p^{(0)} = \frac{1}{4\pi^2 a^4}\int_0^\infty dx\, x\left(\cos x\tilde\tau - \frac{1}{\sqrt{\varepsilon}}\cos(x\tilde\tau/\sqrt{\varepsilon})\right) \times\left[1+\frac{e^{-x\tilde\delta}}{x^2\tilde\delta^4}\left(2x\tilde\delta\left(1+\tilde\delta^2/4\right)+2x^2\tilde\delta^2\left(1-\tilde\delta^2/4\right)\right)\right]. \tag{2.14}$$

Doing the integrals as before, and again setting $\delta = 0$ for simplicity, we have

$$p^{(0)} = \frac{1}{\pi^2 a^4\tilde\tau^4}(1-\varepsilon^{3/2}). \tag{2.15}$$

Recalling that $\tau = a\tilde\tau$ is independent of a, this corresponds to the energy

$$E^{(0)} = -\frac{4}{3}\frac{1}{\pi a\tilde\tau^4}(1-\varepsilon^{3/2}). \tag{2.16}$$

When expanded for small $\varepsilon - 1$, this reduces to Eq. (2.13). The conclusion is that the bulk energy is purely divergent.[1] Appendix A contains a generalization of this result.

It is straightforward to demonstrate explicitly that the bulk subtraction results in the absence from p of any term linear in $\varepsilon - 1$ in the dilute limit. Let

$$p^* = \frac{1}{2a^4}\int_{-\infty}^\infty \frac{dy}{2\pi}e^{iy\tilde\tau}\sum_{l=1}^\infty \frac{2l+1}{4\pi}P_l(\cos\delta)\,x\frac{d}{dx}\ln D_l \tag{2.17}$$

be the pressure prior to the bulk subtraction, so that $p = p^* - p^{(0)}$, and let p_1^* be the part of p^* that is linear in $\varepsilon - 1$ in the dilute limit. Noting that $D_l|_{\varepsilon=1} = 1$ and $\frac{d}{d\varepsilon}D_l\big|_{\varepsilon=1} = -\frac{1}{2}f_l(x)$, where $x = |y|$, we immediately obtain

$$p_1^* = -\frac{\varepsilon-1}{32\pi^2 a^4}\int_{-\infty}^\infty dy\, e^{iy\tilde\tau}\sum_{l=1}^\infty (2l+1)P_l(\cos\delta)\,x\frac{d}{dx}f_l(x), \tag{2.18}$$

which is identical to Eq. (2.7). Thus, $p_1^* - p_1^{(0)} = 0$, and we can conclude that, as a consequence of the bulk subtraction, p has no term linear in $\varepsilon - 1$ in the dilute limit.

3. Stress tensor and comparison with Ref. [7]

Avni and Leonhardt [7] base their calculation on the divergence of the stress tensor. Let us re-examine that here. According to Ref. [11], the time-averaged divergence of the stress tensor is proportional to the inhomogeneity of the permittivity:

$$\overline{\nabla\cdot\langle\mathbf{T}\rangle}(\mathbf{r}) = \frac{1}{2}\int_{-\infty}^\infty \frac{d\zeta}{2\pi}\,\mathrm{tr}\,\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r};\zeta)\nabla\boldsymbol{\varepsilon}(\mathbf{r},\zeta). \tag{3.1}$$

For an isotropic dispersionless homogeneous dielectric ball this gives the radial component of the force density

$$f_r = -\overline{\nabla\cdot\langle\mathbf{T}\rangle}(\mathbf{r})\cdot\hat{\mathbf{r}} = \frac{\varepsilon-1}{2}\int_{-\infty}^\infty\frac{d\zeta}{2\pi}\,\mathrm{tr}\,\boldsymbol{\Gamma}(a,a;\zeta)\delta(r-a). \tag{3.2}$$

[1] The analogy is not true for the δ-function sphere [12]. There the first-order term in the coupling ζ_p is

$$E_1 = \frac{\zeta_p}{\pi}\left(\frac{1}{\tilde\tau^2}+\frac{11}{24}\ln\tilde\tau - 0.345979\right).$$

There is a finite part to this "tadpole contribution", but it is not unique because of the presence of the logarithmic divergence. But even here, this first-order term is usually omitted as unphysical.

The volume integral of this gives the stress, which, divided by the surface area, gives the pressure on the surface of the ball:

$$p = \frac{\varepsilon - 1}{2} \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi} \operatorname{tr} \boldsymbol{\Gamma}(a, a; \zeta). \tag{3.3}$$

This already has one factor of $\varepsilon - 1$. Now if we make the bulk subtraction, it is evident that the Green's dyadic vanishes if $\varepsilon = 1$; there is no way a linear term could arise. This was shown explicitly already nearly 40 years ago [13]; the explicit quadratic term was worked out in Ref. [5]. See also Refs. [14,15].

Ref. [7] computes the stress on the ball not by some process of point-splitting but by computing the difference in the radial stress at radii $a+\Delta$ and $a-\Delta$. Power and logarithmic divergences appear as $\Delta \to 0$. The coefficients of these divergences are determined numerically, and these divergent terms are then simply omitted, even though the logarithmic term can modify the finite remainder, in an arbitrary way. That remainder is regarded as the finite self-energy. In the dilute limit, the divergent terms are proportional to $(\varepsilon - 1)^2$, but, surprisingly, the finite part starts with a linear dependence, with the stress behaving approximately as $F = 0.02(\varepsilon - 1)\hbar c/a^2$. How can this be?

In more detail, Avni and Leonhardt [7] note that the radial component of the force density is not a total divergence:

$$f_r = -(\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \hat{\mathbf{r}} = -\boldsymbol{\nabla} \cdot (\mathbf{T} \cdot \hat{r}) + \frac{1}{r}({T_\theta}^\theta + {T_\phi}^\phi) = \frac{F}{4\pi r^2} \delta(r - a), \tag{3.4}$$

so if the stress tensor were finite at $r = a$, the stress on the sphere could be written as

$$F = 4\pi r_-^2 T_{rr}(r_-) - 4\pi r_+^2 T_{rr}(r_+) + 4\pi \int_{r_-}^{r_+} dr\, r \left[{T_\theta}^\theta + {T_\phi}^\phi\right](r), \tag{3.5}$$

where the integral is over a volume bounded by concentric spheres drawn inside and outside the boundary of the dielectric ball, $r_- < a < r_+$. (This result would also obtain by directly integrating Eq. (15) of Ref. [7].) The stress tensor components inside and outside the boundary are finite, divergent as $r_\pm \to a$,

$$r_\pm^2 T_{rr}(r_\pm) = a_{-3}^\pm \Delta_\pm^{-3} + a_{-2}^\pm \Delta_\pm^{-2} + a_{-1}^\pm \Delta_\pm^{-1} + b_0^\pm \ln \Delta_\pm + a_0^\pm, \tag{3.6}$$

where $\Delta_\pm = |r_\pm - a|$. Ref. [7] omits the divergent terms, claiming the finite force is

$$F_m = 4\pi(a_0^- - a_0^+), \tag{3.7}$$

even though the presence of the logarithmic terms means that the finite part cannot be unique. However, the volume integral over ${T_\theta}^\theta + {T_\phi}^\phi$ is mysteriously omitted. In fact the latter is actually divergent; the integral over r in Eq. (3.5) does not exist. For example, for $\varepsilon - 1$ small,

$${T_\theta}^\theta + {T_\phi}^\phi \sim \mp \frac{\varepsilon - 1}{160\pi^2(r - a)^4}, \tag{3.8}$$

as the surface is approached from the outside (inside). This singularity is non-integrable. (This is discussed in detail in Appendix B.) Only if $\Delta^\pm \to 0$, and the integrals regulated in some other way, such as by a temporal point splitting as we do, can the integral over the angular components of the stress tensor be omitted.[2] Then Eq. (3.5) yields the pressure (2.1). The arguments in Ref. [7] about the invalidity of interchanging the limits $r \to a$ and infinite series on l are questionable. The choice of $\Delta_\pm$ as a putative regulator is a poor one; $\Delta_\pm$ is ineffective in this capacity.

2 For example, using the leading uniform asymptotic approximant given in Eq. (B.7), we find the integral in Eq. (3.5) goes like $(r_+ - r_-)/(a^3 \tilde{\tau}^4) \to 0$ as $r_+ - r_- \to 0$ for fixed $\tilde{\tau}$. Similarly, there would be a divergent contribution to the radial–radial integrated term in Eq. (3.5) as well, if regularization were not supplied.

4. Diaphanous or isorefractive ball

As mentioned above, in the case that the speed of light is the same inside and outside the sphere, $\sqrt{\varepsilon\mu} = \sqrt{\varepsilon'\mu'}$, the bulk term vanishes. So there is no question that the energy begins as $(\varepsilon - \varepsilon')^2$. But Ref. [7] claims small discrepancies with results shown in Ref. [10]. However, the latter was just a first approximation, initially and more fully discussed by Brevik and Kolbenstvedt and other collaborators [16–21]. Recently, this simple problem has been reanalyzed with improved numerical results [9].

In this case, the energy corresponding to the pressure (2.1) reduces to

$$E = -\frac{1}{4\pi a}\int_{-\infty}^{\infty} dy\, e^{iy\tilde{\tau}} \sum_{l=1}^{\infty}(2l+1)P_l(\cos\delta)\, x\frac{d}{dx}\ln(1-\xi^2[(e_l(x)s_l(x))']^2), \tag{4.1}$$

where now $\xi = (\varepsilon - \varepsilon')/(\varepsilon + \varepsilon')$, and $x = |y|\sqrt{\varepsilon\mu}$. Ref. [7] notes that a Δ^{-1} divergence also occurs here; we saw the analog of this previously for a general form of cutoff in Ref. [12]. However, for a purely temporal regulator ($\delta = 0$) the energy is completely finite. We can efficiently and accurately evaluate this by including the first two uniform asymptotic approximants, followed by numerically computed remainders,

$$E = \frac{1}{\sqrt{\varepsilon\mu}}\left(E^{(2)} + E^{(4)} + \sum_{l=1}^{\infty} R_l\right), \tag{4.2}$$

where the speed of light factor has been pulled out (which Ref. [7] correctly notes we inadvertently omitted). Here

$$E^{(2)} = \frac{5\xi^2}{32\pi a}, \quad E^{(4)} = \frac{9\xi^2}{2^{12}a}\left(\frac{\pi^2}{2} - 1\right)(6 - 7\xi^2). \tag{4.3}$$

It is sufficient to include only the first two remainder terms, which gives

$$E(\xi = 0.33) = 0.00380(0.00387)/a; \quad E(\xi = 0.2) = 0.001619(0.00162)/a, \tag{4.4}$$

where the parenthetical numbers are those given in Ref. [7]. There is likely no significant discrepancy here.

5. Discussion

In this paper, we show that the procedure proposed in Ref. [7] is erroneous. One cannot extract the Casimir self-stress on the surface of a homogeneous dielectric ball by taking the difference between the normal–normal components of the stress tensor a finite distance away from the boundary of the ball in the absence of regulation. We have demonstrated that although the stress tensor away from the physical boundary is finite, divergences at the boundary and a divergent integral over the angular parts of the stress tensor have been rather mysteriously omitted. Therefore the result of Ref. [7], that there is a term linear in the susceptibility in the dilute limit, cannot be accepted.

Furthermore, we have re-examined the issue of the bulk subtraction that must be supplied in order to obtain a meaningful Casimir self-energy. Doing so is necessary to preserve the connection between van der Waals or Casimir–Polder forces and the Casimir energy of a body composed of polarizable molecules. This is signified by the energy in the dilute limit being proportional to $(\varepsilon - 1)^2$. A linear term can only arise if the bulk energy were not properly subtracted. We have evaluated the latter and have found that it only consists of divergent terms; no meaningful linear term can be extracted. Thus there is no possible source of a term linear in the susceptibility.

Unfortunately, the bulk subtraction is not sufficient to achieve a finite Casimir self-energy for a dielectric ball. One way of stating this fact is that the a_2 heat-kernel coefficient is nonzero in order $(\varepsilon - 1)^3$ [22]. Only if the speed of light is the same inside and outside a spherical shell of negligible thickness, which includes the case of the perfectly conducting spherical shell in vacuum [8], can a unique, finite energy be extracted. Agreement with numerical results shown in Ref. [7] for such a case reveals no significant discrepancy with our results.

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Appendix A. Scaling behavior

It is easily verified from Eqs. (2.1) and (2.6) that both $p(\varepsilon,\mu,\varepsilon',\mu',a)$ and $p^{(0)}(\varepsilon,\mu,\varepsilon',\mu',a)$ possess the following scaling property, for scale factors $\lambda_\varepsilon,\lambda_\mu>0$:

$$p\left(\lambda_\varepsilon\varepsilon,\lambda_\mu\mu,\lambda_\varepsilon\varepsilon',\lambda_\mu\mu',(\lambda_\varepsilon\lambda_\mu)^{-\frac{1}{2}}\,a\right)=(\lambda_\varepsilon\lambda_\mu)^{\frac{3}{2}}\,p(\varepsilon,\mu,\varepsilon',\mu',a). \tag{A.1}$$

This property can be useful. For example, it may be employed to relate the pressure for the dielectric–dielectric problem to that for the dielectric-vacuum problem: setting $\lambda_\varepsilon=\frac{1}{\varepsilon'}$ and $\lambda_\mu=\frac{1}{\mu'}$ yields

$$p(\varepsilon,\mu,\varepsilon',\mu',a)=(\varepsilon'\mu')^{\frac{3}{2}}\,p\left(\frac{\varepsilon}{\varepsilon'},\frac{\mu}{\mu'},1,1,(\varepsilon'\mu')^{\frac{1}{2}}\,a\right). \tag{A.2}$$

The scaling property may be expressed in the form of a partial differential equation. To this end, it is helpful to work with an overall scale factor of

$$\lambda=\sqrt{\lambda_\varepsilon\lambda_\mu} \tag{A.3}$$

and partition this between λ_ε and λ_μ so that

$$\lambda_\varepsilon=\lambda^\alpha\quad\text{and}\quad\lambda_\mu=\lambda^\beta, \tag{A.4}$$

where $\alpha+\beta=2$. Then Eq. (A.1) becomes

$$p\left(\lambda^\alpha\varepsilon,\lambda^\beta\mu,\lambda^\alpha\varepsilon',\lambda^\beta\mu',\lambda^{-1}a\right)=\lambda^3\,p(\varepsilon,\mu,\varepsilon',\mu',a). \tag{A.5}$$

Differentiating with respect to λ at $\lambda=1$, there results the following partial differential equation representation of the scaling property:

$$a\frac{\partial}{\partial a}p-\alpha\left(\varepsilon\frac{\partial}{\partial\varepsilon}+\varepsilon'\frac{\partial}{\partial\varepsilon'}\right)p-\beta\left(\mu\frac{\partial}{\partial\mu}+\mu'\frac{\partial}{\partial\mu'}\right)p+3p=0, \tag{A.6}$$

where $\alpha+\beta=2$. Note that

$$\left(\varepsilon\frac{\partial}{\partial\varepsilon}+\varepsilon'\frac{\partial}{\partial\varepsilon'}\right)p=\left(\mu\frac{\partial}{\partial\mu}+\mu'\frac{\partial}{\partial\mu'}\right)p, \tag{A.7}$$

because the functional dependence of p on ε, μ, ε', and μ' is through only the following forms: $\varepsilon\mu$, $\varepsilon'\mu'$, $\frac{\varepsilon}{\varepsilon'}$, and $\frac{\mu}{\mu'}$.

The scaling property may also be expressed in the form of a conservation equation. Let

$$E=4\pi a^3p\left(\varepsilon,\mu,\varepsilon',\mu',a\right), \tag{A.8}$$

which would be the energy if there were no cutoff dependence. If, in Eq. (A.8), ε, μ, ε', and μ' are permitted to vary with a according to

$$\varepsilon(a)=\varepsilon_1\,a^{-\alpha}, \tag{A.9a}$$
$$\mu(a)=\mu_1\,a^{-\beta}, \tag{A.9b}$$
$$\varepsilon'(a)=\varepsilon_1'\,a^{-\alpha}, \tag{A.9c}$$
$$\mu'(a)=\mu_1'\,a^{-\beta}, \tag{A.9d}$$

where $\alpha+\beta=2$, then, as is easily verified from Eq. (A.6),

$$\frac{dE}{da}=0, \tag{A.10}$$

that is, E is conserved under such change with a. (Here ε_1, μ_1, etc., are held constant, not the physical values $\varepsilon(a)$, $\mu(a)$.) It follows from Eqs. (A.8) and (A.10) (or from Eq. (A.5), with $\lambda=a$) that

$$p\left(\frac{\varepsilon_1}{a^\alpha},\frac{\mu_1}{a^\beta},\frac{\varepsilon_1'}{a^\alpha},\frac{\mu_1'}{a^\beta},a\right)=\frac{1}{a^3}\,p\left(\varepsilon_1,\mu_1,\varepsilon_1',\mu_1',1\right). \tag{A.11}$$

Eq. (A.11) provides a useful test of the functional form of terms in p.

Let us illustrate this by examining the bulk pressure, which we write as

$$p^{(0)}(\varepsilon,\mu,\varepsilon',\mu',a)=T^{(0)}_{rr}(\varepsilon,\mu,a_-)-T^{(0)}_{rr}(\varepsilon',\mu',a_+), \tag{A.12}$$

where (setting $\delta=0$)

$$T^{(0)}_{rr}(\varepsilon,\mu,a)=-\frac{1}{2a^4}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\,e^{iy\tilde\tau}\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}f_l(x). \tag{A.13}$$

$T^{(0)}_{rr}(\varepsilon,\mu,a)$ is the relevant component of the stress tensor in the case that the interior medium fills all space. It must therefore have dimensions of length^{-4}, yet be independent of a. This can only be achieved if it has a factor of τ^{-4}. It is also a function of ε and μ solely through the product $\varepsilon\mu$. In order to satisfy the scaling property (A.11), it must therefore have a factor of $(\varepsilon\mu)^{\frac{3}{2}}$. These facts suggest that Eq. (A.13) be rewritten as

$$T^{(0)}_{rr}(\varepsilon\mu)=-\frac{(\varepsilon\mu)^{\frac{3}{2}}}{\tau^4}g(\theta), \tag{A.14}$$

where

$$g(\theta)=\theta^4\int_0^\infty\frac{dx}{2\pi}\,\cos(x\theta)\sum_{l=1}^{\infty}\frac{2l+1}{4\pi}f_l(x) \tag{A.15}$$

and

$$\theta=\frac{\tau}{a\sqrt{\varepsilon\mu}}. \tag{A.16}$$

By the above arguments, $g(\theta)$ must be independent of a. It must therefore be independent of its argument, that is, $g(\theta)$ must be a constant, g. To determine g, it suffices to apply Eq. (A.14) in the case of vacuum, which immediately yields $g=\frac{1}{\pi^2}$.

This value can also be confirmed by explicit evaluation of Eq. (A.15). If δ were reinstated, the method used in Eq. (2.14) could be employed. However, here we will use an alternative approach. Let

$$f_l(x,r,s)=2\frac{\partial}{\partial r}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial r}\right)\frac{s_l(rx)\,e_l(sx)}{x}, \tag{A.17}$$

where $s>r>1$, which, by construction, satisfies

$$f_l(x)=\lim_{s\to r\to 1}f_l(x,r,s). \tag{A.18}$$

Then

$$\sum_{l=1}^{\infty}(2l+1)f_l(x,r,s)=2\frac{\partial}{\partial r}\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial r}\right)\left(\frac{rs}{s-r}-\frac{1}{2x}\right)e^{-(s-r)x} \tag{A.19}$$

and

$$g=\frac{\theta^4}{8\pi^2}\lim_{s\to r\to 1}2\frac{\partial}{\partial r}\left[\frac{s-r}{(s-r)^2+\theta^2}+\left(\frac{\partial}{\partial s}-\frac{\partial}{\partial r}\right)\frac{rs}{(s-r)^2+\theta^2}\right]=\frac{1}{\pi^2}. \tag{A.20}$$

Finally, it now follows from Eq. (A.12) that

$$p^{(0)} = \frac{(\varepsilon'\mu')^{\frac{3}{2}} - (\varepsilon\mu)^{\frac{3}{2}}}{\pi^2\tau^4}, \tag{A.21}$$

which generalizes Eq. (2.15).

Appendix B. Calculation of $T_\theta{}^\theta$

Here we will examine the divergence structure of the transverse components of the stress tensor as the surface of the sphere is approached. In a medium with constant ε and μ, the stress tensor is defined by the dyadic

$$\mathbf{T} = \mathbf{1}\frac{1}{2}(\varepsilon E^2 + \mu H^2) - (\varepsilon \mathbf{EE} + \mu \mathbf{HH}), \tag{B.1}$$

which yields for the transverse components

$$T_\theta{}^\theta = \frac{1}{2}\varepsilon(E_r^2 + E_\phi^2 - E_\theta^2) + \frac{1}{2}\mu(H_r^2 + H_\phi^2 - H_\theta^2) \to \frac{1}{2}(\varepsilon E_r^2 + \mu H_r^2), \tag{B.2}$$

where the last uses spherical symmetry. Apart from a contact (δ-function) term, the divergenceless Green's dyadic $\boldsymbol{\Gamma}' = \boldsymbol{\Gamma} + \mathbf{1}/\varepsilon$ can be constructed as follows:

$$\boldsymbol{\Gamma}'(\mathbf{r}, \mathbf{r}') = \sum_{lm}\left[\omega^2\mu F_l(r, r')\mathbf{X}_{lm}(\Omega)\mathbf{X}_{lm}(\Omega')^* - \frac{1}{\varepsilon}\boldsymbol{\nabla} \times G_l(r, r')\mathbf{X}_{lm}(\Omega)\mathbf{X}_{lm}(\Omega')^* \times \overleftarrow{\boldsymbol{\nabla}}'\right], \tag{B.3}$$

where the vector spherical harmonics are defined by $\mathbf{X}_{lm} = (l(l+1))^{-1/2}\mathbf{L}Y_{lm}$, $\mathbf{L} = \mathbf{r} \times (1/i)\boldsymbol{\nabla}$. Using the completeness property of the spherical harmonics we immediately obtain

$$T_\theta{}^\theta = -\frac{1}{16\pi^2 a r^2}\int_{-\infty}^{\infty} dy\, e^{iy\tilde{\tau}}\sum_{l=1}^{\infty}(2l+1)l(l+1)P_l(\cos\delta)[G_l(r, r)+F_l(r, r)], \quad x = |y|\sqrt{\varepsilon\mu}, \tag{B.4}$$

where we have retained the temporal and transverse spatial regulators $\tilde{\tau}$ and δ. This agrees with the formula (A.8) given in Ref. [7]. Now we use the uniform asymptotic expansion as also given in Appendix B of Ref. [7]. The leading approximant gives close to the surface of a homogeneous dielectric–diamagnetic ball in vacuum ($|r-a| \ll a$)

$$F_l(r, r) \sim \pm\frac{R^E(z)}{2\nu r}t(z_\pm r/a)\, e^{-2\nu|r/a-1|/t(z_\pm)}, \tag{B.5}$$

where the upper (lower) signs refer to the outside (inside) of the spherical boundary, $\nu = l + 1/2$, $z_+ = z$, $z_- = z\sqrt{\varepsilon\mu}$, $t(z) = (1+z^2)^{-1/2}$, and the reflection coefficient is

$$R^E(z) = \frac{\sqrt{\mu z_- t(z_-)} - \sqrt{\varepsilon z_+ t(z_+)}}{\sqrt{\mu z_- t(z_-)} + \sqrt{\varepsilon z_+ t(z_+)}}. \tag{B.6}$$

The TM Green's function G_l is given by the same formula with $R^E \to R^H$, obtained by interchanging ε and μ.

If we insert these approximants into Eq. (B.4) and omit the point-splitting parameters, we find outside or inside the sphere

$$T_\theta^{E\theta}(r) + T_\theta^{H\theta}(r) \sim \mp\frac{3}{64\pi^2}\frac{1}{(r-a)^4}\int_0^\infty dz\,\frac{R^E(z)+R^H(z)}{(1+z_\pm^2)^{5/2}}, \tag{B.7}$$

where the remaining integral on z is evidently convergent. So there is a quartic, nonintegrable singularity as $r \to a$. Therefore, the integral in Eq. (3.5) does not exist. Note, for a perfectly conducting sphere, $R^E \to -1$, $R^H \to +1$, so this leading singularity cancels, and the leading surface divergence goes like $(r-a)^{-3}$ in agreement with the well-known results [23–25].

A similar calculation can be carried out for the radial–radial component of the stress tensor. In that case, the leading large-ν contribution cancels, so the surface divergence is only of order $(r-a)^{-3}$

in general. For the perfect conductor case, again the TE and TM modes exhibit a further cancellation, so the divergence is only of order $(r-a)^{-2}$. [These results also follow from Eq. (3.4) for $r \neq a$.] The energy density and the transverse stress tensor components exhibit the most singular behavior, as we would expect.

The same types of cancellations, $R^E + R^H = 0$, occur for the isorefractive ball where $\varepsilon\mu = 1$, so $z_+ = z_-$. There, for example, the energy density behaves as [18]

$$u \sim \mp \frac{\mu - 1}{\mu + 1} \frac{1}{30\pi^2 (r-a)^3}, \tag{B.8}$$

as the surface is approached from the inside (outside).

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