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Scattering with Manning–Rosen potential in all partial waves

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ABSTRACT

Ordinary differential approach together with certain properties of Gaussian hypergeometric functions is exploited to construct exact analytical expressions for regular and irregular solutions in all partial waves for motion in Manning–Rosen potential. The Jost function thus obtained from the near the origin behaviour of the irregular solution is used appropriately to compute scattering phase shifts for the neutron–proton and neutron–deuteron systems for few lower partial waves. Our results are in good agreement with experimental data.

1. Introduction

The conceptual understanding of physical systems through the exact analytical solutions of wave equations in quantum mechanics provides some strong evidences to the correctness of different quantum theory. Since its inception the number of exactly solvable problems in quantum mechanics is limited but these solutions are valuable for cross checking as well as improving different models and numerical methods. Recently, many efforts have been given to the exactly solvable problems by a number of authors. The factorization method [1–5], group theoretical techniques [6–8], super-symmetric quantum mechanics [9–17], exact quantization and proper quantization rule [18–22] and shape invariance [23] are few methods adopted by these authors to solve the quantum mechanical problem. The study of exponential type potentials like the Hulthén potential [24–29], Manning–Rosen potential [30–42], Wood–Saxon potential [43–45], Eckart-type potential [46] are oldest one however, they are still used in many applications of physics to obtain

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different observables quite accurately. We plan to work with the Manning–Rosen potential in the nuclear domain to treat nucleon–nucleon and nucleon–nucleus scattering. The present text is an effort in this direction. In Section 2 we construct Jost solution and function corresponding to the potential under consideration in all partial waves. Section 3 is meant for computation and discussions on phase shifts. Section 4 is devoted for concluding remarks.

2. Jost solution and function

The S-wave Manning–Rosen potential reads as

$$V(r)=b^{-2}\left[\frac{\alpha(\alpha-1)}{(1-e^{-r/b})^2}e^{-2r/b}-\frac{Ae^{-r/b}}{1-e^{-r/b}}\right] \tag{1}$$

where A and α are two dimensionless parameters. However, b has the dimension of length. For all partial wave the above potential can be written as

$$V(r)=b^{-2}\left[\frac{\delta(\delta-1)}{(1-e^{-r/b})^2}e^{-2r/b}-\frac{Ae^{-r/b}}{1-e^{-r/b}}\right] \tag{2}$$

with

$$\delta=\frac{1}{2}\left[1\pm\sqrt{1+4\left\{\alpha(\alpha-1)+\ell(\ell+1)\right\}}\right] \tag{3}$$

and ℓ takes the values $0,1,2,3,\ldots$. Here the centrifugal barrier term [31,47] is considered as $b^{-2}\frac{\ell(\ell+1)}{(1-e^{-r/b})^2}e^{-2r/b}$, for small values of r which behaves as $\frac{\ell(\ell+1)}{r^2}$. In the limit $\hbar^2/2m = 1$, the Schrödinger's wave equation for Manning–Rosen potential in all partial waves is given by

$$\left[\frac{d^2}{dr^2}+k^2-b^{-2}\left\{\frac{\delta(\delta-1)}{(1-e^{-r/b})^2}e^{-2r/b}-\frac{Ae^{-r/b}}{1-e^{-r/b}}\right\}\right]\varphi_\ell(k,r)=0. \tag{4}$$

Here $\varphi_\ell(k,r)$ stands for the on-shell regular solution. Using the transformation

$$\varphi_\ell(k,r)=b^\delta(1-e^{-r/b})^\delta\, e^{ikr}\, g_\ell(k,r) \tag{5}$$

Eq. (4) is converted to

$$\begin{aligned}&b^2(1-e^{-r/b})e^{r/b}\,g_\ell''(k,r)+\left[2\delta b+2ikb^2(1-e^{-r/b})e^{r/b}\right]g_\ell'(k,r)\\&\quad+(2ikb\delta-\delta+A)\,g_\ell(k,r)=0.\end{aligned} \tag{6}$$

By changing the variable $(1-e^{-r/b})=z$ the above equation becomes

$$z(1-z)g_\ell''(k,z)+\left[c'-(a'+b'+1)z\right]g_\ell'(k,z)-a'b'g_\ell(k,z)=0, \tag{7}$$

where

$$a'=\delta-ikb+(\delta^2-k^2b^2-\delta+A)^{1/2}, \tag{8}$$

$$b'=\delta-ikb-(\delta^2-k^2b^2-\delta+A)^{1/2} \tag{9}$$

and

$$c'=2\delta. \tag{10}$$

Comparing Eq. (7) with the standard Gaussian hypergeometric differential equation [48–52]

$$\left[z(1-z)\frac{d^2}{dz^2}+[\gamma-(\alpha+\beta+1)z]\frac{d}{dz}-\alpha\,\beta\right]F(z)=0 \tag{11}$$

in conjunction with Eq. (5) the general solution of Eq. (4) reads as

$$\varphi_\ell(k,r)=b^\delta(1-e^{-r/b})^\delta\, e^{ikr}\ {}_2F_1\left(a',b';\ c';\ 1-e^{-r/b}\right). \tag{12}$$

Substituting $\delta = \omega + 1$ in above equation one has

$$\varphi_\ell(k,r) = b^{\omega+1}(1-e^{-r/b})^{\omega+1} e^{ikr}\ {}_2F_1\left(A', B';\ C';\ 1-e^{-r/b}\right) \tag{13}$$

with

$$A' = 1 + \omega - ikb + (\omega^2 - k^2b^2 + \omega + A)^{1/2}, \tag{14}$$

$$B' = 1 + \omega - ikb - (\omega^2 - k^2b^2 + \omega + A)^{1/2} \tag{15}$$

and

$$C' = 2\omega + 2. \tag{16}$$

Using the analytic continuation of Gaussian hypergeometric [48,49] function

$$\begin{aligned} {}_2F_1(a,b;c;z) \;=\; & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-z) \\ & + (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a,c-b;c-a-b+1;1-z) \end{aligned} \tag{17}$$

and the relation

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\ {}_2F_1(c-a,c-b;c;z) \tag{18}$$

Eq. (13) yields

$$\begin{aligned} \varphi_\ell(k,r) \;=\; & \frac{b^\omega \Gamma(2\omega+2)}{2ik}\Bigg[\frac{\Gamma(1+2ikb)}{\Gamma(A'^*)\Gamma(B'^*)}(1-e^{-r/b})^{-\omega}\, e^{ikr} \\ & \times {}_2F_1\left(A'-1-2\omega, B'-1-2\omega;\ 1-2ikb;\ e^{-r/b}\right) - \frac{\Gamma(1-2ikb)}{\Gamma(A')\Gamma(B')}(1-e^{-r/b})^{-\omega}\, e^{-ikr} \\ & \times {}_2F_1\left(A'^* - 1 - 2\omega, B'^* - 1 - 2\omega;\ 1+2ikb;\ e^{-r/b}\right)\Bigg] \end{aligned} \tag{19}$$

On comparing Eq. (19) with the standard relation between regular and irregular solutions [53]

$$\varphi_\ell(k,r) = \frac{1}{2ik}\left[\Im_\ell(-k) f_\ell(k,r) - \Im_\ell(k) f_\ell(-k,r)\right] \tag{20}$$

One can easily identify the Jost solution (irregular) $f_\ell(k,r)$ and Jost function $\Im_\ell(k)$ as

$$f_\ell(k,r) = (1-e^{-r/b})^{-\omega}\, e^{ikr}\ {}_2F_1\left(A'-1-2\omega, B'-1-2\omega;\ 1-2ikb;\ e^{-r/b}\right) \tag{21}$$

and

$$\Im_\ell(k) = \frac{b^\omega \Gamma(2\omega+2)\Gamma(1-2ikb)}{\Gamma(A')\Gamma(B')}. \tag{22}$$

Here $f_\ell(-k,r) = (f_\ell(k,r))^*$ and $\Im_\ell(-k) = (\Im_\ell(k))^*$. It is well known that near the origin behaviour of the Jost/irregular solution $f_\ell(k,r)$ represents the Jost function $\Im_\ell(k)$ [53–55] according as

$$\Im_\ell(k) = (2\ell+1) \mathop{Lim}_{r\to 0} r^\ell f_\ell(k,r). \tag{23}$$

For the potential under consideration the above relation is modified appropriately to have

$$\Im_\ell(k) = (2\omega+1) b^\omega \mathop{Lim}_{r\to 0} (1-e^{-r/b})^\omega f_\ell(k,r). \tag{24}$$

Now we verify whether $f_\ell(k,r)$ in Eq. (21) reproduces $\Im_\ell(k)$ as given in Eq. (22) under the limiting condition in Eq. (24). From Eqs. (21) and (24) one gets Eq. (22) by using [48,49]

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{25} $$

Thus, our constructed Jost solution and function are in order.

We shall make a check on our expressions for $f_\ell(k,r)$ and $\Im_\ell(k)$. Under the limiting condition $\alpha \to 0$ the effective Manning–Rosen potential [30–42] in Eq. (2) is converted to pure Hulthén-like (HL) one [26] in all partial waves with $-(A+\ell(\ell+1))/b^2 = V_{0eff}$, the effective strength of the potential. Therefore, from Eqs. (21) and (22) we obtain

$$ \underset{\alpha\to 0}{Lim}\, f_\ell(k,r) = f_{\ell HL}(k,r) = (1-e^{-\mathrm{r}/\mathrm{b}})^{-l}\, e^{ikr}\, {}_2F_1\left(-l+P, -l+Q, 1-2ikb, e^{-\mathrm{r}/\mathrm{b}}\right) \tag{26} $$

and

$$ \underset{\alpha\to 0}{Lim}\, \Im_l(k) = \Im_{lHL}(k) = \frac{b^l\Gamma(1-2ikb)\Gamma(2\ell+2)}{\Gamma(l+1+P)\Gamma(l+1+Q)} \tag{27} $$

with

$$ P = -ikb + i\,b(V_{0eff}+k^2)^{1/2};\;\; Q = -ikb - i\,b(V_{0eff}+k^2)^{1/2}. \tag{28} $$

For $\alpha = 0$ Eq. (3) may take the value $\delta = -\ell$ *or* $\ell+1$ so that $\omega = -(\ell+1)$ *or* ℓ. When $\omega = \ell$ Eqs. (14)–(16) yield

$$ A' = \ell + 1 - ikb + ib(k^2 + V_{0eff})^{1/2} = \ell + 1 + P, \tag{29} $$

$$ B' = \ell + 1 - ikb - ib(k^2 + V_{0eff})^{1/2} = \ell + 1 + Q \tag{30} $$

and

$$ C' = 2\ell + 2. \tag{31} $$

But when $\omega = -(\ell+1)$. The value of C' becomes a negative integer. Thus, it does not produce an acceptable solution. These equations are similar to Hulthén potential in all partial waves [26] except $V_{0eff} = -(A+\ell(\ell+1))/b^2$ is replaced by V_0, the strength of the conventional Hulthén potential. For pure Hulthén potential $V_0 = -A/b^2$, while V_{0eff} contains a term involving angular momentum. This arises due to the choice of centrifugal barrier as $b^{-2}\frac{\ell(\ell+1)}{(1-e^{-r/b})^2}e^{-2r/b}$ instead of $b^{-2}\frac{\ell(\ell+1)}{(1-e^{-r/b})^2}e^{-r/b}$. If one assumes the screened centrifugal barrier like that has been considered in case of Manning–Rosen potential [30–42], the proper Hulthén limits are obtained. In Appendix we present the solution of the Hulthén potential with the centrifugal barrier $b^{-2}\frac{\ell(\ell+1)}{(1-e^{-r/b})^2}e^{-2r/b}$. However, for s-wave case Eqs. (26) and (27) reproduce the desired results for Hulthén potential, reported in a number of publications [24–29].

The Jost function plays a very important role in examining the properties of both bound and scattering states for quantum mechanical systems [54–56]. The Jost function is in general a complex quantity. The zeros of the Jost function in the upper-half of the complex momentum space reproduce the bound state energies and phase of the same is the negative of the scattering phase shifts [52–56]. In a recent article [42] we have studied the (n-p) and (n-d) systems and found excellent agreement in binding energies and scattering phase shifts for s-wave Manning–Rosen potential. In the present text we apply our expression for Jost function in all partial waves to compute neutron–proton and neutron–deuteron phase shifts up to $\ell = 2$ by fixing the parameters of the potential.

From Eq. (22) the Jost function $\Im_\ell(k)$ has zeros at the poles of the gamma functions $\Gamma(A')$ or $\Gamma(B')$ with $k = i\kappa$. Thus, from the condition $A' = -n$ at $k = i\kappa$ we have

$$ \kappa = \frac{A-(n+1)^2-\omega(2n+1)}{2b(n+\omega+1)};\;\; n = 0, 1, 2, \ldots \tag{32} $$

Table 1
List of parameters for the potential.

System	State	A	b (fm)	α
n-p	$^1S_0$	0.952	1.152	−0.0043
	$^3S_1$	1.57	1.2135	0.005
	$^1P_1$	1.4505	0.83	0.005
	$^3P_0$	1.9045	0.32	0.005
	$^3P_1$	1.4505	0.78	0.005
	$^3P_2$	2.0505	0.141	0.005
	$^1D_2$	3.35	0.07	0.005
	$^3D_1$	2.90	0.44	0.005
	$^3D_2$	4.15	0.12	0.005
	$^3D_3$	3.20	0.07	0.005
n-d	$1/2^{(+)}$	2.725	2.1441	0.005
	$1/2^{(-)}$	1.40	1.71	0.005
	$3/2^{(-)}$	2.40	0.53	0.005
	$3/2^{(+)}$	8.043	1.70	0.005

and

$$E_n = -\frac{\hbar^2}{8mb^2}\left[\frac{A-(n+1)^2-\omega(2n+1)}{(n+\omega+1)}\right]^2; \quad n = 0, 1, 2, \ldots \tag{33}$$

For $\ell = 0$, $\omega = \alpha - 1$ and our Eq. (33) reproduces the results of Refs. [32,42].

3. Results and discussions

Using the parameters given in Table 1 we have computed the scattering phase shifts for different states of n-p and n-d systems and portrayed them in Figs. 1–4.

Looking closely into Fig. 1 it is noticed that our $^1S_0$ & $^3S_1$ scattering phase shifts for n-p system are in good agreement with that of Ref. [57] up to $E_{Lab} = 50$ MeV. Beyond that our results gradually increase with energy from the stand data [57].

For P- and D-states our phase shifts values for n-p systems as depicted in Figs. 2 and 3 are in reasonable agreement with the results of Gross and Stadler [57]. For $^3P_0$ state our results started increasing with energy beyond $E_{Lab} = 125$ MeV, on the other hand, the same for $^3P_1$ state agree well beyond $E_{Lab} = 125$ MeV and differs slightly in the low energy regions. The scattering phase shifts for $^3D_1$ and $^3D_3$ states portrayed in Fig. 3 are in exact agreement with that of standard data [57]. For $^1D_2$ state our results are in closed agreement with that of standard data [57] up to $E_{Lab} = 175$ MeV. Beyond 175 MeV, however, our results differ slightly from those of ref. [57]. The phase shift values for $^3D_2$ state differ at low and intermediate energies but reproduce better results beyond $E_{Lab} = 175$ MeV. The scattering phase shifts for n-d system for different states under consideration plotted in Fig. 4 are in excellent agreement with those of Ref. [58] in the entire energy range.

The associated potentials for different states of various systems are also presented in Figs. 5–8.

The nature of our phase shifts are fully consistent with those of the potentials for the concerned systems.

4. Conclusion

In one [42] of our recent articles we have computed S-wave scattering phase shifts for n-p and n-d systems using the Manning–Rosen potential up to 50 MeV and 10.5 MeV respectively and found a close agreement with the standard results [57,58]. However, for n-d system a slight difference in phase shift values was observed up to $E_{Lab} = 5$ MeV. In the present text we have used the same parameters for $^1S_0$ and $^3S_1$ states of n-p system but some new parameters, reproducing the correct binding energy of n-d system, are used to compute the scattering phase shifts for $1/2^+$ state of n-d

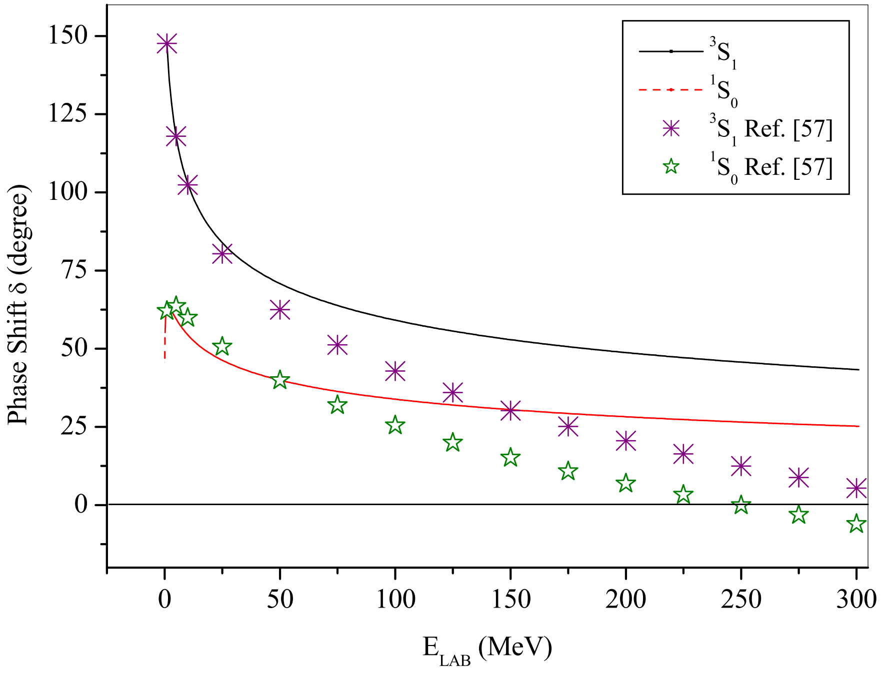

Fig. 1. (n-p) scattering phase shifts ($^1S_0$ and $^3S_1$) as a function of laboratory energy.

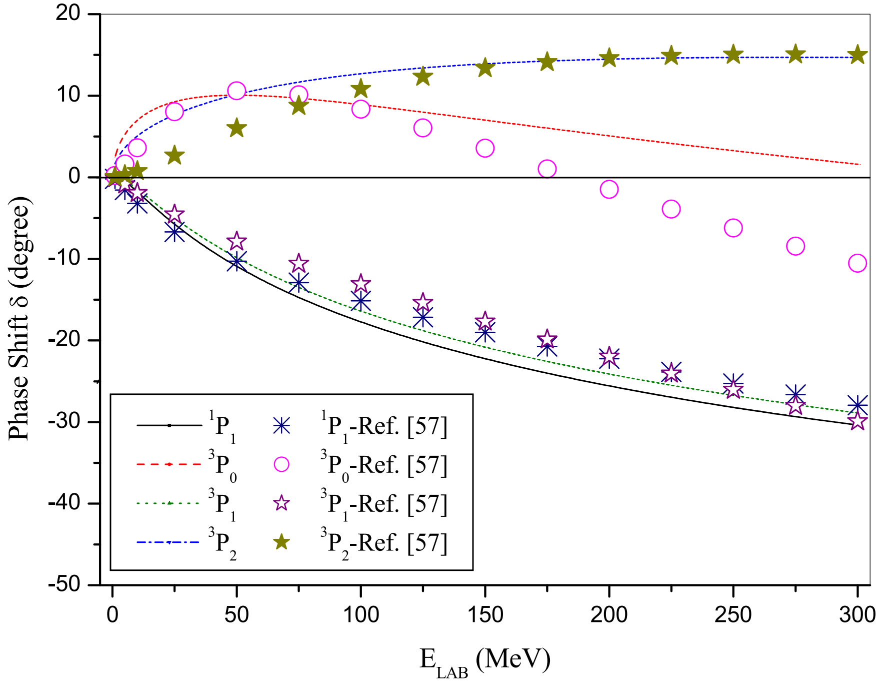

Fig. 2. (n-p) scattering phase shifts ($^1P_1$, $^3P_0$, $^3P_1$ and $^3P_2$) as a function of laboratory energy.

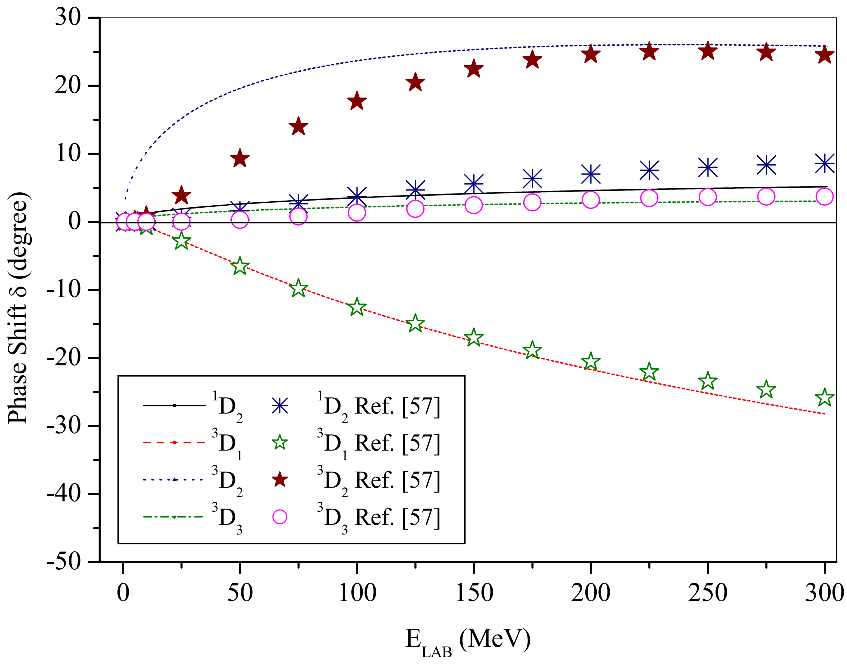

Fig. 3. (n-p) scattering phase shifts ($^1D_2$, $^3D_1$, $^3D_2$ and $^3D_3$) as a function of laboratory energy.

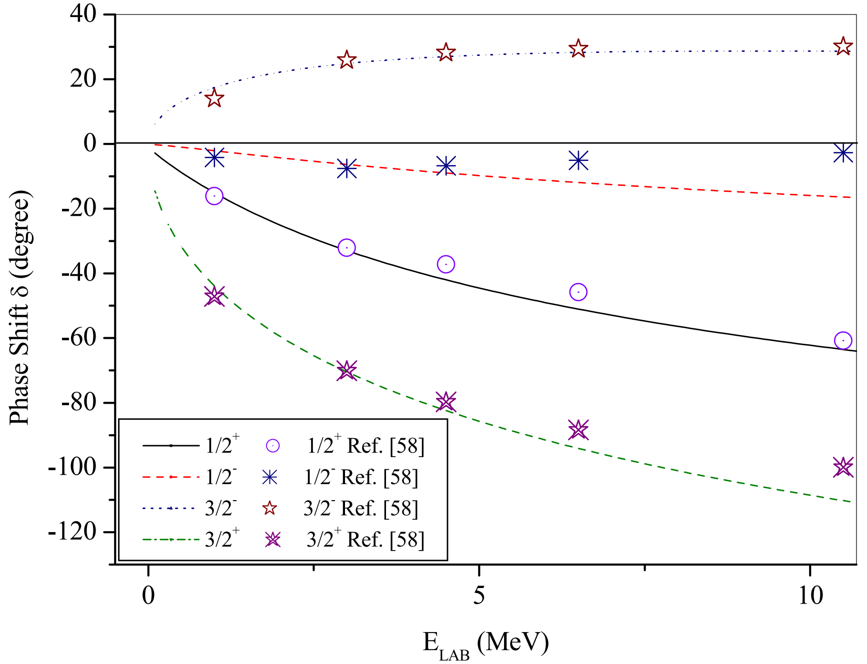

Fig. 4. (n-d) scattering phase shifts as a function of laboratory energy.

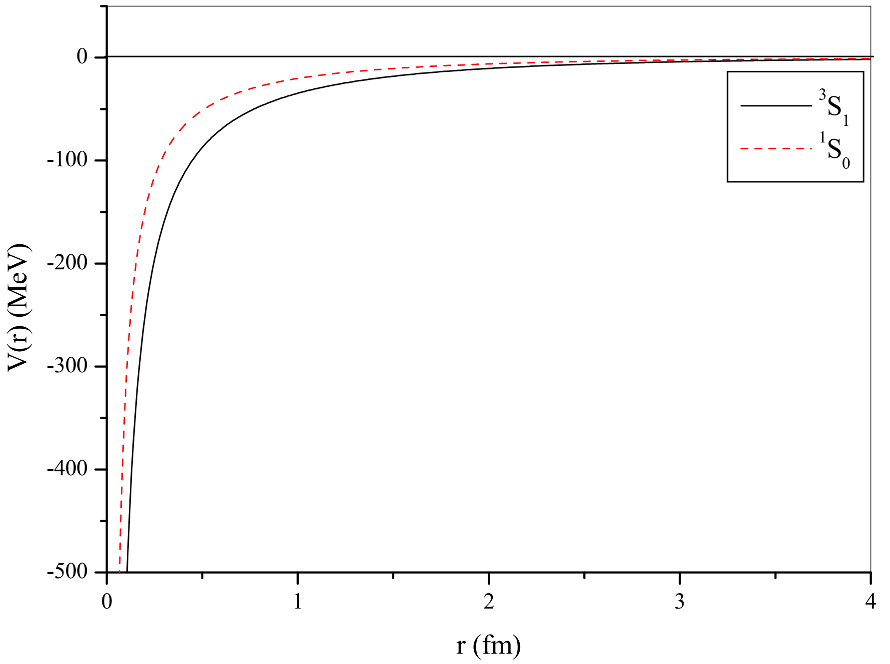

Fig. 5. S-wave (n-p) potentials as a function of r.

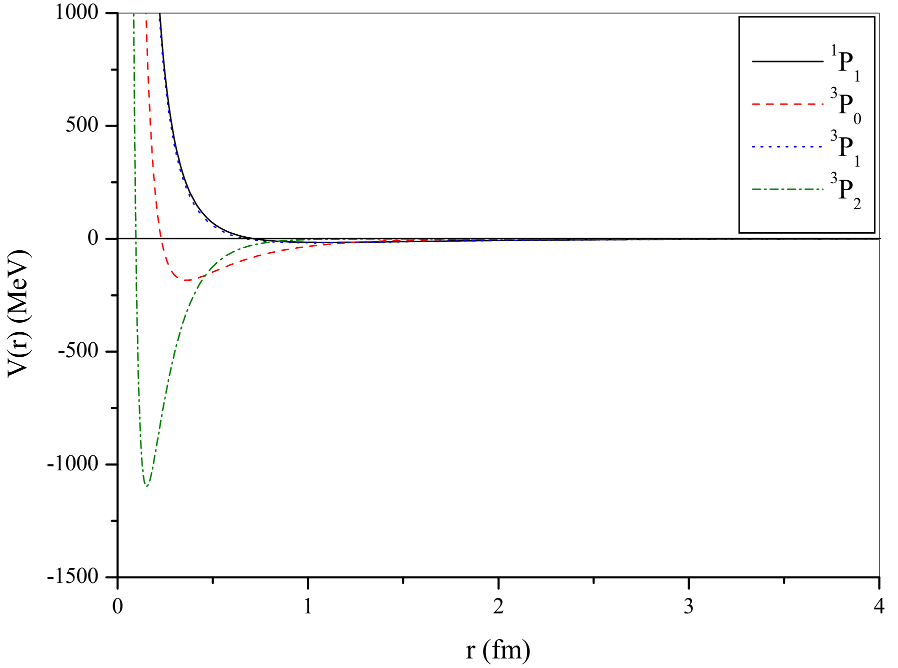

Fig. 6. P-wave (n-p) potentials as a function of r.

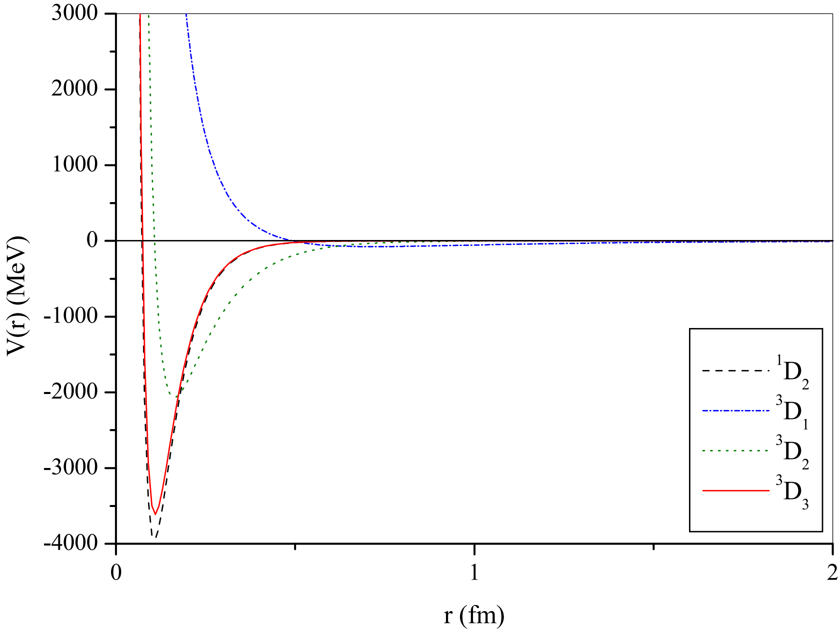

Fig. 7. D-wave (n-p) potentials as a function of r.

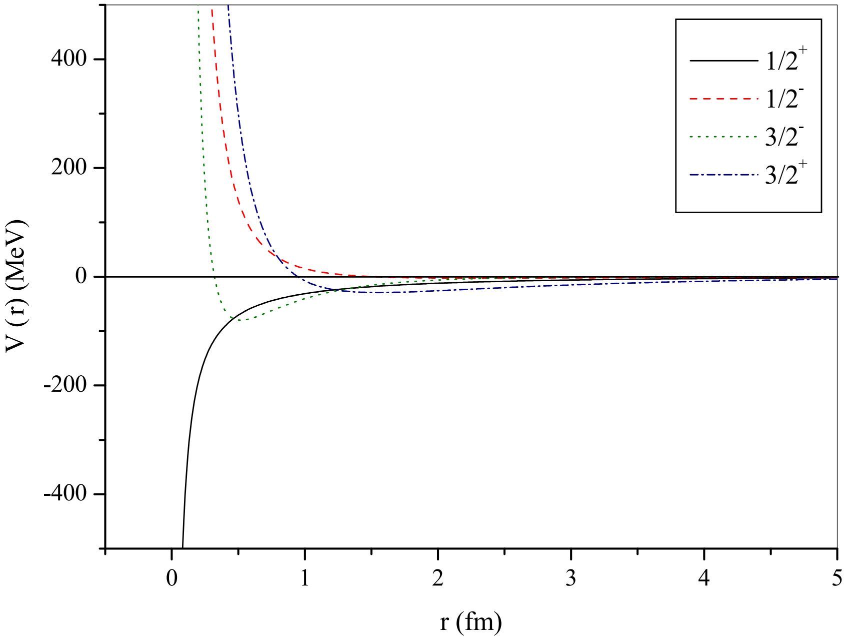

Fig. 8. (n-d) potentials as a function of r.

system. The expressions for the Jost functions are exploited for computing the scattering phase shifts for different states of n-p and n-d systems up to $\ell = 2$. With the parameters given in Table 1 we obtain good agreement in the phase shifts over the entire energy range under consideration for n-p and n-d systems. Hüber et al. [58] calculated elastic n-d scattering phases up to 19 MeV using Faddeev equations in the momentum space formalism with the realistic Bonn-B NN potential. In the current text we treat the n-d system within the two-body model of interaction and achieve excellent agreement in phase shift values with the sophisticated calculation of Hüber et al. [58]. From the foregoing discussions it is clear that three parameter Manning–Rosen potential, although used as a molecular potential, also has the ability to reproduce the various physical observables for nuclear systems. Further, it may be extended for studying complex nucleus–nucleus systems. However, to treat charged hadron system one has to construct an expression for the Jost function with Manning–Rosen Plus electromagnetic interaction. This problem is in our active consideration and will be addressed in the future. Thus, it is our belief that the present treatment with the Manning–Rosen potential may be quite interesting to nuclear physicists.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

The Schrödinger's equation for Hulthén potential with the screened centrifugal term $\xi^2 \frac{\ell(\ell+1)e^{-2\xi\,\mathrm{r}}}{\left(1-e^{-\xi\,\mathrm{r}}\right)^2}$ reads as

$$\left(\frac{d^2}{dr^2} + k^2 - V_0 \frac{e^{-\xi\,\mathrm{r}}}{1-e^{-\xi\,\mathrm{r}}} - \delta^2 \frac{\ell\,(\ell+1)\,e^{-2\xi\,\mathrm{r}}}{\left(1-e^{-\xi\,\mathrm{r}}\right)^2}\right)\varphi_{\ell H}(k,r) = 0. \tag{A.1}$$

Using the following transformation

$$\varphi_{\ell H}(k,r) = e^{ikr}\frac{\left(1-e^{-\xi\,\mathrm{r}}\right)^{(\ell+1)}}{\xi^{(\ell+1)}} F_\ell(k,r) \tag{A.2}$$

Eq. (A.1) converts to

$$\begin{aligned} &e^{\xi r}\left(1-e^{-\xi r}\right)\frac{d^2F_\ell(k,r)}{dr^2} + \left\{2ike^{\xi r}\left(1-e^{-\xi r}\right) + 2\,(\ell+1)\,\xi\right\}\frac{dF_\ell(k,r)}{dr} \\ &\quad + \left\{2ik\,(\ell+1)\,\xi - (\ell+1)\,\xi^2 - V_0\right\}F_\ell(k,r) = 0. \end{aligned} \tag{A.3}$$

By changing the variable $1 - e^{-\xi r} = z$, the above equation becomes

$$z\,(1-z)\frac{d^2F_\ell(k,z)}{dz^2} + \left[c' - \left(a' + b' + 1\right)z\right]\frac{dF_\ell(k,z)}{dz} - a'b'F_\ell(k,z) = 0, \tag{A.4}$$

where

$$a' = 1 + \ell - \frac{ik}{\xi} + \left(\ell^2 + \ell - \frac{k^2}{\xi^2} - \frac{V_0}{\xi^2}\right)^{1/2}, \tag{A.5a}$$

$$b' = 1 + \ell - \frac{ik}{\xi} - \left(\ell^2 + \ell - \frac{k^2}{\xi^2} - \frac{V_0}{\xi^2}\right)^{1/2} \tag{A.5b}$$

and

$$c' = 2\ell + 2. \tag{A.5c}$$

Comparing Eq. (A.4) with the standard Gaussian hypergeometric differential equation [48,49]

$$\left[z(1-z)\frac{d^2}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{d}{dz} - \alpha\,\beta\right]F(z) = 0 \tag{A.6}$$

along with the Eq. (A.2) the general solution of Eq. (A.1) reads as

$$\varphi_{\ell H}(k,r) = \frac{(1-e^{-\xi r})^{\ell+1}}{\xi^{\ell+1}} e^{ikr}\ {}_2F_1\left(a',b';\ c';\ 1-e^{-\xi r}\right). \tag{A.7}$$

Using the analytic continuation of Gaussian hypergeometric function [48,49]

$$\begin{aligned}{}_2F_1(\alpha,\beta;\gamma;z) =\ & \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} {}_2F_1(\alpha,\beta;\alpha+\beta-\gamma+1;1-z) \\ & + (1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z)\end{aligned} \tag{A.8}$$

and the relation

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\ {}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z) \tag{A.9}$$

Eq. (A.7) yields

$$\begin{aligned}\varphi_{\ell H}(k,r) =\ & \frac{1}{2ik}\xi^{-\ell}\left[\frac{\Gamma(2\ell+2)\Gamma\left(1+\frac{2ik}{\xi}\right)}{\Gamma(a'^*)\Gamma(b'^*)}(1-e^{-\xi r})^{-\ell}\, e^{ikr}\right. \\ & \times\ {}_2F_1\left(a'-2\ell-1, b'-2\ell-1; 1-\frac{2ik}{\xi};\ e^{-\xi r}\right) - \frac{\Gamma(2\ell+2)\Gamma\left(1-\frac{2ik}{\xi}\right)}{\Gamma(a')\Gamma(b')} \\ & \left.\times\ (1-e^{-\xi r})^{-\ell}\, e^{-ikr}\ {}_2F_1\left(a'^*-2\ell-1, b'^*-2\ell-1; 1+\frac{2ik}{\xi};\ e^{-\xi r}\right)\right].\end{aligned} \tag{A.10}$$

On comparing Eq. (A.10) with the standard relation between regular and irregular solutions [53]

$$\varphi_{\ell H}(k,r) = \frac{1}{2ik}\left[\Im_{\ell H}(-k) f_{\ell H}(k,r) - \Im_{\ell H}(k) f_{\ell H}(-k,r)\right] \tag{A.11}$$

one can easily identify the Jost solution (irregular) $f_{\ell H}(k,r)$ and Jost function $\Im_{\ell H}(k)$ as

$$f_{\ell H}(k,r) = (1-e^{-\xi r})^{-\ell}\, e^{ikr}\ {}_2F_1\left(a'-2\ell-1, b'-2\ell-1;\ 1-\frac{2ik}{\xi};\ e^{-\xi r}\right) \tag{A.12}$$

and

$$\Im_{\ell H}(k) = \xi^{-\ell}\frac{\Gamma(2\ell+2)\Gamma\left(1-\frac{2ik}{\xi}\right)}{\Gamma(a')\Gamma(b')}. \tag{A.13}$$

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