Probabilistic Engineering Mechanics 59 (2020) 103015

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

A copula-based Gaussian mixture closure method for stochastic response of nonlinear dynamic systems

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ARTICLE INFO

Keywords:
Moment equation
Non-Gaussian closure
Copula
Gaussian mixture model
Random vibration

ABSTRACT

Gaussian closure method is commonly used in the analysis of nonlinear stochastic systems. However, Gaussian closure may lead to unacceptable errors when system response is very much different from being Gaussian, and accuracy of the method decreases as the nonlinearity of the system increases. The need for better accuracy in strongly non-linear problems has caused the development of non-Gaussian closure schemes. In this paper, we develop a new copula-based Gaussian mixture closure method for randomly excited nonlinear systems. Our method relies on the assumption of marginal PDF of response in terms of finite Gaussian mixture model, and the derivation of joint PDF with aid of dependence modeling of Gaussian copula. By substituting the non-Gaussian PDF representation into moment equations of nonlinear system, we further develop an optimization-based closure scheme for the solution of the unknown parameters in joint PDF. In this way, PDF and thus, moments of response of highly nonlinear system can be described in a more flexible and robust way. Effectiveness of the new closure method is demonstrated by a nonlinear and a Duffing oscillator that are subjected to Gaussian white noise. The results are compared with the Gaussian closure and exact solution. It has been shown that Gaussian closure is a special case of the new closure method, and accuracy of Gaussian closure is the lower bound of that of the new closure method.

1. Introduction

The study of behavior of non-linear dynamic systems under random excitations has attracted considerable attention due to their numerous applications in various fields of engineering and physical sciences [1–4]. Generally, the response of many real mechanical and structural systems to external random excitations can be described by non-linear stochastic differential equations. Since only in a very few cases can an exact closed form solutions be obtained, a variety of approximate techniques have been developed to solve such equations [5]. Among them, the Gaussian equivalent linearization is presently the most widely used tool because the method allows to use up the available analytic results from stochastic linear systems [6–8]. An alternative class of methods relies on the derivation of moment equations, which describe the evolution of the response statistical moments. However, the difficulty encountered in using this method is that the equations for the statistical moments of the response are not closed, and they form an infinite hierarchy which cannot be solved exactly. This requires the adoption of closure schemes, which essentially truncate the infinite system of moment equations to a finite one. One of the simplest schemes along this line is Gaussian closure. It has been shown that there is an equivalence between the Gaussian equivalent linearization method and the moment equation method applied in conjunction with Gaussian closure scheme [9,10]. This is due to the shared hypothesis of Gaussianity of the response process, as a result of which both methods lead to the same resolving equations and, consequently, to the same results.

Unfortunately, accuracy of the Gaussian closure, or equivalently, the Gaussian equivalent linearization, decreases as the nonlinearity increases and may lead to unacceptable errors in the second moments [11–13]. Further, when the system response is very much different from being Gaussian, both methods do not perform as well and may result in erroneous predictions. In practice, it is sometimes inevitable that one is concerned with strong non-linear systems whose non-linearities play the dominant roles. Different from those weak non-linear systems, the response of strong non-linear systems is far from being Gaussian. Thus there are usually some great errors resulting from a Gaussian assumption. The need for better accuracy in strongly non-linear problems has caused the development of non-Gaussian closure scheme, that do not rely on a Gaussian assumption, as evidenced in the literature [14]. The basic idea of the scheme is to assume a non-Gaussian probability density function with adjustable parameters for the response and to use the moment relations derived from system equations to obtain equations for the unknown parameters. The

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https://doi.org/10.1016/j.probengmech.2019.103015
Received 18 August 2019; Received in revised form 14 October 2019; Accepted 26 November 2019
Available online 2 December 2019
0266-8920/

truncation of moments and semi-moments (cumulants) for the non-Gaussian closure were most extensively investigated. Non-Gaussian closure scheme based on truncated Gram–Charlier or Edgeworth expansion was developed for improving stationary moment accuracy [15,16]. However, the method is only valid for nonlinear systems with linear damping. In addition, negative values in PDF of response may result with this method, which is inconsistent with probability theory. In order to satisfy the non-negativity condition, multi-Gaussian closure method was subsequently developed, in which the approximate PDF is constructed with the linear superposition of some two-dimensional Gaussian PDFs [17]. However, this PDF assumption produces the correlation between stationary displacement response and velocity response, which is inconsistent with random vibration theory. Further, the multi-Gaussian closure method also produces a large amount of nonlinear algebraic equations, and the solution of these highly nonlinear algebraic equations is a tedious problem. Although exponential closure method was further developed to alleviate this problem by the same author, huge amount of computational cost still limits the application of the method [18,19]. In a very recent paper, a moment-equation-copula-based method was developed, in which copula function was introduced to establish the correlated structures between response quantities and excitation at different time points [20]. Since the joint distribution was approximated through the assumed correlation structure, the equivalent linearization model of the nonlinear system can be determined according to the joint distribution. No wonder, the choice of trial density functions is of course open and important, and as a result, various non-Gaussian closure schemes are still underdeveloped in light of capturing the complex behavior of response accuracy and stability in robustness [21,22].

The goal of this paper is the development of a copula-based Gaussian mixture closure methodology to overcome limitations of the present non-Gaussian closure schemes. In the proposed method, we start from assuming the marginal PDF of response in terms of Gaussian mixture model as this model has shown strong capacity in representing a general probability density with complex shape. With this assumption, a *bona fide* estimated density can be guaranteed just by imposing simple conditions on the weights of Gaussian mixture model. Next, by virtue of the superior performance of copula function in dependence modeling, we formulate the joint PDF of response of nonlinear system based on the assumed marginal PDF and Gaussian copula. Different from [20], copula function is employed to establish the correlation between displacement response and velocity response at the same time point so that the joint PDF of response can be conveniently represented. Based on the property of zero mean stationary response in most nonlinear systems, the formulated joint PDF representation is further simplified to decrease the computational cost. Lastly, by substituting the assumed PDF representation into the moment equations of nonlinear system, we develop an optimization-based closure procedure for the solution of the unknown parameters in joint PDF. In this way, the number of unknown parameters in PDF of response is not limited to that of moment equations, and the resulting PDF is flexible in that it has the potential to converge to a wide class of non-Gaussian distributed PDF. In contrast to most of the existing non-Gaussian closure schemes, the procedure for solving sets of highly nonlinear algebra equations can be avoided.

The remainder of the paper is organized as follows. Section 2 first briefly introduces the problem of moments of response of nonlinear system. A new copula-based Gaussian mixture closure scheme is then developed in Section 3. Numerical examples are finally given to demonstrate the proposed method. Comparisons of the developed method with the Gaussian closure and the FPK method are made.

2. Problem formulation

Consider a randomly excited non-linear multi-dimensional system whose dynamic behavior, described by the $n \times 1$ vector of state variables $\mathbf{X}(t)$, is governed by the following Itô stochastic differential equation:

$$d\mathbf{X}(t) = \mathbf{F}(\mathbf{X}(t), t)\, dt + \mathbf{G}(\mathbf{X}(t), t)\, d\mathbf{W}(t) \tag{1}$$

where $\mathbf{F}(\mathbf{X}(t), t)$ is an $n \times 1$ vector representing the deterministic influences in the model, and $\mathbf{W}(t)$ is a zero-mean $m \times 1$ vector of mathematical idealization of Gaussian white noise processes which influence the model through the $n \times m$ matrix $\mathbf{G}(X(t), t)$. Function $\mathbf{F}$ and $\mathbf{G}$ are generally non-linear; however, their functional forms are assumed to be deterministic. Intensity of $\mathbf{W}(t)$ is given as

$$E\left[d\mathbf{W}(t)\, d\mathbf{W}^T(t+\tau)\right] = \mathbf{D}\delta(\tau) \tag{2}$$

where $\delta(\cdot)$ denotes the Dirac's delta function, and $\mathbf{D} = 2\pi\mathbf{S}$ is the constant strengths of the white noise processes, $\mathbf{S}$ being the cross-power spectral density matrix of $\mathbf{W}(t)$.

It is known that the response $\mathbf{X}(t)$ of the dynamic system as described in Eq. (1) is a Markov vector and the probability density of the Markov vector is governed by the FPK equation. Further, the statistic information characterized by the FPK equation can be expressed through the response governed by the following moment propagation equations

$$\frac{\partial E[\gamma(\mathbf{x})]}{dt} = \sum_{i=1}^{n} E\left[f_i(\mathbf{X}, t)\frac{\partial \gamma(\mathbf{x})}{\partial X_i}\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left\{\left[\mathbf{GDG}^T\right]_{ij}\frac{\partial^2 \gamma(\mathbf{x})}{\partial X_i \partial X_j}\right\} \tag{3}$$

where the Kth-order moment of state-variables is written as $\gamma(X) = \prod_{i=1}^{n} X_i^{k_i}$ for nonnegative integers $k_i (i = 1, 2, \ldots, n)$, K being the summations of k_i, and $f_i(\mathbf{X}, t)$ is the ith component of $\mathbf{F}(\mathbf{X}(t), t)$. In general, the nonlinear model formula in Eq. (3) will generate an infinite hierarchy of coupled moment equations, which can be illustrated by re-expressing Eq. (3) in a more suggestive form as

$$\dot{M}_K = F_K\left(M_1, M_2, \cdots, M_K, M_{K+1}, \cdots\right) \tag{4}$$

where $\left\{M_1, M_2, \cdots\right\}$ denote the infinite set of moments. As a result, suitable closure schemes are required to truncate the infinite set of coupled moment equations in the determination of approximate solution of nonlinear stochastic differential equation in Eq. (1).

3. The copula-based Gaussian mixture closure scheme

3.1. Copula and Sklar's theorem

Copula is a multivariate probability distribution function with uniformly distributed marginals. It offers a flexible way of describing nonlinear dependence among multi-variate data in isolation from their marginal probability distributions [23]. Mathematically, a copula function $C(u_1, \ldots, u_k)$ is the k-dimensional probability distribution on a unit hypercube $[0, 1]^k$ with uniform marginal probability distributions on $[0, 1]$, and is defined as [24]

$$C(u_1, \ldots, u_k) = \Pr[U_1 \le u_1, \ldots, U_k \le u_k] \tag{5}$$

where u_i represents a sample of a standard uniform random variable U_i $(i = 1, \ldots, k)$, and $\Pr(\cdot)$ represents the probability. In this study, we deal with bivariate distributions – the formulas are given for that case – though they can be extended to higher dimensions.

Let us consider the joint CDF of two random variables X_1 and X_2, $F(x_1, x_2) = \Pr[X_1 \le x_1, X_2 \le x_2]$, continuous marginal probability distributions of which are denoted by $F_1(x_1)$ $(= u_1)$ and $F_2(x_2)$ $(= u_2)$, respectively. The copula function uniquely describes the dependence structure and it is independent of the marginals if those are continuous. Sklar's theorem establishes the connection among $F(x_1, x_2)$, $F_1(x_1)$, and $F_2(x_2)$ by using the copula function $C(u_1, u_2)$ as [25]

$$F(x_1, x_2) = C\left[F_1(x_1), F_2(x_2)\right] = C(u_1, u_2) \tag{6}$$

It indicates that the joint probability distribution of the two random variables can be characterized by a copula function in terms of their marginal distributions. An important implication of the Sklar's theorem is that marginal modeling and dependence modeling can be carried out separately. From Eq. (6), the joint PDF of X_1 and X_2, $p(x_1, x_2)$, can be obtained as

$$p\left(x_1, x_2\right) = p_1\left(x_1\right) p_2\left(x_2\right) c\left[F_1(x_1), F_2(x_2)\right] \tag{7}$$

where $p_1(x_1)$ and $p_2(x_2)$ are the marginal PDFs of X_1 and X_2, respectively, and $c\left[F_1(x_1), F_2(x_2)\right]$ is the copula density function, which is given by

$$c\left[F_1(x_1), F_2(x_2)\right] = c(u_1, u_2) = \partial^2 C(u_1,u_2)/\partial u_1 \partial u_2 \tag{8}$$

Theoretically, the joint CDF and PDF of X_1 and X_2 can be determined by Eqs. (6) and (7) as long as the marginal distributions and the copula function are known.

Since copulas fully describe multivariate dependencies, it is natural to introduce dependence measures based on the copula only, and not on the marginals. Such dependence measures include Pearson correlation coefficient, Spearman's correlation coefficient, and Kendall's correlation coefficient, etc. Among them, the most popular one is the classical Pearson linear correlation coefficient since this measure is strongly related to the prevalence of linear correlation model in the simulation of multivariate data. Other copula based measures of pairwise concordance exist, as well as multivariate extensions [26].

Many copulas such as Gaussian copula, t copula, Frank copula, Gumble copula and Clayton copula, etc, can be used to describe the dependence between random variables. A list of classical families of copulas and their properties can be found in [26]. In the framework of Pearson dependence modeling, the use of Gaussian copula and t copula within a class of elliptical copulas is prevalent. For a bi-variate case, the Gaussian copula with the linear correlation coefficient $-1 \le \rho \le 1$, which represents the strength of correlation, is given by

$$c_\rho^N\left(u_1, u_2\right) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(\frac{2\rho\Phi^{-1}\left(u_1\right)\Phi^{-1}\left(u_2\right) - \rho^2\left(\Phi^{-1}\left(u_1\right)^2 + \Phi^{-1}\left(u_2\right)^2\right)}{2(1-\rho)^2}\right) \tag{9}$$

where $c_\rho^N\left(u_1, u_2\right)$ is the density of Gaussian copula, and $\Phi(\cdot)$ represents the standard cumulative distribution function. The bi-variate t copula with the correlation coefficient ρ and the degree-of freedom parameter λ, $c_{\rho,\lambda}^t\left(u_1, u_2\right)$, is given by

$$c_{\rho,\lambda}^t\left(u_1, u_2\right) = \rho^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\lambda+2}{2}\right)\Gamma\left(\frac{\lambda}{2}\right)}{\left[\Gamma\left(\frac{\lambda+2}{2}\right)\right]^2} \frac{\left[1 + \frac{\zeta_1^2+\zeta_2^2-2\rho\zeta_1\zeta_2}{\lambda(1-\rho)^2}\right]^{-\frac{\lambda+2}{2}}}{\prod_{i=1}^2\left(1+\frac{\zeta_i^2}{\lambda}\right)^{-\frac{\lambda+2}{2}}} \tag{10}$$

where $\Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt$ is the Gamma function satisfying $\Gamma(p+1) = p\Gamma(p)$, $\zeta_i = t_\lambda^{-1}\left(u_i\right)(i=1,2)$, $t_\lambda(\cdot)$ represents the cumulative distribution function of the unary standard t distribution. It has been shown that Gaussian copula is symmetrical, and is a limiting case of t copula function [24].

3.2. Copula-based PDF representation

It has been mentioned that the choice of trial density of response is an important issue in various version of non-Gaussian closure schemes. As shown in [16], when an inappropriate choice is made the approximate solution may become worse although the order of the method goes higher. In this study, the non-Gaussian marginal probability density of response is expressed in terms of finite mixture Gaussian distribution as this model offers a flexible and robust tool in representing general probability densities that show complex shapes, especially for dynamic responses of nonlinear systems [27].

Mathematically, PDF of a Gaussian mixture model is given by [28]

$$p_{GM}(x, \mathbf{v}) = \sum_{k=1}^{K} \phi_k f_{\mathcal{N}}\left(x \mid \mu_k, \sigma_k^2\right) \tag{11}$$

where K denotes the number of Gaussian densities in the mixture, ϕ_k, $k = 1, \ldots, K$ are relative weights of the Gaussian densities satisfying nonnegative and sum-to-one, i.e., $\sum_{k=1}^K \phi_k = 1$, and $\phi_k \ge 0$ for $\forall k$, and $f_{\mathcal{N}}\left(x \mid \mu_k, \sigma_k^2\right)$ denotes the Gaussian PDF with mean μ_k and standard deviation σ_k. Thus, the distribution parameters of the Gaussian mixture model are summarized as $\mathbf{v} = \left\{\phi_1, \ldots, \phi_K, \mu_1, \ldots, \mu_K, \sigma_1, \ldots, \sigma_K\right\}$. Through an appropriate choice of its components and weights, a Gaussian mixture model is sufficiently able to model quite complex distributions.

By means of the Gaussian mixture model in Eq. (11), marginal probability densities of response of nonlinear system in Eq. (1), i.e., PDF of displacement response $X(t)$, and PDF of velocity response $\dot{X}(t)$, can be expressed in the form

$$f_X(x) = \sum_{i=1}^{m} \phi_i f_{\mathcal{N}}\left(x \mid 0, \sigma_i^2\right) \tag{12}$$

and

$$f_{\dot{X}}(\dot{x}) = \sum_{j=1}^{n} \phi_j f_{\mathcal{N}}\left(\dot{x} \mid 0, \sigma_j^2\right) \tag{13}$$

respectively, where i and j are the number of Gaussian densities. Since stationary response of most nonlinear system has zero mean, without loss of generality, mean value of each Gaussian component in Eqs. (12) and (13) can be assumed to be zero. According to Sklar's theorem, once the marginal PDF has been determined, the joint PDF of response of the nonlinear system can be further represented with aid of the dependence structure through Gaussian copula as

$$f(x, \dot{x}) = f_X(x) f_{\dot{X}}(\dot{x})\, c\left(F_X(x) F_{\dot{X}}(\dot{x})\right) \tag{14}$$

where the marginal CDF $F_X(x)$ and $F_{\dot{X}}(\dot{x})$ can be directly obtained from PDF of displacement response $X(t)$ and velocity response $\dot{X}(t)$, respectively. It has to be noted that the joint PDF of response depends on the selection of copula functions since the dependence structure between the displacement response $X(t)$ and the velocity response $\dot{X}(t)$ has been fully modeled in the copula function $c\left(F_X(x) F_{\dot{X}}(\dot{x})\right)$, as shown in Eq. (14).

3.2.1. The selection of copula

Theoretically, any available copula functions can be used in the representation of the joint PDF of response as shown in Eq. (14), and various copula functions may lead to different accuracy of approximations. Although the effect of the selection of copula functions on the accuracy of the results can be eliminated to some extent by a subsequent parameter optimization procedure, as will be shown in the following, the criteria for the selection of copula in the context of nonlinear random vibration analysis still needs to be examined. In order to accommodate the classical linear correlation structure between displacement response and velocity response in random vibration theory, the Pearson dependence measure is preferred to be adopted. In this context, Gaussian copula and t copula are the only left suitable candidates among various available copulas in the estimation of PDF of response of nonlinear systems as they readily describe the linear dependence model, and the values of the correlation coefficients can approach one. Further, since these two copula functions involve different number of unknown parameters, the copula should be selected so as the unknown parameters could be determined according to the available knowledge of the nonlinear system. In addition, copula function should make the form of joint PDF of response as concise as possible so that the subsequent copula-based closure method can be conveniently generalized.

It is worth mentioning that the only available information of an arbitrary randomly excited nonlinear system is that the stationary response $X(t)$ and $\dot{X}(t)$ are statistical uncorrelated random variables at arbitrary time point, i.e., $E\left(X\dot{X}\right) = E\left(\dot{X}X\right) = 0$. Under this condition, we firstly consider the Gaussian copula in the context of representation of joint PDF of response. By substituting (9) into Eq. (14), the joint PDF of response yields

$$\begin{aligned} f(x,\dot{x}) &= f_X(x) f_{\dot{X}}(\dot{x}) c_\rho^N \left(F_X(x) F_{\dot{X}}(\dot{x})\right) \\ &= \frac{f_X(x) f_{\dot{X}}(\dot{x})}{\sqrt{1-\rho^2}} \\ &\times \exp\left(\frac{2\rho\Phi^{-1}\left(F_X(x)\right)\Phi^{-1}\left(F_{\dot{X}}(\dot{x})\right) - \rho^2\left(\Phi^{-1}\left(F_X(x)\right)^2 + \Phi^{-1}\left(F_{\dot{X}}(\dot{x})\right)^2\right)}{2\left(1-\rho^2\right)}\right) \end{aligned} \tag{15}$$

Obviously, there exists only one unknown parameter, known as linear correlation coefficient ρ of $X(t)$ and $\dot{X}(t)$, in the representation of joint PDF of response as long as the marginal PDFs of response can be determined. In this case, coefficient ρ can be derived by considering the fact that $E\left(X\dot{X}\right) = E\left(\dot{X}X\right) = 0$. Note that covariance matrix of displacement response $X(t)$ and velocity response $\dot{X}(t)$ can be expressed as

$$\sum = \begin{bmatrix} E\left(X^2\right) & E\left(X\dot{X}\right) \\ E\left(\dot{X}X\right) & E\left(\dot{X}^2\right) \end{bmatrix} = \begin{bmatrix} E\left(X^2\right) & 0 \\ 0 & E\left(\dot{X}^2\right) \end{bmatrix} \tag{16}$$

which indicates that linear correlation coefficient between $X(t)$ and $\dot{X}(t)$ is zero, or equivalently, the density function of Gaussian copula in the joint PDF of response $c_\rho^N\left[F_X(x), F_{\dot{X}}(\dot{x})\right] = 1$. In this way, the dependence structure between the marginal PDFs $f_X(x)$ and $f_{\dot{X}}(\dot{x})$ can be conveniently determined via the Gaussian copula, and correspondingly, the joint PDF of response of nonlinear system in Eq. (15) can be further simplified as

$$f(x,\dot{x}) = f_X(x) f_{\dot{X}}(\dot{x}) = \sum_{i=1}^{m} \phi_i f_{\mathcal{N}}\left(x \mid 0, \sigma_i^2\right) \cdot \sum_{j=1}^{n} \phi_j f_{\mathcal{N}}\left(\dot{x} \mid 0, \sigma_j^2\right) \tag{17}$$

From the joint PDF representation in Eq. (17), it can be deduced that displacement response and velocity response are statistical independent variables. This is a direct result from the fact that linear correlation coefficient in Gaussian copula equals zero. However, this independence structure between $X(t)$ and $\dot{X}(t)$ cannot be assumed a *priori* because only uncorrelation condition can be imposed on these two variables, and one has to further determine the correlation coefficients according to some posterior knowledge. In this regard, it is rational to firstly assume a general PDF formulation and then to derive the correlation structure via Gaussian copula to represent joint PDF of the response.

We next examine the case when another candidate, t copula, is used in the representation of joint PDF of response. By substituting (10) into Eq. (14), we then have

$$\begin{aligned} f(x,\dot{x}) &= f_X(x) f_{\dot{X}}(\dot{x}) c_{\rho,\lambda}^t\left(F_X(x), F_{\dot{X}}(\dot{x})\right) \\ &= f_X(x) f_{\dot{X}}(\dot{x}) \rho^{-\frac{1}{2}} \frac{\Gamma\left(\frac{\lambda+2}{2}\right)\Gamma\left(\frac{\lambda}{2}\right)}{\left[\Gamma\left(\frac{\lambda+2}{2}\right)\right]^2} \frac{\left[1 + \frac{\zeta_1^2+\zeta_2^2-2\rho\zeta_1\zeta_2}{\lambda(1-\rho)^2}\right]^{-\frac{\lambda+2}{2}}}{\prod_{i=1}^2\left(1+\frac{\zeta_i^2}{\lambda}\right)^{-\frac{\lambda+2}{2}}} \end{aligned} \tag{18}$$

Note that there exist two unknown parameters, i.e., the linear correlation coefficient ρ and the degree-of freedom parameter λ, in the representation of t copula-based joint PDF of response. In this case, ρ and λ cannot be determined simultaneously with the only condition $E\left(X\dot{X}\right) = E\left(\dot{X}X\right) = 0$, and as a result, one has to assume addition condition to determine these two parameters, which may lead to errors. More importantly, even if ρ and λ can be determined, the joint PDF of response in Eq. (18) cannot be expressed separately as that in Eq. (17). Consequently, the form of the t copula-based joint PDF of response will be too complicated to develop a subsequent truncation scheme.

We emphasize that, although copulas from different families can all be used to model the dependence structure between displacement response and velocity response theoretically, one has to determine the relations between the linear correlation and other dependence measures, i.e., Spearman's correlation, or Kendall's correlation, which may require additional assumptions. Therefore, the Gaussian copula is selected in this study since the Gaussian copula-based PDF representation in Eq. (17) is a justified and a general choice in the context of non-Gaussian closure scheme for response estimation of nonlinear systems.

3.3. The new closure scheme

Based on the Gaussian copula-based joint PDF representation in Eq. (17), the joint moment of the response of order $(s+t)$ can be computed as

$$\begin{aligned} E\left(X^s\dot{X}^t\right) &= \iint_\infty x^s\dot{x}^t f(x,\dot{x}) dx d\dot{x} \\ &= \iint_\infty x^s\dot{x}^t \sum_{i=1}^{m}\phi_i f\left(x\left|0,\sigma_i^2\right.\right) \cdot \sum_{j=1}^{n}\phi_j f\left(\dot{x}\left|0,\sigma_j^2\right.\right) dx d\dot{x} \\ &= \int_\infty x^s \sum_{i=1}^{m}\phi_i f\left(x\left|0,\sigma_i^2\right.\right) dx \int_\infty \dot{x}^t \sum_{j=1}^{n}\phi_j f\left(\dot{x}\left|0,\sigma_j^2\right.\right) d\dot{x} \\ &= E\left(X^s\right) E\left(\dot{X}^t\right) \end{aligned} \tag{19}$$

Obviously, according to Eq. (19), joint moment of the response can be obtained by separately determining the moment of displacement response and velocity response, respectively. From probability theory, k-order moment of Gaussian-distributed variable, Y, with zero mean and variance of σ^2, can be derived as

$$E\left(Y^k\right) = \sigma^r E\left(Z^r\right) \tag{20}$$

where $E(Z^r)$ is the r-order moment of standard normal variable, Z, given by

$$E\left(Z^r\right) = \begin{cases} 0 & r = 1,3,5,\cdots \\ (r-1)(r-3)\cdots 3\cdot 1 & r = 2,4,6,\cdots \end{cases} \tag{21}$$

Thus, moment of displacement response $X(t)$ of order s can be computed as

$$E\left(X^s\right) = \begin{cases} 0 & s = 1,3,5,\cdots \\ (s-1)(s-3)\cdots 3\cdot 1\cdot \sum_{i=1}^{m}\phi_i\sigma_i^s & s = 2,4,6,\cdots \end{cases} \tag{22}$$

and moment of velocity response $\dot{X}(t)$ of order r yields

$$E\left(\dot{X}^t\right) = \begin{cases} 0 & t = 1,3,5,\cdots \\ (t-1)(t-3)\cdots 3\cdot 1\cdot \sum_{j=1}^{n}\phi_j\sigma_j^t & t = 2,4,6,\cdots \end{cases} \tag{23}$$

By substituting Eqs. (22) and (23) into Eq. (19), the joint moments of response, $E\left(X^s\dot{X}^t\right)$, can be represented in a very concise form. With sets of formulated moments of response, moment equation of nonlinear systems can be rewritten in the form

$$\dot{M}_K = F_K\left(E\left(X^2\right), E\left(X^4\right), \cdots, E\left(\dot{X}^2\right), E\left(\dot{X}^4\right), \cdots\right) \quad K = 1,2,\cdots \tag{24}$$

by following the suggestive form of moment equation as in Eq. (4). For the stationary response, since the joint moment M_k is a time-independent constant, moment equation in Eq. (24) can be further reduced to

$$0 = F_K\left(E\left(X^2\right), E\left(X^4\right), \cdots, E\left(\dot{X}^2\right), E\left(\dot{X}^4\right), \cdots\right) \quad K = 1,2,\cdots \tag{25}$$

It is worth mentioning that only even moments of response are involved in the moment equation since the odd moment of response vanishes in the computation of displacement response or velocity response, as shown in Eqs. (22) and (23). As a result, the computational cost can be significantly reduced in the procedure of closure.

Similar as the conventional moment equations, the developed moment equation also form an infinite coupled system, and closure scheme is therefore required to truncate Eq. (25) to a closed system of equations. For finite number of Gaussian densities that constitute the marginal PDF of response, the prediction of moment of response depends on the determination of the set of coefficients $\{\phi_i, \phi_j, \sigma_i, \sigma_j\}_{i,j=1}^{i=m,j=n}$. For this purpose, we firstly define the cost function in terms of the developed moment equation as

$$\Gamma\left(\phi_i, \phi_j, \sigma_i, \sigma_j\right) = \sum_{l=1}^{K} F_l^2\left(E\left(X^2\right), E\left(X^4\right), \ldots, E\left(\dot{X}^2\right), E\left(\dot{X}^4\right), \ldots\right)$$
$$K = 1, 2, \ldots \tag{26}$$

Then, the set of couples $\{\phi_i, \phi_j, \sigma_i, \sigma_j\}_{i,j=1}^{i=m,j=n}$ are determined from minimizing $\Gamma\left(\phi_i, \phi_j, \sigma_i, \sigma_j\right)$ by substituting the formulation of joint moments of response in Eq. (19) into the cost function in Eq. (26). Thus, the estimation of moment of response from the new closure scheme is equivalent to

$$\begin{cases} \text{Find}: \{\phi_i, \sigma_i\}_{i=1}^{m},\ \{\phi_j, \sigma_j\}_{j=1}^{n} \\ \text{Minimize}: \Gamma\left(\phi_i, \phi_j, \sigma_i, \sigma_j\right) \\ \text{subjectto}: \phi_i \geq 0,\ \sum_{i=1}^{m} \phi_i = 1; \phi_j \geq 0,\ \sum_{j=1}^{n} \phi_j = 1 \end{cases} \tag{27}$$

Since the optimized set of couples $\{\phi_i, \phi_j, \sigma_i, \sigma_j\}_{i,j=1}^{i=m,j=n}$ can be obtained during the above procedure, estimation of the moment of response can thus be accomplished. In this way, the number of unknown parameters $\{\phi_i, \phi_j, \sigma_i, \sigma_j\}_{i,j=1}^{i=m,j=n}$ is not limited to that of moment equations in the proposed closure scheme, and the resulting PDF is thus flexible in that it has the potential to converge to a wide class of non-Gaussian distributed PDF. More importantly, in contrast to the present non-Gaussian closure methods, the procedure for solving set of highly nonlinear algebra equations is avoided. No wonder, the more the number of moment equations are involved, the more statistical information of the response can be estimated. However, the number of moment equations adopted in the optimization, i.e., value of K, directly influences the computational complexity. On the other hand, the number of Gaussian densities in marginal PDF should also be chosen carefully to achieve a balance between the accuracy and efficiency. A smaller value of m and n may reduce the ability of represent the details in the PDF of response. Obviously, with the combination of $m = n = 1$, and $K = 2$, the new closure will be reduced to the well-known Gaussian closure scheme, and in this case, highly non-Gaussian distributed response cannot be accurately estimated. According to the author's experience, the combinations of small values of m, n, and K (i.e., $m = 2$, $n = 2$, and $K = 2$) is generally sufficient for the target accuracy of problems of interests. The choice of m, n, and K will be discussed in more details in the following numerical examples.

4. Numerical examples

4.1. Non-linear oscillator under Gaussian white noise

The first example considers the following oscillator with nonlinear damping and stiffness, which is subjected to Gaussian white noise excitation. Unlike the nonlinear system with linear damping, it is difficult to determine the steady-state probability density of nonlinear damped system even if the stiffness of the system is linear [1]. The equation of motion of such a system is described as

$$m\ddot{x} + q(E)\dot{x} + g(x) = F(t) \tag{28}$$

where $F(t)$ is the Gaussian white noise with spectral density S_0, E represents the total energy of system, expressed as

$$E = \frac{1}{2}m\dot{x}^2 + \int_0^x g(t)dt \tag{29}$$

and $q(E)$ is the function of E. The exact PDF of response can be obtained by FPK method, i.e,

$$f(x, \dot{x}) = C\exp\left[-\frac{1}{\pi S_0}\int_0^E q(u)du\right] \tag{30}$$

where C is the normalization constant of PDF. In this study, $q(E)$ and $g(x)$ are adopted as those in [29], then the nonlinear system becomes

$$\ddot{x} + 4h\left(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2 + \frac{\varepsilon}{4}x^4\right)\dot{x} + \omega_0^2 x + \varepsilon x^3 = \sigma F(t) \tag{31}$$

where parameters of the nonlinear oscillator are given as $h = 0.1$, $\sigma = 1$, and $\omega_0^2 = 1$. Spectral intensity S_0 are selected as 1 and 10, and the nonlinear intensity ε varies from 1 to 10, respectively, to examine the effect of level of excitation intensity and nonlinearity of system to the performance of the developed method.

In this example, order of joint moments of response, K, is fixed as 2 in all analyses, and the number of Gaussian densities in marginal PDF of displacement response and velocity response, i.e., values of m and n, are equally adopted as $m = n = 1, 2$ and 3, respectively, to independently investigate the effect of the selection of number of Gaussian densities on the performance of the developed method. Under low level of excitation intensity, i.e., $S_0 = 1$, Figs. 1(a) and 1(b) show the estimated variance of displacement response and velocity response with different values of ε from FPK, Gaussian closure, and the developed copula-based Gaussian mixture closure method, respectively. Note that since the stochastic excitation is Gaussian distributed, the results from FPK method are the exact solution of the problem. Obviously, the conventional Gaussian closure yields the worst results in both cases. For example, variance of the velocity response from Gaussian closure is beyond two times of the exact one. It is also seen that results from Gaussian closure are identical to those from the developed method with the case $m = n = 1$, and $K = 2$, illustrating that Gaussian closure is the special case of the new closure method. On the other hand, with increasing the number of Gaussian densities in marginal PDF, i.e., $m = n = 2$ and 3, accuracy of the new closure method obviously improves even for the case of high level of nonlinearity (i.e., $\varepsilon = 10$) for both displacement response and velocity response. In particular, the estimated variance of displacement response from the new closure method is almost in accordance with the exact one for high level of nonlinearity, e.g., $\varepsilon > 5$. This means that accuracy of Gaussian closure is the lower bound of that of the developed closure scheme. It is also worth mentioning that further increasing the value of m and n (i.e., from 2 to 3) does not significantly improve accuracy of the solution. This is the direct result from the optimization procedure as it has been found that the optimized Gaussian mixture model with two components is very similar as that with three Gaussian components. Therefore, m and n are chosen as 2 for simplicity in the following of this paper.

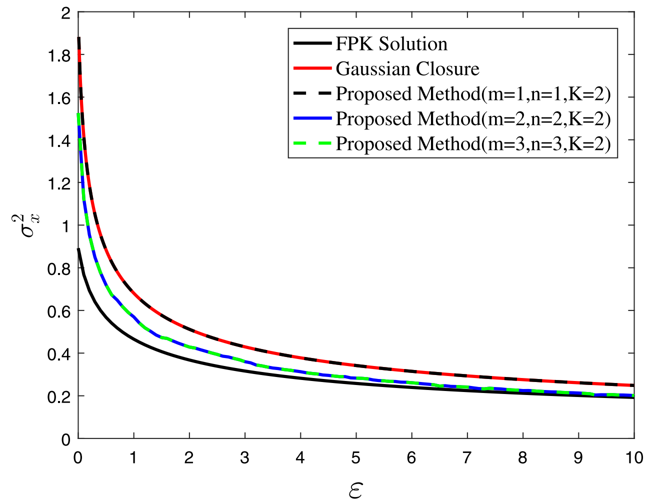
When the strength of random excitation is increased, i.e., $S_0 = 10$, the estimated variance of displacement response and velocity response with different values of ε from FPK, Gaussian closure, and the developed method are described in Figs. 2(a) and 2(b). In this case, we only examine the performance of the developed method under the condition $K = 2$, and $m = n = 2$. Compared with the results in Fig. 1, the new closure method consistently yields more accurate predictions with respect to the conventional Gaussian closure, illustrating that performance of the new method is insensitive to the excitation intensity.

4.2. Duffing oscillator subject to Gaussian white noise

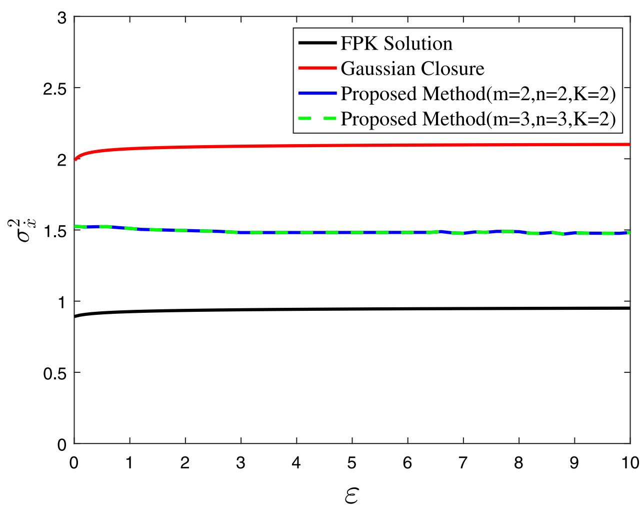
We next consider a Duffing nonlinear SDOF oscillator with linear damping and cubic nonlinear spring, which has been applied to model many mechanical systems. The equation of motion of is given by

$$\ddot{x} + \beta\dot{x} + \omega_0^2\left(x + \varepsilon x^3\right) = F(t) \tag{32}$$

where parameters of the nonlinear system are given as $\beta = 1$, and $\omega_0^2 = 1$. In order to examine the performance of the new closure method to different level of nonlinearity and excitation intensity, ε is adopted from 1 to 5, and the intensity of Gaussian white noise $F(t)$, S_0, is adopted as $1/\pi$ and $5/\pi$, respectively. In this example, the number of Gaussian densities in both marginal PDFs are fixed as two, i.e., $m = n = 2$, and order of joint moments of response, K, is specified as 2 and

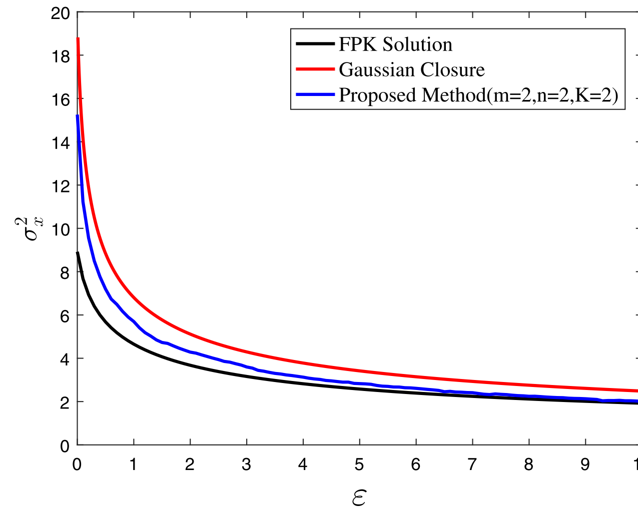


(a) The variance of displacement response

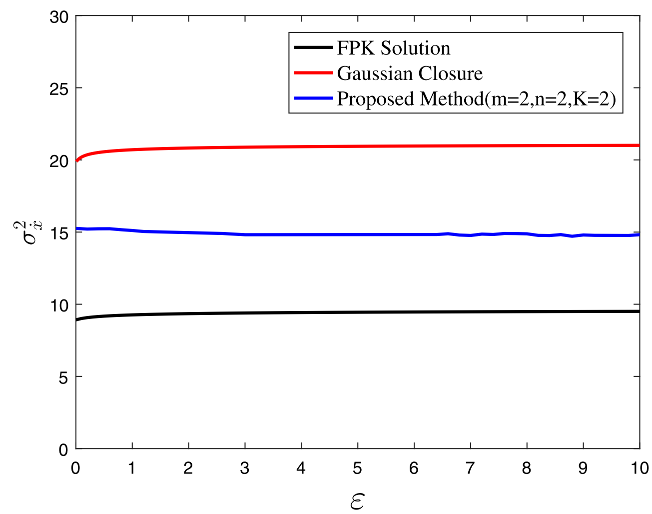


(b) The variance of velocity response

Fig. 1. The variance of response of nonlinear oscillator under Gaussian white noise ($S_0 = 1$).



(a) The variance of displacement response



(b) The variance of velocity response

Fig. 2. The variance of response of nonlinear oscillator under Gaussian white noise ($S_0 = 10$).

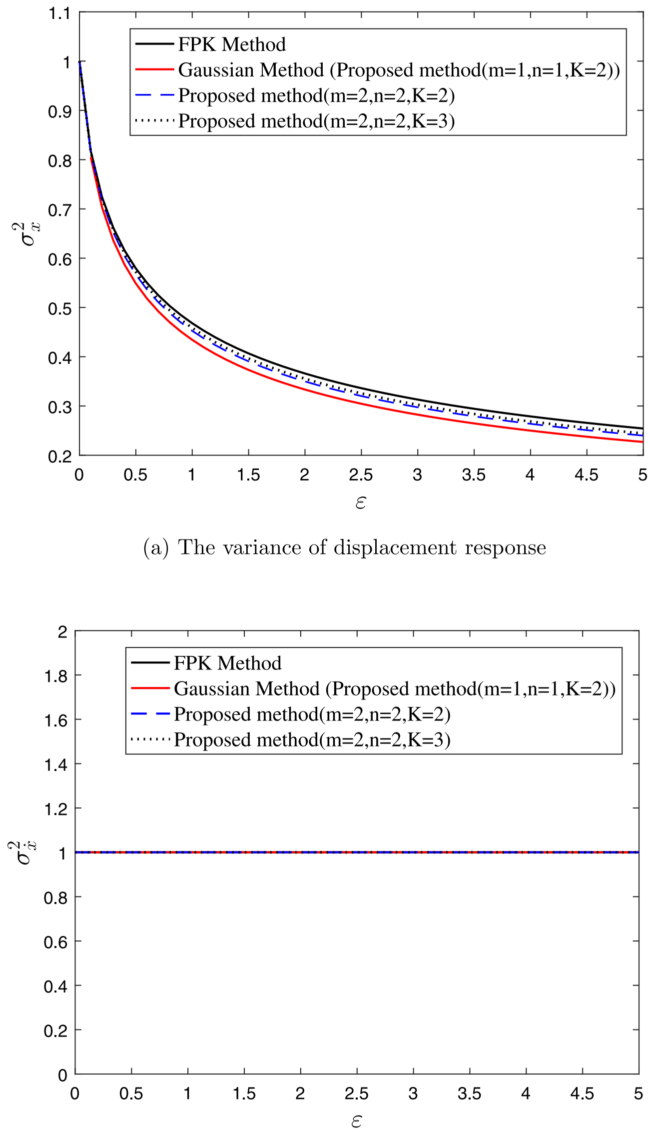
3, respectively, to investigate the effect of the selection of K on the performance of the new closure method.

With low level of excitation intensity, i.e., $S_0 = 1/\pi$, Figs. 3(a) and 3(b) describe the estimated variance of displacement response and velocity response with different values of K from FPK, Gaussian closure, and the developed copula-based Gaussian mixture closure method, respectively. The exact analytic solution of the FPK equation is used to check the accuracy of the methods. For the nonlinear oscillator with linear damping, since the stationary displacement response and velocity response are statistical independent variables each with zero mean, both Gaussian closure and the new closure method could yield the exact variance of velocity response, as shown in Fig. 3(b). In this case, only variance of displacement response is considered. Similar observations can be made as in the previous example. Accuracy of Gaussian closure, or equivalently the new closure scheme with the case $m = n = 1$ and $K = 2$, is obviously improved by increasing the number of Gaussian densities in marginal PDF, values of m and n, from 1 to 2 with the fixed order of moments $K = 2$. On the other hand, when the number of Gaussian densities is kept the same, i.e., $m = n = 2$, it can be found that accuracy of the new closure method increases with the order of moments K. However, further increasing the order of moments, i.e., K varies from 2 to 3, does not significantly improve the accuracy of the new closure method, as shown in Fig. 3(a). This means that, in order to obtain satisfactory results for problems of interests, one can choose relatively small values of m, n, and K for efficiently implementing the developed closure method.

When the intensity of Gaussian white noise is increased to $S_0 = 5/\pi$, the estimated variance of displacement response and velocity response with different values of ε from FPK, Gaussian closure, and the developed method are described in Figs. 4(a) and 4(b). Again, the new method achieves better accuracy than the Gaussian closure method with the combination of high level of excitation intensity and nonlinearity.

5. Conclusion

In this paper, a novel copula-based Gaussian mixture closure method is developed for stochastic response analysis of nonlinear systems. We firstly assume the marginal PDF of response in terms of Gaussian mixture model, and then formulate the joint PDF of response of nonlinear system based on the assumed marginal PDF and Gaussian copula.

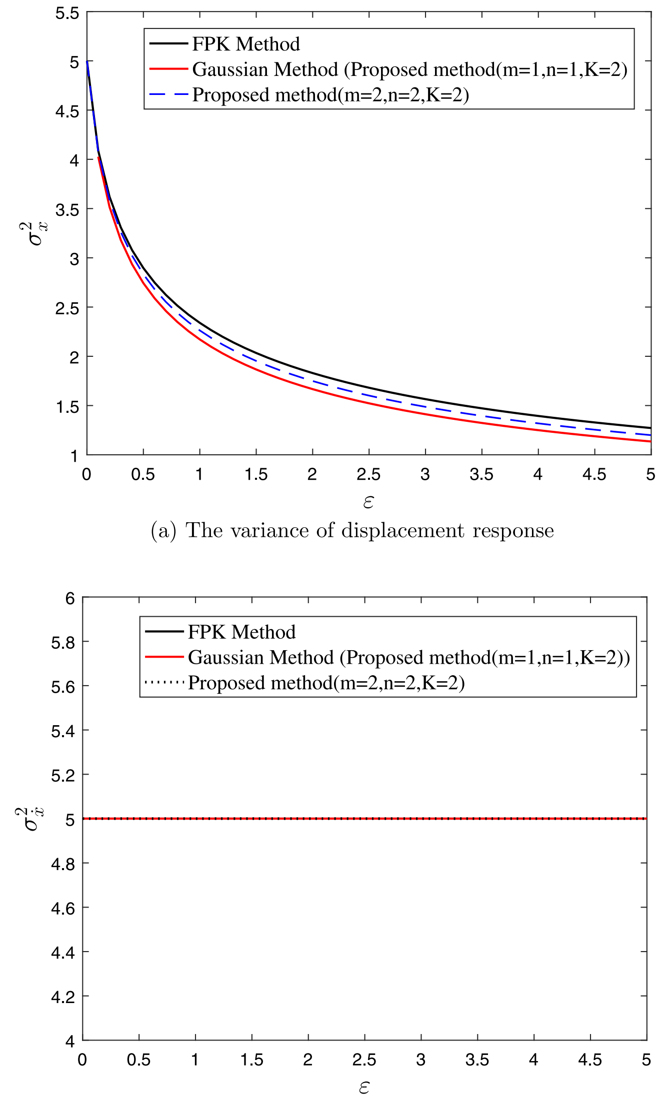


(a) The variance of displacement response

(b) The variance of velocity response

Fig. 3. The variance of response of Duffing oscillator under Gaussian white noise ($S_0 = 1/\pi$).

By substituting non-Gaussian PDF representation into the moment equations of nonlinear system, we further develop an optimization-based closure scheme for the solution of the unknown parameters in joint PDF. In this way, the number of unknown parameters in PDF of response is not limited to that of moment equations, and the resulting PDF is flexible in that it has the potential to converge to a wide class of non-Gaussian distributed PDF. Also, in contrast to the existing non-Gaussian closure schemes, the procedure for solving sets of highly nonlinear algebra equations can be avoided, and as a result, the computational cost of the new closure method can be significantly decreased. Effectiveness of the new closure method is finally demonstrated by a nonlinear and a Duffing oscillator that are subjected to Gaussian white noise. Comparisons with the exact results demonstrate the superior accuracy of the new closure method even for a high level of system nonlinearity. We note that Gaussian closure is a special case of the new closure method, and accuracy of Gaussian closure is the lower bound of that of the new closure method. In the future work, the developed copula-based Gaussian mixture closure method will be generalized to



(a) The variance of displacement response

(b) The variance of velocity response

Fig. 4. The variance of response of Duffing oscillator under Gaussian white noise ($S_0 = 5/\pi$).

problems that involve multi-degree-of freedom nonlinear systems and color noise excitations.

Acknowledgments

This research was supported by Grant from the National Natural Science Foundation of China (Project 11672091, Project 11972009 and Project 51609054). These supports are gratefully acknowledged.

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