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Analysis of periodicity-induced attenuation effect in a nonlinear waveguide by means of the method of polynomial system resultants

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ABSTRACT

This paper addresses the application of the novel method of polynomial system resultants for solving two problems governed by systems of cubic equations. Both problems emerge in analysis of stationary dynamics of a periodic waveguide, which consists of linearly elastic continuous rods with nonlinear springs between them. The first one is the classical problem of finding "backbone curves" for free nonlinear vibrations of a symmetric unit periodicity cell of the waveguide. The second one is the problem of finding the Insertion Losses for a semi-infinite waveguide with several periodicity cells. Similarly to the canonical linear case, a very good agreement between boundaries of high attenuation frequency ranges and eigenfrequencies of a unit cell is demonstrated.

1. Introduction

The periodicity-induced attenuation of the wave energy transmission in linear systems is a classical subject explored in many details since the pioneering work of Brillouin [1]. Discrete spring-mass lattices and continuous periodic beams, plates and cylindrical shells are standard models for the analysis of the Bloch–Floquet waves in linear mechanical systems. We do not dwell on any literature survey on these studies.

The obvious advance in modelling the periodicity effects is to take into account the nonlinearity of a periodic system. However, this step challenges the applicability of the Floquet theory, which has been developed under the fundamental assumption of linearity of a governing system of differential equations with periodic coefficients. Therefore, most of the work in the analysis of nonlinear periodic systems has been done so far in the "weakly nonlinear formulation", which facilitates the application of traditional asymptotic methods with the method of multiple scales usually being the first choice [1–5]. Alternatively, the harmonic balance method has been used in [4,6], whereas an emerging method of varying amplitudes has been used in [6].

In the vast majority of papers dealing with non-linear periodic systems, the discrete spring-mass 1D and 2D lattices are considered and the weak nonlinearity (quadratic or cubic) is introduced in spring elements [1–5,7,8]. Much fewer papers are concerned with weakly nonlinear continuous components of a periodic structure [6,9].

We consider a periodic system, which consists of identical elastic rods supporting linear plane dilatation waves interconnected with identical nonlinear springs. This system may be perceived as a reduced-order model of a multilayered structure with its relatively thick plies connected to each other by a thin layer of glue. It is realistic to assume that the behavior of glue departs from conventional Hooke's law and hence introduces nonlinear interfacial forces, whereas the behavior of each lamina is perfectly linear. Multilayered structures of this type are broadly used, for example, in wind turbines, and in aeronautics.

In what follows, the problems' formulations are reduced to systems of algebraic equations, but, due to the nonlinearity of springs' stiffness, these equations are cubic polynomials. Analysis of systems of the polynomial equations is the classical topic, which originates from the string theory. The theory of systems of polynomial equations is similar to the canonical theory of systems of linear algebraic equations. However, a generalization of the fundamental concept of the determinant of the system of linear algebraic equations to polynomial equations has been formalized only recently [10,11]. The counterpart of a determinant is called the polynomial system resultant. The analytical method of the polynomial system resultants [11] is an emerging convenient tool to solve these equations without any a priori assumptions (e.g. weak nonlinearity). In its relatively simple form, for the single polynomial equation, the resultants method, was used in [12]

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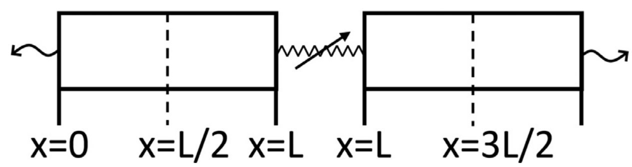

Fig. 1. Periodic structure (boundaries of the symmetrical unit cell – dashed vertical lines).

analyse the Hopf bifurcations. We are unaware of any publications, where this method has been used for nonlinear eigenfrequency or wave propagation analysis in the mechanical systems. We believe that this emerging method will soon find a broad scope of applications in the non-linear theory of dynamical systems.

As a final remark, we notice that as soon as the plies are considered as infinitely stiff and the springs are linear, the discrete mass-spring model proposed by Newton in 1686 is readily recovered, which does not predict any stop-bands. On the other hand, as is also well known, if a layer of glue is considered as a linearly elastic continuum, then stop-bands emerge.

The paper is organized as follows. In Section 2 the mathematical model of a periodic structure is formulated. In Section 3 the polynomial system resultants method is used to solve the problem of finding a “backbone curve” for a unit periodicity cell (i.e. the nonlinear eigenfrequency problem). In Section 4 the forcing (energy flow) problem for a semi-infinite structure with a variable number of nonlinear periodic inserts is solved also by means of the polynomial system resultants method. The findings reported in this paper and the intended future work are summarized in conclusions (Section 5).

2. The problem formulation

We consider a periodic structure, which consists of identical elastic rods with nonlinear springs between them as shown in Fig. 1. If it is not stated otherwise, we assume that the structure is infinite. It means, that the spring connections are spread to the left, i.e there are the springs at the points $x = -L, -2L, \dots,$ and to the right at $x = 2L, 3L, \dots$.

Each elastic element supports a single plane axial wave governed by the standard linear 1D wave equation Eq. (1) (u is axial displacement, c is sound speed, $c = \sqrt{\frac{E}{\rho}}$ with E as Young's modulus and ρ as the density of the material).

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{1}$$

To introduce the spring connection, we formulate the continuity conditions at each interface (as an example we show one at the point $x = L$). The continuity conditions are concerned with force Eq. (2(a)) and force-displacement Eq. (2(b)):

$$\begin{aligned} &EAu'_1(L,t) = EAu'_2(L,t) \qquad (a) \\ &EAu'_1(L,t) + K_L[u_1(L,t) - u_2(L,t)] \\ &\qquad + K_{NL}[u_1(L,t) - u_2(L,t)]^3 = 0 \qquad (b) \end{aligned} \tag{2}$$

Each spring has the linear stiffness, K_L and the non-linear one K_{NL}. In the absence of a spring, $K_L \to \infty$, $K_{NL} \to \infty$ and the conventional displacement continuity is recovered.

Also, we impose periodicity conditions in the standard form, i.e. we assume, that periodicity multiplier Λ is not depended on the coordinate and thus the periodicity conditions could be written in a form Eq. (3).

$$\begin{aligned} u(L/2) &= \Lambda u(3L/2) \\ u'(L/2) &= \Lambda u'(3L/2) \end{aligned} \tag{3}$$

To convert the problem formulation to a non-dimensional form, the length scale is chosen as L. Thus, the non-dimensional axial coordinate is $\bar{x} = \frac{x}{L}$ and the non-dimensional axial displacement is $\bar{u} = \frac{u}{L}$. In what follows, bars over non-dimensional quantities are omitted and all quantities are considered as dimensionless. The linear spring has the non-dimensional stiffness parameter $\kappa = \frac{K_L L}{EA}$ (that is, obviously, a ratio of stiffness of discrete and continuous components of a periodic structure), the nonlinear stiffness parameter is defined as $N = \frac{K_{NL} L^2}{K_L}$.

This structure is, probably, the simplest possible periodic nonlinear continuous system, and we will assess the influence of the nonlinearity on vibrations of a finite periodic structure (a symmetric unit periodicity cell), on wave propagation in a semi-infinite periodic structure under time-harmonic force applied at its edge. In the case, then the nonlinearity stiffness parameter is set to zero, $N = 0$, each of the three above-mentioned problems has a well-known elementary solution, which serves as a convenient reference.

It is convenient first to find eigenfrequencies of a finite structure, to understand, how the wave propagation should work in the infinite case.

3. Eigenfrequencies of a symmetrical periodicity cell

As is known for linear periodic structures, the boundaries between pass- and stop-bands can be easily found as the eigenfrequency spectra of a finite counterpart of an infinite waveguide, the symmetric unit periodicity cell shown in Fig. 1 with vertical dashed lines as the boundaries of the finite structure. In the linear case, $N = 0$, the response is time-harmonic, $u_j(x,t) = U_j(x)\cos\omega t$, $j = 1, 2$. Then the so-called A-type boundary conditions (Eq. (4)) define fixed edges

$$U_1(1/2) = U_2(3/2) = 0 \tag{4}$$

The B-type (Eq. (5)) conditions define free edges

$$U'_1(1/2) = U'_2(3/2) = 0 \tag{5}$$

The solution of the linear homogeneous equation Eq. (1) is shown in Eq. (6).

$$U_j(x) = B_{1j}\sin\Omega x + B_{2j}\cos\Omega x,\ \Omega = \frac{\omega L}{c},\ j = 1, 2 \tag{6}$$

It is a straightforward matter to formulate boundary problems Eqs. (2)–((4)) and (Eqs. (2) and (5)) and obtain the eigenfrequency equations for $N = 0$ (Eq. (7)).

$$\begin{aligned} D_A^0(\Omega) &= -4\Omega\cos\left(\frac{\Omega}{2}\right)\left(\Omega\cos\left(\frac{\Omega}{2}\right) + 2\kappa\sin\left(\frac{\Omega}{2}\right)\right) = 0 \\ D_B^0(\Omega) &= 4\Omega^3\sin\left(\frac{\Omega}{2}\right)\left(-2\kappa\cos\left(\frac{\Omega}{2}\right) + \Omega\sin\left(\frac{\Omega}{2}\right)\right) = 0 \end{aligned} \tag{7}$$

Each of these equations introduces two sub-spectra: free vibrations without deformation of the spring (eigenfrequencies found from conditions $\cos\frac{\Omega}{2} = 0$ and $\sin\frac{\Omega}{2} = 0$, respectively), and free vibrations which involve deformation of the spring (these eigenfrequencies are found by equating to zero expressions in the brackets).

As soon as the nonlinearity is taken into account, $N \neq 0$, the response should be sought in the form of truncated expansion (Eq. (8)).

$$u_j(x,t) = \sum_{m=1}^{M} U_{j,m}(x)\cos m\omega t,\ j = 1, 2 \tag{8}$$

However, since the equation Eq. (1) remains valid, the displacement field in each spatial component has the form

$$U_{j,m}(x) = b_{j,1}^m \sin m\Omega x + b_{j,2}^m \cos m\Omega x \tag{9}$$

To assess the influence of nonlinearity on the eigenfrequencies Eq. (7), we retain only one term in the decomposition Eq. 8, $M = 1$. Then we substitute the approximate solution Eq. (9) into the system of equations Eqs. (2)–(4) for Class A boundary conditions and into the system of equations Eqs. (2) and (5) for Class B ones. In each case, we obtain the following equation written in vector form Eq. (10).

$$L * \cos(\omega t) + C * \cos^3(\omega t) = 0 \quad (10)$$

The vectors L, C are written explicitly in form Eq. (11).

$$L = \begin{pmatrix} \sin(\Omega)(\kappa(b^1_{1,1}\text{-}b^1_{2,1})\text{-}\Omega b^1_{1,2}) + \cos(\Omega)(\kappa(b^1_{1,2}\text{-}b^1_{2,2})\text{+}\Omega b^1_{1,1}) \\ \Omega((b^1_{2,2} - b^1_{1,2})\sin(\Omega) + (b^1_{1,1} - b^1_{2,1})\cos(\Omega)) \\ b^1_{1,1}\sin\left(\frac{\Omega}{2}\right) + b^1_{1,2}\cos\left(\frac{\Omega}{2}\right) \\ b^1_{2,1}\sin\left(\frac{3\Omega}{2}\right) + b^1_{2,2}\cos\left(\frac{3\Omega}{2}\right) \end{pmatrix};$$

$$C = \kappa N \begin{pmatrix} ((b^1_{1,1}\text{-}b^1_{2,1})\sin(\Omega)\text{+}(b^1_{1,2}\text{-}b^1_{2,2})\cos(\Omega))^3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (11)$$

We apply the conventional harmonic balance method and eliminate the cubic term using the identity $\cos^3(\omega t) = \frac{1}{4}(3\cos(\omega t) + \cos(3\omega t))$. Then balancing terms with $\cos(\omega t)$ only we obtain the equations Eq. (12).

$$L + \frac{3}{4}C = 0 \quad (12)$$

The system of equations Eq. (12) is a homogeneous system of cubic equations with respect to the unknown coefficients $b^m_{i,j}$. To find eigenfrequencies, an analog of the canonical determinant of a system of linear algebraic equations should be formulated and equated to zero. The generalization of the determinants in the case of a system of polynomial equation case is the resultant of the system [10,11].

First, we define the resultant of two polynomials. Let $A(x_1, \ldots, x_n)$ and $B(x_1, \ldots, x_n)$ be the polynomials in variables $x_1, \ldots, x_n$. The polynomials are written in the form (we assume that all variables except x_k are kept constant) $A = \sum_{i=0}^{n} a_i x_k^i$ and $B = \sum_{j=0}^{m} b_j x_k^j$. The polynomial A has the roots λ_i with respect to the variable x_k and the polynomial B has the roots μ_j with respect to the same variable x_k. Then we can define the resultant of the two polynomials as Eq. (13).

$$res_{x_k}(A(x_1, \ldots, x_n), B(x_1, \ldots, x_n)) = a_0^e b_0^d \prod_{\substack{1 \le i \le d \\ 1 \le j \le e}} (\mu_j - \lambda_i) \quad (13)$$

With index x_k we denote that resultant is taken with respect to the variable x_k, i.e. other variables are assumed to be fixed as constants. Resultant of two polynomials can also be computed as the determinant of the Sylvester matrix, which consists of the polynomial term coefficients a_i, b_j. The exact form of the Sylvester matrix is easily found in the textbooks and therefore we will not reproduce it in the article.

The theory of resultants of systems of this type is presented, among others, in the text [10]. In what follows, we apply this theory with minimum mathematical details, which may be found in the above reference. To find the resultant of the system composed as more than two equations it is expedient to enumerate equations. Thus, for the system Eq. (12), we obtain the vector of four equations which is shown in Eq. (14).

$$L + \frac{3}{4}C = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = 0 \quad (14)$$

System Eq. (14) is the system of the polynomial equations with respect to the unknown amplitudes of the displacements $b^1_{1,1}$, $b^1_{1,2}$,$b^1_{2,1}$, $b^1_{2,2}$. To find the resultant of the system Eq. (14) we have to define all permutations of the unknown variables of size $n - 1 = 3$. The set of all permutations is $V = \{(b^1_{1,1}, b^1_{1,2}, b^1_{2,1}), \ldots, (b^1_{2,2}, b^1_{2,1}, b^1_{1,2})\}$. The set V contains $A^4_3 = 24$ permutations, where $A^i_j = \frac{i!}{(i-j)!}$. In order to find the resultant for every entry in the set V, we use the following procedure.

For brevity, we consider the only permutation $(b^1_{1,1}, b^1_{1,2}, b^1_{2,1})$. First, we take resultants of an arbitrary pair of equations from Eq. (14) with respect to the first variable $b^1_{1,1}$.

$$\begin{aligned} r_{1,1} &= res_{b^1_{1,1}}(e_1, e_2) \\ r_{1,2} &= res_{b^1_{1,1}}(e_1, e_3) \\ r_{1,3} &= res_{b^1_{1,1}}(e_1, e_4) \end{aligned} \quad (15)$$

The second step is to take resultants from Eq. (15) with another variable, $b^1_{1,2}$. It is written as Eq. (16).

$$\begin{aligned} r_{2,1} &= res_{b^1_{1,2}}(r_1, r_2) \\ r_{2,2} &= res_{b^1_{1,2}}(r_1, r_3) \end{aligned} \quad (16)$$

The final step is the resultant of the resultants from the previous step with respect to the yet unused variable, $b^1_{2,1}$, which is written as Eq. (17).

$$r_{3,1} = res_{b^1_{2,1}}(r_{2,1}, r_{2,2}) \quad (17)$$

The resulting equation Eq. (17) contains the unknown amplitude $b^1_{2,2}$ and the other variables κ, N, Ω stated above as the parameters. The resultant of the system Eqs. (2)–(4) is the least common part among the all possible final steps $r_{3,1}$ for the all possible permutations in V. For the problem Eqs. (2)–(4), resultant Eq. (17) contains the single parameter $\beta = (b^1_{2,2})^2 N$, which is the product of the scaled unknown amplitude and the nonlinear stiffness. In the paper, we assume that unknown amplitude is equal to 0.1, for example $(b^1_{2,2})^2 = 0.1$, and we change only nonlinear stiffness as $\beta = 0.1N$. However, we note that for any prescribed value of the parameter β there is an infinite number of possible combinations of the amplitudes $b^1_{2,2}$ and nonlinear stiffness N.

Thus, the equations Eq. (7) may be used as the calibration condition for the single resultant procedure for arbitrary permutation of the variables, i.e. we normalize Eq. (17) such that $r_{3,1} = D^0_A(\Omega)$ (Eq. (7)) at $\beta = 0$. Therefore, the procedure Eqs. (15)–(17) is not need to be done among all permutations of the unknown displacements. It should be noted that the same result, in this case, is obtained with the full procedure described above.

Using normalization procedure or the complete resultant method, one can obtain the backbone eigenfrequency curves for a symmetrical periodicity cell, shown in Fig. 1. The final form of the non-linear eigenfrequency equation for A- and B-type boundary

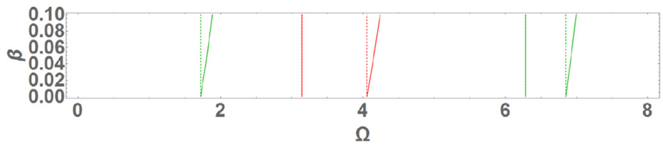

Fig. 2. Eigenfrequencies of a symmetrical cell both type A boundary conditions and type B boundary for different parameters $\beta = (b^1_{2,2})^2 N = N$ (vertical axis) and eigenfrequencies of a linear symmetrical periodicity cell (vertical dashed lines).

conditions is Eq. (18).

$$
\begin{aligned}
D_A(\beta,\Omega) = &\ -4\Omega\cos\left(\frac{\Omega}{2}\right)\left(2\kappa\sin\left(\frac{\Omega}{2}\right)\right.\\
&\left.+\Omega\cos\left(\frac{\Omega}{2}\right) - 24\beta\kappa\sin^3\left(\frac{\Omega}{2}\right)\right) = 0\\
D_B(\beta,\Omega) = &\ 4\Omega^3\sin\left(\frac{\Omega}{2}\right)\left(-2\kappa\cos\left(\frac{\Omega}{2}\right)\right.\\
&\left.+\Omega\sin\left(\frac{\Omega}{2}\right) - 24\beta\kappa\cos^3\left(\frac{\Omega}{2}\right)\right) = 0
\end{aligned}
\tag{18}
$$

As is proved in [13] for the linear case, $\beta = 0$, the eigenfrequencies found from these equations, D^0_A=0 and D^0_B=0, see also Eq. (7), coincide with all boundaries between stop- and pass-bands in an infinite periodic structure composed of the cells shown in Fig. 1. Therefore, it is reasonable to assume that, in the non-linear case, eigenfrequencies found from equations Eq. (18) may be considered as educated guesses for positions of these boundaries. Naturally, a proof of this statement requires a generalization of the Floquet theory, and this task lies beyond the scope of this paper. However, the reader may be referred to [14], where it has been shown that stop-band boundaries for an infinite radially periodic membrane also match fairly well spectra of symmetric periodicity cells with the A- and B-type boundary conditions – although the Floquet theory is not directly applicable for the problems formulated in polar coordinates.

The first five eigenfrequencies found from equations Eq. (18) are shown in Fig. 2 as functions of the parameter β for $\kappa = 1$. It is seen, the structure of eigenfrequency spectra is the same as in the linear case: the eigenfrequencies found from equations $\cos\frac{\Omega}{2} = 0$, $\sin\frac{\Omega}{2} = 0$ correspond to the eigenmodes with no deformation of the spring. Thus, these are independent upon the nonlinearity parameter β (vertical lines in Fig. 2). The second part of the spectra is the purely non-linear eigenfrequencies which are dependent on a non-linear parameter β, which merges their linear counterparts in Eq. (7) as soon as $\beta = 0$.

It should be noted, that the eigenfrequency curve cannot be obtained directly with other methods unless the 'weak nonlinearity' assumption is used.

The eigenfrequency spectra analysis suggests that the attenuation zones, similar to the linear stop-bands, should exist in the non-linear case. To verify this statement, the forcing problem for a semi-infinite periodic nonlinear structure should be solved.

4. Forcing problem

The standard "virtual experiment" for verification of positions of pass- and stop-bands predicted by the Floquet theory in the linear case is the calculation of Insertion Losses (IL) for a semi-infinite structure, which accommodates a variable number of periodicity cells near its edge (where a driving force is applied). The "rest" of the structure (its part extended to infinity) is homogeneous. In pass-bands, the energy flow is insensitive to the amount of inserted periodicity cells and remains of the same order as it would be in a homogeneous structure. In stop-bands, the energy flow is heavily suppressed due to the periodicity, and IL grow as the amount of inserted periodicity cells increases.

We hypothesize that eigenfrequencies defined by equations Eq. (18) in the nonlinear case preserve their properties known in the linear case. To verify this statement, we need to solve the forcing problem for the semi-infinite periodic structure shown in Figure 1 and perform numerical energy flow analysis. Energy is transmitted by the travelling waves, thus, for the forcing problem the sine and the cosine functions in Eq. (9) are replaced with the $\exp(\pm i\Omega x)$. The boundary conditions are replaced with the excitation condition (a harmonic force) at the left boundary and Sommerfeld condition at the infinity. Additionally, we assume that the structure is semi-infinite and exists only if $x \geq 0$. The conditions have the form Eq. (19).

$$
\begin{aligned}
&f_1(0,t) = 1 * \cos(\omega t)\\
&\sum_m b^m_{last,2}\cos(m\omega t) = 0
\end{aligned}
\tag{19}
$$

The formulation of the forcing problem makes the system of cubic equations inhomogeneous, thus, the problem becomes inhomogeneous and can be written in the form Eq. (20)

$$
\sum_{m=1}^{M} L_m\cos(m\omega t) + \kappa\beta\left[\sum_{m=1}^{M} C_m\cos(m\omega t)\right]^3 = -F_1\cos\omega t \tag{20}
$$

Using again the harmonic balance method we obtain the final inhomogeneous system of coupled polynomial equations with respect to the unknown amplitudes of the displacements $b^m_{i,j}$ in each time-harmonic mode. For the inhomogeneous system, the resultant may be found analogously to the homogeneous case, however, it is required to introduce an auxiliary variable to homogenize the system.

For every permutation $V = \{(b^1_{1,1}, b^1_{1,2}, b^1_{2,1}, \ldots), \ldots, (b^1_{2,2}, b^1_{2,1}, b^1_{1,2}, \ldots)\}$, we introduce the auxiliary variable a and compose the system of the homogeneous equations with respect to the unknowns $a, b^1_{1,1}, b^1_{1,2}, \ldots, b^k_{n,2}$ in the following way.

First, we multiply each equation by a^{k_1}, where k_1 is the highest power of the first variable in a permutation (for example, $b^1_{1,1}$) in the equation at hand. Second, we replace every unfixed variable following the rule $b^1_{1,2} \to \frac{b^1_{1,2}}{a}, \ldots, b^k_{n,2} -> \frac{b^k_{n,2}}{a}$. The obtained system has two properties. First, it is homogeneous and second, its solution at $a = 1$ is the solution of the original system. The resulting system for the fixed variable $b^1_{1,1}$ is written as Eq. (21)

$$
S(b^1_{1,1}; a, b^1_{1,2}, \ldots, b^k_{n,2}) = 0 \tag{21}
$$

The resultant of the system is taken as Eqs. (15)–(17). It should be noted that in this case the resultant is taken with respect to the all possible permutations in the set $\tilde{V}(b^1_{1,1}) = \{a, b^1_{1,2}, \ldots, b^k_{n,2}\}$.

As soon as the coefficients $b^m_{i,j}$ are found, the performance of the semi-infinite structure with the given number of periodicity cells is fully defined. We are interested in the assessment of the periodicity-induced vibro-isolation effect. It is a common practice to quantify it by the Insertion Losses function defined as Eq. (22).

$$
IL_n(x,\Omega) = 10\log_{10}\left(\frac{E_0(\Omega)}{E_n(x,\Omega)}\right) \tag{22}
$$

In Eq. (22) $E_0(\Omega)$ is the energy flow through the homogeneous rod and $E_n(x, \Omega)$ is the energy flow through the structure with n periodic inserts at the same frequency Ω and the same excitation conditions. In the linear case, this quantity is constant along the structure. In the nonlinear case, it becomes dependent on the coordinate x. It occurs because the energy injected into the structure at the directly excited frequency Ω leaks at each non-linear

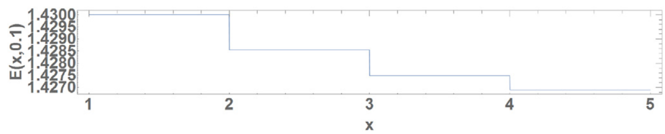

Fig. 3. The energy flow through the five ($n = 5$) consequent periodicity cells.

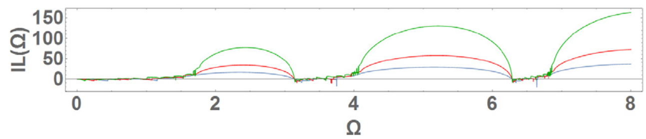

Fig. 4. Insertion Losses for $N = 0.3$ for a different number of periodic insertions $n = 3, 5, 10$ (blue, red, green). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

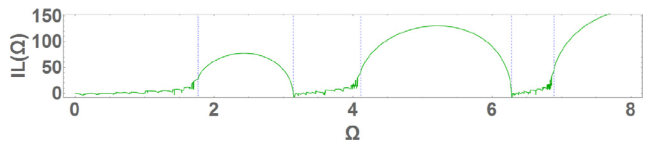

Fig. 5. Insertion Losses for $n = 10$ periodic insertions (green) and eigenfrequencies of the unit symmetrical cell (vertical lines, blue, dashed), $N = 0.3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

interface to high-order harmonics $m\Omega$. Naturally, within each segment, the energy is preserved, but each interface contributes to the energy loss at the frequency Ω. In the Fig. 3 the energy flow $E_5(x, 0.1)$ through the five periodicity cells at the frequency $\Omega = 0.1$ is shown.

Thus, computations of Insertion Losses are done at the point $x = n + 1/2,$ where the multi-segment periodic insert ends and merges the uniform semi-infinite waveguide. The insertion losses for a number of cells $n = 3, 5, 10$ is shown in Fig. 4.

We note that the higher order time-harmonics do not introduce significant changes in energy flow and, thus, in Insertion Losses picture and stop-band position.

The positions of the eigenfrequencies of the symmetrical unit cell versus Insertion Losses are shown in the Fig. 5.

As clearly seen from this graph, the non-linear periodic structure features pass- and stop bands similar to the linear one. However, it is hard to claim that the eigenfrequencies of a unit periodicity cell precisely define boundary frequencies separating pass- and stop-bands. Therefore, the nonlinear counterpart of the canonical Floquet theory for linear periodic systems is required to assess the validity of the use of eigenfrequencies of the unit cell to predict the location of pass- and stop-bands. Such an approximate Floquet theory may be formulated for the system considered in this paper using the resultant method along with the conventional Floquet periodicity conditions. The similar task for a radially periodic membrane (i.e., in polar coordinated) has been accomplished in [14] in the linear formulation of the problem. The nonlinear Floquet theory approximations, nevertheless are out of the scope of the paper.

5. Conclusion

We have demonstrated that the novel method of polynomial system resultants may reliably be used for solving nonlinear eigenfrequency and wave propagation problems for continuous mechanical systems. In contrast to existing methods, the assumption of a weak nonlinearity is not a pre-requisite for its applicability.

Using this method, we have computed the Insertion Losses for a semi-infinite structure with a variable number of periodic inserts and found that there are frequency ranges, where Insertion Losses are high and sensitive to the number of periodic inserts and the frequency ranges, where they are not. We have also computed the eigenfrequencies (backbone curves) for nonlinear free vibrations of a unit symmetric periodicity cell and found a good agreement between the two solutions.

These results much resemble the properties of linear periodic structures and their unit periodicity cells and suggest that the Floquet theory may be generalized for non-linear waveguides. This is a challenging task for future studies.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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