# An iterative improving procedure based on the first approximation of the linearization method above the phase curve

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## ABSTRACT

A general (applicable to arbitrary nonlinearity) linearization method above the phase curve is detailed in this paper. It is shown that the proposed method – based on the first harmonic approximation provided by the method itself – is capable of establishing the amplitudes of higher-order harmonics in order to improve the accuracy of the first approximation. The applicability of our method is demonstrated through the analytical–numerical analysis of well-known (cubic truly-nonlinear (TNL) and van der Pol) single-degree-of-freedom (SDOF), harmonically-forced nonlinear systems. The obtained results are compared with solutions in the literature that are considered to be exact. This comparison demonstrates the accuracy, efficiency and applicability of the proposed linearization procedure to arbitrary periodically-forced nonlinear vibrations. Thus, on the whole, the relevance of the method is justified.

## 1. Introduction

Due to the increasing industrial demand for machines and structures, complicated vibration phenomena often need to be dealt with. The investigation of such problems usually can be launched through the analysis of a nonlinear dynamical model with the concomitant nonlinear governing equation or equation system. Although several worthwhile publications have been put forward so far (e.g. [1–7]), the theory of nonlinear differential equations, even in the case of single-degree-of-freedom (SDOF) 2nd-order differential equations, is still incomplete. This may raise impassable barriers to handling these problems with an analytical approach. Although the closed-form solutions of such nonlinear differential equations can be set up in few cases only [3,5,6], still there is a strong demand from the side of the engineers for solving these nonlinear problems, even with the application of certain approximation methods.

In engineering practice, several approximation methods are applied that share one common feature: each method includes at least one step that cannot be justified exactly by means of mathematics, at least not at this point in time. These methods are referred to as "engineering methods" by some publications [8–11]. Reviewing the approximation methods that are applied and prevalent in the field of the engineering, it is clearly seen that the great majority of these methods are based on approximation by finite or infinite series of mathematical functions. Two sets of methods are worth contrasting with each other. One, which contains pure numerical procedures, provides a numerical solution within a predefined tolerance range with the disadvantages of providing a discrete solution. This solution belongs to only one fixed set of parameters, and these procedures may sometimes be needlessly time-consuming. The other set contains analytical approximation procedures which, as a result of a series expansion, provide analytical expressions as the function of the parameters [12–14]. Sometimes even the first steps of these methods are enough to reveal the main nonlinear features of the dynamic system being investigated.

Due to the drawbacks mentioned above, in some cases there have been attempts to replace a nonlinear system with a linear one, which yields what is called an equivalent linear system. This brought about the evolution of a particular set of procedures, the methods of linearization. They do not focus on establishing the exact solution, but rather on setting up a first approximation solution of the problem by assigning a considered equivalent linear equation to the original one. In particular, the nonlinear term is approximated by a so-called "equalizing" plane in such a way that the difference between the two surfaces is minimized by a certain principle. Hence, the solution of the equivalent linear system is assumed to be the approximated solution of the nonlinear one. The difference between the several methods of linearization can be distinguished by the instructions on how the equivalent linear equation is assigned to the nonlinear equation. Consequently, a wide variety of linearization methods can be mentioned, such as the simplest and long-existing tangent-line linearization method [15], harmonic linearization [16,17], which is considered to be a special type of the method of harmonic balances (HBM) [18–20], as well as the linearization method based on the equivalence of energies [21,22], which is the physical interpretation of harmonic

linearization, the equivalent linearization procedures [23–25], which can be constructed on the basis of the first approximation of the Krylov–Bogoliubov-type asymptotic method, the direct linearization method of Ponovko [26–29], the partial-corresponding correlation method [30], the different types of global linearization methods [31–33], and the Blaquiere-type optimal linearization method [34,35], which was improved by Iwan [36–40]. Some additional earlier works about the different subtypes of linearization methods are mentioned in [41–43].

Besides the SDOF deterministic systems that our investigations are restricted to in this paper, the class of stochastic linearization techniques, which determine the nonlinear vibration response in the presence of uncertainties, also needs to be mentioned. These procedures are particularly useful in the random vibration field, where there are inherent uncertainties in the excitation (e.g. earthquake loading and wind loading) [44–47].

The obvious advantage of the above-mentioned methods over nonlinearizing ones is that the analytical handling of a linearized system is well-established, the analytical forms of the response functions can be set up in most cases and, moreover, the relative simplicity and the illustrative presentation of such methods are coherent with the engineering mindset. In this context the Ponovko method can be mentioned as the very first and only linearization method which is based on the geometrical considerations that are close to the engineering way of thinking. A drawback of the Ponovko method is its way of approximation of a nonlinear characteristic surface by the equalizing plane. This takes place in such a way that each point of the considered range of arguments is taken into account. A particular solution is however evolved from only the nonlinear characteristic surface, which covers the phase curve projectively, while the other points of the surface are neglected; hence, it is sufficient to take only the surface curve above the phase curve into account when the equalizing plane is defined during any linearization process. Since some publications assume the Ponovko method to be a generalized procedure of some other linearization methods [41], then the problem concerning the range of arguments of linearization may emerge in case of several other linearization methods. In order to eliminate these difficulties, some procedures of linearization above the phase curve have been developed [48,49]. These methods, however, would be considered semi-numerical/semi-analytical procedures rather than pure analytical ones. As regards the Ponovko method, another common drawback of some existing linearization methods can be broached: some equivalent linearization methods [50–52], like that of Ponovko, require the application of weight functions, which are selected arbitrarily, lacking any kind of physical meaning. The optimal selection of such functions requires practical experience, thus they are mathematically inexact. The above-mentioned disadvantages indicate that the line of thought of these methods includes more than one mathematically inexact step. So in order to establish the optimal solution by these methods, several trials are required, which, in some cases, actually are lacking any kind of physical meaning.

In this article a general (applicable to arbitrary nonlinearity) linearization method above the phase curve is proposed. This method is a pure analytical procedure that requires no arbitrary weighing functions lacking physical meaning. It provides an arbitrarily exact approximating solution of the nonlinear problem in such a way that only the points constructing the phase curve are taken into account. The proposed method does not assume any linear term included by the nonlinear equation; therefore, both "strongly" nonlinear systems [5] and truly-nonlinear (TNL) systems [4] are also capable of being analyzed by our method. Our experience has shown that the applied minimizing principle can be taught almost effortlessly. In addition, some other linearizing procedures, which nowadays might still be refused by mechanical engineers intuitively, can be illustrated by our method. Moreover, on the basis of the regular first approximation and beyond, an improved approximation including higher-order harmonic terms can also be established, giving rise to solving the inverse problem (establishing the parameters of a nonlinear vibration).

In the following sections, our method, a linearization above the phase curve, is detailed first. Then the applicability, accuracy and efficiency of the method are demonstrated through analytical investigations of harmonically forced TNL cubic Duffing and van der Pol systems. The investigation of each system begins by setting up the equivalent linearized equation, which is followed by the calculation of the first harmonic approximation. On the basis of the first harmonic approximation provided by the method itself, the computation of the first few higher-order harmonics with a successive refining procedure is performed. Then the results of our method are compared with those obtained by computation with error calculation in [53]. Besides, the applicability and the convenience of our method for solving nonlinear equations from the fields of mechanical engineering and physics are also demonstrated.

## 2. The method of linearization above the phase curve

Our investigations are restricted to deterministic SDOF heteronomous systems whose governing equation takes the general form of

$$x'' + f\left(x', x\right) = g\left(\tau\right), \quad (2.1)$$

where $x$, $\tau$, $(\ldots)'$, in that order, are denoted to the dimensionless displacement, time, and the derivative with respect to $\tau$. $g\left(\tau\right)$ is the harmonic forcing term taking the form

$$g\left(\tau\right) = g_0 \cos \eta\tau, \quad (2.2)$$

where $\eta$, $g_0 > 0$ are constants and denote the dimensionless forcing angular velocity and amplitude, respectively. The term $f\left(x, x'\right)$ is called the nonlinear characteristic surface and its partial differentials in respect with $x$ and $x'$ are assumed to be at least piecewise-continuous within the considered range of arguments. The detailed theoretical background of the method can be found in [41–43,54]. Ref. [41] details the method of linearization above the phase curve, such as the procedure itself, historical review of the methods of linearization, and gives an example (SDOF Duffing system) for the application. Solely the establishment of the first-order approximation ($x_1 = a_1 \cos \psi$) is mentioned and detailed. Ref. [42] highlights the importance of the transformation of the independent variable in case of the proposed linearization procedure above the phase curve. Then it compares the first approximation established by the proposed method to that of other methods. Ref. [43] gives the details (mathematical background and literature review) of the proposed linearization method, giving an example for a possible "a priori" way of judgment of the measure of nonlinearity, which facilitates the "a priori" judgment of the quality of the linear approximation. Ref. [43] mentions a possible way of improving the actual approximation with higher-order harmonics, but without any examples or details. Ref. [54] is a short summary of the author's working life for the habilitation procedure, where a short extract of the proposed method is also included and, in addition, an attempt was made to extend the linearization method to MDOF systems.

The forcing term is assumed to be a harmonic expression, thus (2.1) takes the form

$$x'' + f\left(x, x'\right) = g_0 \cos \eta\tau, \quad (2.3)$$

where $\eta > 0$ constant and is called the dimensionless circular frequency of the excitation, while $g_0$ is a dimensionless forcing amplitude.

The linearization of a nonlinear differential equation (2.3) will generally come to the point where the nonlinear term of $f\left(x, x'\right)$ is replaced with a linear expression of $x$ and $x'$. That is to say, an equivalent linear differential equation

$$x'' + bx' + cx + d = g_0 \cos \eta\tau \quad (2.4)$$

is assigned to (2.3) in such a way that the periodic solution of (2.4) is a "good" approximation of the periodic solution of (2.3). Eq. (2.3) is
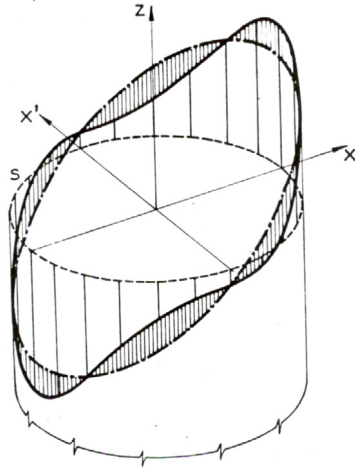
**Fig. 1.** A nonlinear characteristic surface (continuous) and its equalizing plane (dashed with dots) above the phase curve (dashed) [43].

assumed to have a known dominant frequency ($\eta$), thus the steady-state solution of (2.4) assumes the form

$$x(\tau) = a_0 + a_1 \cos(\eta\tau - \vartheta_1). \tag{2.5}$$

The unknown $a_0$, $a_1$ and $\vartheta_1$ constants in (2.5) can be expressed as the functions of $b$, $c$ and $d$ coefficients as follows:

$$a_0 = -\frac{d}{c}, \quad a_1 = \frac{g_0}{\sqrt{(c - \eta^2)^2 + (b\eta)^2}}, \quad \vartheta_1 = arctg\left(\frac{b\eta}{c - \eta^2}\right). \tag{2.6}$$

When (2.3) is linearized, the nonlinear characteristic surface $f(x, x')$ is approximated by an equalizing plane of $z = bx' + cx + d$ over the considered range of arguments. Thus, the problem is reduced to establishing the unknown $b$, $c$ and $d$ coefficients of the approximating plane. In order to obtain the equalizing plane, the method of linearization along the phase-curve applies the following procedure: let the quadratic integration of the difference $e = \left| f(x, x') - (bx' + cx + d) \right|$ along the phase-curve $s$ of the vibrating system be minimal, which then can be formulated as

$$J_1 = \oint_{(s)} \left[ f(x, x') - (bx' + cx + d) \right]^2 ds = \min. \tag{2.7}$$

Fig. 1 displays the geometrical approach of the proposed method. From an engineering and educational point of view, this might be the most important advantage of our method.

The continuous curve displays the intersection curve of the nonlinear characteristic surface and the cylinder that is extruded from the circular phase curve, while the dotted line indicates the intersection of the same circular cylinder and the equalizing plane. The equalizing plane is selected in accordance with (2.7), i.e., the solution of the extremum problem (2.7) yields the unknown $b$, $c$ and $d$ parameters as the functions of the vibrational amplitude and frequency. The solution procedure of (2.7) can be simplified significantly if the calculation is performed on the phase plane. This requires a new dimensionless variable to be introduced, $\varphi = \eta\tau$. Thus (2.3) and (2.4) are transformed into the forms

$$\eta^2 x'' + f(x, \eta x') = g_0 \cos\varphi \tag{2.8}$$

and

$$\eta^2 x'' + b\eta x' + cx + d = g_0 \cos\varphi, \tag{2.9}$$

respectively, where $'$ denotes the derivatives by $\varphi$. Using these notations, the solution of (2.5) and its derivative by $\varphi$ take the forms

$$x(\varphi) = a_0 + a_1 \cos(\varphi - \vartheta_1) \tag{2.10}$$

and

$$x'(\tau) = -a_1 \sin(\varphi - \vartheta_1), \tag{2.11}$$

respectively, whose constant coefficients $a_0$, $a_1$ and $\vartheta_1$ are established also by (2.6). It is clearly seen by (2.10) and (2.11) that the $(x, x')$ phase-curve of (2.9) is a circle with the origin of $a_0$ situated along the $x$ axis and with the radius $a_1$. Since the dimensions of (2.10) and (2.11) are identical, an element arc of $ds$ can be defined along the circular phase curve, which implies the following form of (2.7):

$$J_1 = \oint_{(s)} \left[ f(x, \eta x') - (b\eta x' + cx + d) \right]^2 ds = \min. \tag{2.12}$$

After introducing $\psi = \varphi - \vartheta_1$, the $ds$ element arc takes the form $ds = a_1 d\psi$, and (2.12) is transformed into

$$J_1 = \int_0^{2\pi} \left[ f(x, \eta x') - (b\eta x' + cx + d) \right]^2 a_1 d\psi = \min. \tag{2.13}$$

From (2.13) the equations $\frac{\partial J_1}{\partial b} = 0$, $\frac{\partial J_1}{\partial c} = 0$, $\frac{\partial J_1}{\partial d} = 0$ are yielded, from which the unknown $b$, $c$ and $d$ coefficients are obtained as follows:

$$
\begin{aligned}
b &= -\frac{1}{\pi a_1 \eta} \int_0^{2\pi} f\left(a_0 + a_1 \cos\psi; -a_1 \eta \sin\psi\right) \sin\psi \, d\psi, \\
c &= \frac{1}{\pi a_1} \int_0^{2\pi} f\left(a_0 + a_1 \cos\psi; -a_1 \eta \sin\psi\right) \cos\psi \, d\psi, \\
d &= \frac{1}{2\pi} \int_0^{2\pi} f\left(a_0 + a_1 \cos\psi; -a_1 \eta \sin\psi\right) d\psi \\
&\quad - \frac{a_0}{\pi a_1} \int_0^{2\pi} f\left(a_0 + a_1 \cos\psi; -a_1 \eta \sin\psi\right) \cos\psi \, d\psi.
\end{aligned}
\tag{2.14}
$$

By (2.14) the approximating $a_1(\eta)$, $\vartheta_1(\eta)$ amplitude–frequency and amplitude–phase functions of (2.3) can be established. In cases where $f(x, x')$ is symmetric about the origin of the coordinate system, i.e.,

$$f(x, x') = -f(-x, -x') \tag{2.15}$$

is fulfilled, $a_0 = 0$ is obtained, and the expressions (2.14) are reduced to the following forms

$$
\begin{aligned}
b &= -\frac{1}{\pi a_1 \eta} \int_0^{2\pi} f\left(a_1 \cos\psi; -a_1 \eta \sin\psi\right) \sin\psi \, d\psi, \\
c &= \frac{1}{\pi a_1} \int_0^{2\pi} f\left(a_1 \cos\psi; -a_1 \eta \sin\psi\right) \cos\psi \, d\psi.
\end{aligned}
\tag{2.16}
$$

Thus, the equivalent linear equation of (2.3) and the corresponding zeroth and first approximations of the nonlinear vibrations have been established. The zeroth approximation is meant to be the shift of the mean line of the vibration from the time axis. As was shown in [43], a proper selection of transformation on the phase plane can also give a geometrical interpretation of the first step of other methods.

Based on the first approximation (2.10), the next section will detail the approximation with additional higher-order harmonics and the successive refining procedure.

## 3. The improvement of linear approximation with higher order harmonics and the successive refining procedure

Here and below the partial differentials with respect with $x$ and $x'$ of $f(x, x')$ are still assumed to be continuous, or at least piecewisely, within the considered range of arguments, and the equalizing plane of $z = bx' + cx + d$ differs slightly from $f(x, x')$ along the phase curve. From this point, we no longer assume that $f(x, x')$ may take the form of (2.15), but the dependency of quantities of $b$, $c$ on $\eta$ and $a_1$ are still emphasized. The amplitudes of the higher harmonics are computed on the basis of the first approximation (2.5). In particular, we still assume that the nonlinear vibration governed by the equation of

$$\eta^2 x'' + f(x, \eta x') = g_0 \cos\varphi \tag{3.1}$$

which is approximated with (2.5) within an acceptable accuracy range, where (2.5) is the stationary periodic solution of the equivalent linear equation of

$$\eta^2 x'' + b\eta x' + cx + d = g_0 \cos\varphi, \tag{3.2}$$

where $\varphi = \eta\tau$. Based on practical observations (see [55,56]), this is likely to occur when (3.1) is "weakly" nonlinear and $\eta$ slightly differs from the undamped linear natural angular frequency of (3.1), i.e., when the nonlinear terms of (3.1) are ignored. In line with the above, $f(x, \eta x')$, along the phase curve and in its neighborhood, assumes the form

$$f(x, \eta x') = b(a_1, \eta)\eta x' + c(a_1, \eta)x + \varepsilon h(a_1, \eta), \tag{3.3}$$

where $\varepsilon$ is a "small" parameter, while the quantities $b(a_1, \eta)$ and $c(a_1, \eta)$ are calculated by (2.14). The approximating solution of (3.1) is still assumed in the form

$$x_1(\psi) = a_1 \cos\psi, \tag{3.4}$$

where $\psi = \eta\tau - \vartheta_1$, $a_1$ and $\vartheta_1$ fulfill the expressions (2.6). The periodic solution of (3.1), which contains higher order harmonics, is taken to be

$$x_{1,\Delta}(\psi) = x_1(\psi) + \Delta x_1(\psi), \tag{3.5}$$

where the Fourier series of $\Delta x_1(\psi)$ can be written as

$$\Delta x_1(\psi) = C_{0,1} + \sum_{n=2}^{\infty}(C_{n,1}\cos n\psi + D_{n,1}\sin n\psi). \tag{3.6}$$

Since $f(x, \eta x')$ takes the form of (3.3), and the system is forced at almost its natural frequency, we may assume that $\|\Delta x_1\| \ll \|x_1\|$, which is emphasized by this short form

$$\Delta x_1 = \varepsilon u_1. \tag{3.7}$$

Substituting (3.5)–(3.7) into (3.1), we obtain

$$\eta^2 x_1'' + \eta^2\varepsilon u_1'' + f[x_1 + \varepsilon u_1, \eta(x_1' + \varepsilon u_1')] = g_0\cos(\psi + \vartheta_1), \tag{3.8}$$

where $x_1 = a_1\cos\psi$ and $(\dots)' = \frac{d}{d\psi}$. Let us generate the linearized form of $f[x_1 + \varepsilon u_1, \eta(x_1' + \varepsilon u_1')]$ of (3.8) at $(x_1, \eta x_1')$, i.e., within the neighborhood of the phase curve. This results in

$$f[x_1 + \varepsilon u_1, \eta(x_1' + \varepsilon u_1')] \approx f(x_1, \eta x_1') + \frac{\partial f}{\partial x_1}\Big|_{(x_1, \eta x_1')}\varepsilon u_1$$
$$+ \frac{\partial f}{\partial(\eta x_1')}\Big|_{(x_1, \eta x_1')}\eta\varepsilon u_1', \tag{3.9}$$

where, from (3.3), the expressions

$$\frac{\partial f}{\partial x_1} = c(a_1, \eta) + \varepsilon\frac{\partial h}{\partial x_1}$$
$$\frac{\partial f}{\partial(\eta x_1')} = b(a_1, \eta) + \varepsilon\frac{\partial h}{\partial(\eta x_1')} \tag{3.10}$$

are yielded. If the expressions (3.9)–(3.10) are substituted into (3.8), and the terms which includes $\varepsilon^2$ are ignored, then

$$\eta^2 x_1'' + \eta^2\varepsilon u_1'' + f(x_1, \eta x_1') + b\eta\varepsilon u_1' + c\varepsilon u_1 = g_0\cos(\psi + \vartheta_1) \tag{3.11}$$

is obtained. If the expressions (3.4) and (3.6) are substituted into (3.11), then

$$-\eta^2 a_1\cos\psi + \eta^2\sum_{n=2}^{\infty}\left[n^2(C_{n,1}\cos n\psi + D_{n,1}\sin n\psi)\right]$$

$$+ f(a_1\cos\psi, -a_1\eta\sin\psi)$$

$$+ \eta b\sum_{n=2}^{\infty}\left[n(-C_{n,1}\sin n\psi + D_{n,1}\cos n\psi)\right] + cC_{01} \tag{3.12}$$

$$+ c\sum_{n=2}^{\infty}(C_{n,1}\cos n\psi + D_{n,1}\sin n\psi)$$

$$= g_0(\cos\psi\cos\vartheta_1 - \sin\psi\sin\vartheta_1)$$

is obtained, where $f(a_1\cos\psi, -a_1\eta\sin\psi)$ is expanded into the Fourier series

$$f = G_{0,1} + \sum_{n=1}^{\infty}(G_{n,1}\cos n\psi + H_{n,1}\sin n\psi) \tag{3.13}$$

and yields the coefficients

$$G_{0,1} = \frac{1}{2\pi}\int_0^{2\pi} f(a_1\cos\psi, -a_1\eta\sin\psi)\,d\psi$$
$$G_{n,1} = \frac{1}{\pi}\int_0^{2\pi} f(a_1\cos\psi, -a_1\eta\sin\psi)\cos n\psi\,d\psi \tag{3.14}$$
$$H_{n,1} = \frac{1}{\pi}\int_0^{2\pi} f(a_1\cos\psi, -a_1\eta\sin\psi)\sin n\psi\,d\psi.$$

Substituting (3.13) into (3.12), and matching the coefficients of the appropriate harmonic terms, the relations

$$\begin{aligned}
&c(a_1, \eta)C_{0,1} + G_{0,1} = 0\\
&-\eta^2 a_1 + G_{1,1} = g_0\cos\vartheta_1\\
&H_{1,1} = -g_0\sin\vartheta_1\\
&[c(a_1, \eta) - n^2\eta^2]C_{n,1} + n\eta b(a_1, \eta)D_{n,1} + G_{n,1} = 0\\
&-n\eta b(a_1, \eta)C_{n,1} + [c(a_1, \eta) - n^2\eta^2]D_{n,1} + H_{n,1} = 0
\end{aligned} \tag{3.15}$$

are obtained. The 2nd and 3rd lines of (3.15) are identical equations, while the remaining equations yield the Fourier coefficients of $\Delta x_1$, which can be formulated as follows:

$$C_{0,1} = -\frac{G_{0,1}}{c(a_1, \eta)},$$
$$C_{n,1} = \frac{-[c(a_1, \eta) - n^2\eta^2]G_{n,1} + n\eta b(a_1, \eta)H_{n,1}}{[c(a_1, \eta) - n^2\eta^2]^2 + [n\eta b(a_1, \eta)]^2}, \tag{3.16}$$
$$D_{n,1} = \frac{n\eta b(a_1, \eta)G_{n,1} + [c(a_1, \eta) - n^2\eta^2]H_{n,1}}{[c(a_1, \eta) - n^2\eta^2]^2 + [n\eta b(a_1, \eta)]^2}.$$

In "weakly" damped systems, when $b(a_1, \eta)$ is "small" enough, and the relations

$$|n\eta b H_{n,1}| \ll |(c - n^2\eta^2)G_{n,1}| \tag{3.17}$$

$$|n\eta b G_{n,1}| \ll |(c - n^2\eta^2)H_{n,1}|$$

hold at the same time, expressions (3.16) will lead to the following simpler forms

$$C_{n,1} = -\frac{G_{n,1}}{c(a_1, \eta) - n^2\eta^2},$$
$$D_{n,1} = -\frac{H_{n,1}}{c(a_1, \eta) - n^2\eta^2}. \tag{3.18}$$

At resonance, namely when $c = \eta^2$, the expressions (3.18) takes the forms

$$C_{n,1} \approx -\frac{G_{n,1}}{(1 - n^2)c(a_1, \eta)}$$
$$D_{n,1} \approx -\frac{H_{n,1}}{(1 - n^2)c(a_1, \eta)}, \tag{3.19}$$

hence $\Delta x_1$ of (3.5) can be written as

$$\Delta x_1 = -\frac{G_{0,1}}{c} + \frac{1}{c}\sum_{n=2}^{\infty}\left(\frac{G_{n,1}\cos n\psi + H_{n,1}\sin n\psi}{n^2 - 1}\right). \tag{3.20}$$

The $G_{0,1}$, $G_{n,1}$ and $H_{n,1}$ coefficients of (3.20) can be calculated from (3.14). Using the expressions (3.4) and (3.20), the improved approximation solution of (3.1), which already contains the higher-order harmonics, can be formulated as follows:

$$\begin{aligned}
x_{1,\Delta}(\psi) &= x_1(\psi) + \Delta x_1(\psi)\\
&= a_1\cos\psi - \frac{G_{0,1}}{c} + \frac{1}{c}\sum_{n=2}^{\infty}\left(\frac{G_{n,1}\cos n\psi + H_{n,1}\sin n\psi}{n^2 - 1}\right).
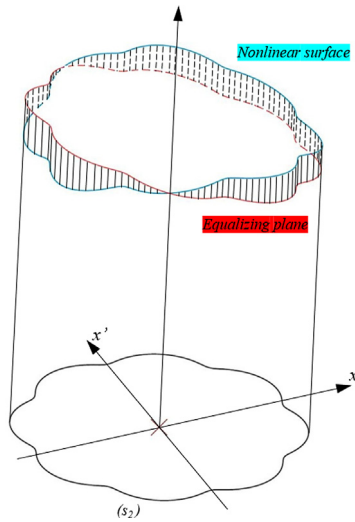\end{aligned} \tag{3.21}$$

**Fig. 2.** The refining process along the phase curve of the improved solution.

Now we turn to the refining process and apply it to (3.21). This procedure is based on the idea of finding a new equalizing plane that approximates $f\left(x_{1,\Delta}, \eta x'_{1,\Delta}\right)$ along the new phase curve that now belongs to the improved (3.21) solution. Now let the equivalent linear equation

$$\eta^2 x'' + b_1 \eta x' + c_1 x = g_0 \cos \varphi \qquad (3.22)$$

be assigned to (3.1), whose particular solution shall take the form $a_2 \cos \psi$. Hence, the approximation solution of (3.1) takes the form

$$x_2(\psi) = a_2 \cos \psi + \Delta x_1, \qquad (3.23)$$

where $a_2$ is unknown for the time being, while $\Delta x_1$ still assumes the form of (3.20). During the refining process of approximation solution (3.21), the $b_1(a_2, \eta)$ and $c_1(a_2, \eta)$ parameters of the equalizing plane $z_1 = b_1 \eta x' + c_1 x$ are calculated in such a way to provide a satisfactory approximation of the nonlinear surface $f\left(x_{1,\Delta}, \eta x'_{1,\Delta}\right)$ along the $s_{1,\Delta}$ phase curve of (3.22) (Fig. 2).

Accordingly, the following extremum problem along the phase curve can be set up:

$$\oint_{(s_{1,\Delta})} \left[ f\left(a_2 \cos \psi + \Delta x_1, -a_2 \sin \psi + \Delta x'\right) \right.$$
$$\left. - \left(b_1 \eta x'_{1,\Delta} + c_1 x_{1,\Delta} + d_1\right) \right]^2 ds = \min. \qquad (3.24)$$

The element arc along the phase-curve $s_{1,\Delta}$ can be defined again. If we wrote it in the exact form, even in simple cases this would transform (3.24) into a complicated elliptic integral. To make the treatise simpler, the $ds_2 = a_2 d\psi$ approximation is assumed, i.e., the $s_{1,\Delta}$ phase curve is replaced by a circle having the same center and with the radius $a_2$. Hence, the simplified extremum calculation yields the expressions

$$b_1(a_2, \eta) = \frac{\int_0^{2\pi} f\left(a_2 \cos \psi + \Delta x_1, -a_2 \sin \psi + \Delta x'_1\right)\left(-a_2 \sin \psi + \Delta x'_1\right) d\psi}{\eta \int_0^{2\pi} \left(-a_2 \sin \psi + \Delta x'_1\right)^2 d\psi}$$

$$c_1(a_2, \eta) = \frac{\int_0^{2\pi} f\left(a_2 \cos \psi + \Delta x_1, -a_2 \sin \psi + \Delta x'_1\right)\left(a_2 \cos \psi + \Delta x_1\right) d\psi}{\eta \int_0^{2\pi} \left(a_2 \cos \psi + \Delta x_1\right)^2 d\psi},$$
$$\qquad (3.25)$$

where the $b_1(a_2, \eta)$ and $c_1(a_2, \eta)$ parameters of the equalizing plane are still the functions of the amplitude and angular frequency. $a_2$ and $\vartheta_2$, included by (3.23), are calculated from the equivalent linear equation (3.22) by the expressions (2.6). Based on the improved and

refined solution, another improved approximation can be established including even higher order harmonics. For further improvement and refinement, this procedure can be continued up to an arbitrary number of new higher-order terms. The flowchart below (Table 1) outlines some steps of the improving–refining procedure

## 4. Nonlinear systems - examples

This section demonstrates the applicability, accuracy and efficiency of the proposed method through the analysis of well-known and frequently referenced nonlinear forced systems. In order to demonstrate the suitability of the proposed method, our results are compared to those of [53], which were obtained with the application of the Galerkin method. Section 4.1 gives details about the application of the proposed linearization method and the improving steps of adding higher-order harmonics. The successive refining process following each improving step is also detailed. The subsequent subsections, without detailed calculations, give only the final results of the linearization and the steps of the successive improvement and refining procedures.

### 4.1. A truly nonlinear (TNL) system

A possible dimensionless equation of motion of a truly nonlinear system may take the form of

$$x'' + x^3 = \cos \tau \qquad (4.1)$$

[3,5], where, as per (2.2), $\eta = 1$. The characteristic surface of (4.1) cannot be linearized at $x = 0$, hence (4.1) is a truly nonlinear equation of motion [3]. Since the system under investigation is undamped, numerical computation, for example the method of Runge–Kutta, may encounter difficulties originating from the integrals. As regards (4.1), or specific $x'' + x^3 = \sin \tau$, it has been pointed out that one and only one isolated periodic solution exists. This was established by Urabe and Reiter with the application of the Galerkin method [53]. Their solution was calculated to 6 decimal places, resulting in the numerical expression

$$x(\tau) = 1.431189 \sin \tau - 0.126915 \sin 3\tau + 0.009754 \sin 5\tau$$
$$- 0.000763 \sin 7\tau + 0.000059 \sin 9\tau. \qquad (4.2)$$

Below, the approximation solution of (4.1), also to 6 decimal places, is established by the proposed method of linearization above the phase curve, and then our results are compared to those of [53].

Since no damping is assumed, the equivalent linearized equation (4.1) takes the form

$$x'' + k_e x = \cos \tau. \qquad (4.3)$$

The closed-form first approximating solution of (4.3) assumes the form

$$x_1(\psi) = a_1 \cos \psi, \qquad (4.4)$$

where $\psi = \tau - \vartheta_1$. Since the $f(x, x') = x^3$ nonlinear characteristic surface of (4.1) is symmetric about the origin of the phase space, and thus (4.4) and (2.16) can be exploited, the equivalent spring stiffness included by (4.3) takes the form

$$k_e = \frac{3}{4} a_1^2. \qquad (4.5)$$

Taking into account the expressions (4.5) and (2.6), the unknown $a_1$ amplitude of the first approximation can be expressed as

$$a_1 = \frac{1}{|k_e - 1|} = \frac{1}{\left|\frac{3}{4} a_1^2 - 1\right|}. \qquad (4.6)$$

Solving (4.6) for $a_1$, the calculation leads to the result of $a_1 = 1.4922$.

Based on (4.6), the 1st improved approximation expression of (4.1), which includes higher harmonics, can now be established, taking the form of (3.21). Due to the origin symmetry of $f(x, x') = x^3$,

**Table 1**
The first few and the general $(i-1)$th and $i$th steps of the improving–refining procedure.

---

**1st approximation**

$x_1(\psi) = a_1 \cos \psi$ (obtained as detailed in Chapter 2)

---

**1st improvement** ($\Delta x_1$ is calculated for improving $x_1$, hence $x_{1,\Delta}$ is obtained)

$x_{1,\Delta}(\psi) = a_1 \cos \psi + \Delta x_1$

$\Delta x_1 = C_{01} + \sum_{n=2}^{\infty} \{ C_{n1} \cos(n\psi) + D_{n1} \sin(n\psi) \}$

$C_{01} = -\dfrac{G_0}{c(a_1, \eta)}, \quad C_{n1} = \dfrac{-\left[c(a_1,\eta) - n^2\eta^2\right] G_n + n\eta b(a_1,\eta) H_n}{\left[c(a_1,\eta) - n^2\eta^2\right]^2 + \left[n\eta b(a_1,\eta)\right]^2}, \quad D_{n1} = \dfrac{n\eta b(a_1,\eta) G_n + \left[c(a_1,\eta) - n^2\eta^2\right] H_n}{\left[c(a_1,\eta) - n^2\eta^2\right]^2 + \left[n\eta b(a_1,\eta)\right]^2}$

$G_{01} = \dfrac{1}{2\pi} \int_0^{2\pi} f(a_1 \cos \psi, -a_1 \eta \sin \psi)\, d\psi, \quad G_{n1} = \dfrac{1}{\pi} \int_0^{2\pi} f(a_1 \cos \psi, -a_1 \eta \sin \psi) \cos n\psi\, d\psi, \quad H_{n1} = \dfrac{1}{\pi} \int_0^{2\pi} f(a_1 \cos \psi, -a_1 \eta \sin \psi) \sin n\psi\, d\psi$

---

**2nd approximation** ($a_2$ is calculated to refine $a_1$, hence $x_2$ is obtained)

$x_2(\psi) = a_2 \cos \psi + \Delta x_1$

$\oint_{(s_2)} \left[ f(a_2 \cos \psi + \Delta x_1, -a_2 \eta \sin \psi + \Delta x_1') - (b_1 \eta x_2' + c_1 x_2 + d_1) \right]^2 ds = \min.$

$b_1(a_2, \eta) = \dfrac{\int_0^{2\pi} f(a_2 \cos \psi + \Delta x_1, -a_2 \eta \sin \psi + \Delta x_1')(-a_2 \eta \sin \psi + \Delta x_1')\, d\psi}{\eta \int_0^{2\pi} (-a_2 \eta \sin \psi + \Delta x_1')^2\, d\psi},$

$c_1(a_2, \eta) = \dfrac{\int_0^{2\pi} f(a_2 \cos \psi + \Delta x_1, -a_2 \eta \sin \psi + \Delta x_1')(a_2 \cos \psi + \Delta x_1)\, d\psi}{\eta \int_0^{2\pi} (a_2 \cos \psi + \Delta x_1)^2\, d\psi}$

$a_2 = \dfrac{g_0}{\sqrt{\left[c_1(a_2, \eta) - \eta^2\right]^2 + \left[b_1(a_2, \eta)\eta\right]^2}}, \quad \vartheta_2 = \text{arctg}\left(\dfrac{b_1 \eta}{c_1 - \eta^2}\right), \quad a_2 = \cdots, \quad \vartheta_2 = \cdots$

---

**$(i-1)$th improvement** ($\Delta x_{i-1}$ is calculated for improving $x_{i-1}$, hence $x_{i-1,\Delta}$ is obtained)

$x_{i-1,\Delta}(\psi) = a_{i-1} \cos \psi + \sum_{j=1}^{i-2} \Delta x_j + \Delta x_{i-1}$

$\Delta x_{i-1} = C_{0,i-1} + \sum_{n=2}^{\infty} \{ C_{n,i-1} \cos(n\psi) + D_{n,i-1} \sin(n\psi) \}$

$C_{0,i-1} = -\dfrac{G_{0,i-1}}{c_{i-1}}$

$C_{n,i-1} = \dfrac{-\left[c_{i-1} - n^2\eta^2\right]\left[n^2\eta^2 \sum_1^{i-1} C_{n,j} - G_{n,i}\right] + n\eta b_{i-1}\left[n^2\eta^2 \sum_1^{i-1} D_{n,j} - H_{n,i}\right]}{\left[c_{i-1} - n^2\eta^2\right]^2 + \left[n\eta b_{i-1}\right]^2},$

$D_{n,i-1} = \dfrac{n\eta b_{i-1}\left[n^2\eta^2 \sum_1^{i-1} C_{n,j} - G_{n,i}\right] + \left[c_{i-1} - n^2\eta^2\right]\left[n^2\eta^2 \sum_1^{i-1} D_{n,j} - H_{n,i}\right]}{\left[c_{i-1} - n^2\eta^2\right]^2 + \left[n\eta b_{i-1}\right]^2},$

*where*

$G_{0,i-1} = \dfrac{1}{2\pi} \int_0^{2\pi} f\left(a_{i-1} \cos \psi + \sum_{j=1}^{i-2} \Delta x_j - a_{i-1}\eta \sin \psi \sum_{j=1}^{i-2} \Delta x_j'\right) d\psi$

$G_{n,i-1} = \dfrac{1}{\pi} \int_0^{2\pi} f\left(a_{i-1} \cos \psi + \sum_{j=1}^{i-2} \Delta x_j - a_{i-1}\eta \sin \psi \sum_{j=1}^{i-2} \Delta x_j'\right) \cos n\psi\, d\psi, \quad H_{n,i-1} = \dfrac{1}{\pi} \int_0^{2\pi} f\left(a_{i-1} \cos \psi + \sum_{j=1}^{i-2} \Delta x_j - a_{i-1}\eta \sin \psi \sum_{j=1}^{i-2} \Delta x_j'\right) \sin n\psi\, d\psi$

*and*

$c_{i-1} = c_{i-1}(a_{i-1}, \eta)$
$b_{i-1} = b_{i-1}(a_{i-1}, \eta)$

---

**$i$th approximation** ($a_i$ is calculated to refine $a_{i-1}$, hence $x_i$ is obtained)

$x_i(\psi) = a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j$

$\oint_{(s_i)} \left[ f\left(a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j, -a_i \eta \sin \psi + \sum_{j=1}^{i-1} \Delta x_j'\right) - (b_{i-1}\eta x' + c_{i-1} x + d_{i-1}) \right]^2 ds = \min.$

$b_{i-1} = \dfrac{\int_0^{2\pi} f\left(a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j, -a_2 \eta \sin \psi + \sum_{j=1}^{i-1} \Delta x_j'\right)\left(-a_2 \eta \sin \psi + \sum_{j=1}^{i-1} \Delta x_j'\right) d\psi}{\eta \int_0^{2\pi} \left(-a_2 \eta \sin \psi + \sum_{j=1}^{i-1} \Delta x_j'\right)^2 d\psi},$

$c_{i-1} = \dfrac{\int_0^{2\pi} f\left(a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j, -a_2 \eta \sin \psi + \sum_{j=1}^{i-1} \Delta x_{j\,1}'\right)\left(a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j\right) d\psi}{\eta \int_0^{2\pi} \left(a_i \cos \psi + \sum_{j=1}^{i-1} \Delta x_j\right)^2 d\psi}$

$a_i = \dfrac{g_0}{\sqrt{\left[c_{i-1}(a_i, \eta) - \eta^2\right]^2 + \left[b_{i-1}(a_i, \eta)\eta\right]^2}}, \quad \vartheta_i = \text{arctg}\left(\dfrac{b_{i-1}\eta}{c_{i-1} - \eta^2}\right), \quad a_i = \cdots, \quad \vartheta_i = \cdots$

---

the $G_{0,1}$ included by (3.21) fulfills $G_{0,1} = 0$, while, as the consequence of the trigonometric identity of $f(x, x') = x^3 = a_1^3 \cos^3 \psi = a_1^3 \left(\frac{3}{4} \cos \psi + \frac{1}{4} \cos 3\psi\right)$, $G_{n,1}$, $H_{n,1}$ are obtained as $G_{1,1} = \frac{3}{4} a_1^3$, $G_{3,1} = \frac{1}{4} a_1^3$ and $H_{n,1} = 0$. On the basis of expressions (2.6) and (3.15)–(3.16), expression (3.21) of the 2nd approximation includes only one term of higher harmonics, which takes the form $\Delta x_1 = C_{3,1} \cos 3\psi = \dfrac{a_1^3}{4\left(9 - \frac{3}{4}a_1^2\right)} \cos 3\psi$. Hence, after the computation, the form of the second approximation solution of (4.1) can be expressed as

$$x_{1,\Delta}(\psi) = a_1 \cos \psi + \Delta x_1 = a_1 \cos \psi + \frac{a_1^3}{4\left(9 - \frac{3}{4}a_1^2\right)} \cos 3\psi$$

$$= 1.4922 \cos \psi + 0.1133 \cos 3\psi. \tag{4.7}$$

Based on (4.7), the refined second approximation solution of (4.1) is assumed in the form

$$x_2(\psi) = a_2 \cos\psi + \Delta x_1 = a_2 \cos\psi + \frac{a_1{}^3}{4\left(9 - \frac{3}{4}a_1{}^2\right)} \cos 3\psi, \qquad (4.8)$$

where $a_2$ is the amplitude. $a_2$ is obtained by the successive linearization procedure over the phase curve of (4.7). The actual phase curve is approximated now with a circle having the radius of $a_2$. From (4.8) the nonlinear characteristic surface of (4.1) is yielded as follows:

$$f(x) = x^3 = \left( a_2 \cos\psi + \frac{a_1{}^3}{4\left(9 - \frac{3}{4}a_1{}^2\right)} \cos 3\psi \right)^3. \qquad (4.9)$$

During the refining process, which is in effect a successive linearization procedure, the (4.9) nonlinear "surface" is now approximated with a new $z_1 = b_1 x' + k_2 x$ equalizing plane. The $b_1$, $k_1$ parameters can be calculated by the expressions (3.25), which, since no damping is applied, lead to

$$b_1 = 0$$

$$k_1 = \frac{\frac{3}{4}a_2{}^2 + Ka_2 + 3K^2\left(1 + \frac{1}{4}K^2\right)}{1 + \frac{K^2}{a_2{}^2}}, \qquad (4.10)$$

where $K = \frac{a_1{}^3}{4\left(9 - \frac{3}{4}a_1{}^2\right)}$. From expressions (4.10) and (2.6) $a_2$ is obtained in a form of expression which is similar to (4.6), and which yields

$$a_2 = \frac{1}{\left| k_1(a_2) - 1 \right|}. \qquad (4.11)$$

Taking into consideration that $k_1 = k_1(a_2)$ and $a_2 = a_2(a_1)$, and solving (4.11) numerically, the refined expression of (4.8) takes the form

$$x_2(\psi) = 1.421176 \cos\psi + 0.113337 \cos 3\psi. \qquad (4.12)$$

Applying the algorithm (3.5)–(3.21) again, the next stage of approximation based on (4.12) can be established. It now includes additional terms of higher harmonics, and can be written as follows:

$$x_{2,\Delta}(\psi) = 1.421176 \cos\psi + 0.119097 \cos 3\psi + 0.007957 \cos 5\psi + 0.000289 \cos 7\psi + 0.000004 \cos 9\psi. \qquad (4.13)$$

Due to the forcing term $\cos\tau$ included by (4.1), the coordinate transformation of $\psi - \frac{\pi}{2}$ is applied to (4.13). Due to the lack of the damping term, and from (2.6), which results in $\vartheta_2 = 0$, and also as a consequence of the introduced new variable of $\psi = \tau - \vartheta_2$, (4.13) takes the modified and the more accurate form of

$$x_{2,\Delta}(\tau) = 1.421176 \sin\tau - 0.119097 \sin 3\tau + 0.007957 \sin 5\tau - 0.000289 \sin 7\tau + 0.000004 \sin 9\tau. \qquad (4.14)$$

(4.14) now can be directly compared to the results of [53], whose solutions were obtained by the application of the Galerkin method. The table below summarizes the expressions of the sequential steps of the approximation procedure carried out above (Table 2). The expressions involved in the table are obtained as the result of the $\psi - \frac{\pi}{2}$ coordinate transformation applied to the expressions (4.4), (4.7), (4.12) and (4.13).

Table 3 compares the computational results (obtained by (4.14)) of our proposed linearization method to those calculated by Urabe and Reiter [53].

Besides the computational results represented in Table 2, the accuracy of our method is also demonstrated in Fig. 3, which represents the time history of the proposed method and of [53]. Fig. 4 shows the phase curves of both methods. From the comparison, it can be concluded that the 2nd-improved approximation of our method, despite not being refined, still provides a very good approximation of the results of Urabe and Reiter.

**Table 2**
The approximation steps of the proposed method when $\eta = 1$.

| The successive steps of the approximation procedure: | $x(\tau)$ |
|---|---|
| 1st approximation | $x_1(\tau) = 1.4922 \sin\tau$ |
| 1st approximation — improved | $x_{1,\Delta}(\tau) = 1.4922 \sin\tau - 0.1133 \sin 3\tau$ |
| 2nd approximation | $x_2(\tau) = 1.421176 \sin\tau - 0.113337 \sin 3\tau$ |
| 2nd approximation — improved | $x_{2,\Delta}(\tau) = 1.421176 \sin\tau - 0.119097 \sin 3\tau + 0.007957 \sin 5\tau - 0.000289 \sin 7\tau + 0.000004 \sin 9\tau$ |
| Urabe and Reiter solution [53] | $x_{UR}(\tau) = 1.431189 \sin\tau - 0.126915 \sin 3\tau + 0.009754 \sin 5\tau - 0.000763 \sin 7\tau + 0.000059 \sin 9\tau$ |

**Table 3**
Comparison of results of Urabe and Reiter [53] and the proposed solutions when $\eta = 1$.

| $\tau$ | Urabe & Reiter solution [53] | Proposed solution (4.14) | R. error [%] |
|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 0.000000 |
| $0.1\pi$ | 0.348737 | 0.350540 | 0.517087 |
| $0.2\pi$ | 0.721221 | 0.722351 | 0.156608 |
| $0.3\pi$ | 1.108687 | 1.104909 | 0.340738 |
| $0.4\pi$ | 1.435235 | 1.421448 | 0.960550 |
| $0.5\pi$ | 1.568650 | 1.548523 | 1.283078 |
| $0.6\pi$ | 1.435235 | 1.421448 | 0.960550 |
| $0.7\pi$ | 1.108687 | 1.104909 | 0.340738 |
| $0.8\pi$ | 0.721221 | 0.722351 | 0.156608 |
| $0.9\pi$ | 0.348737 | 0.350540 | 0.517087 |
| $\pi$ | 0.000000 | 0.000000 | 0.000000 |
| $1.1\pi$ | −0.348737 | −0.350540 | 0.517087 |
| $1.2\pi$ | −0.721221 | −0.722351 | 0.156608 |
| $1.3\pi$ | −1.108687 | −1.104909 | 0.340738 |
| $1.4\pi$ | −1.435235 | −1.421448 | 0.960550 |
| $1.5\pi$ | −1.568650 | −1.548523 | 1.283078 |
| $1.6\pi$ | −1.435235 | −1.421448 | 0.960550 |
| $1.7\pi$ | −1.108687 | −1.104909 | 0.340738 |
| $1.8\pi$ | −0.721221 | −0.722351 | 0.156608 |
| $1.9\pi$ | −0.348737 | −0.350540 | 0.517087 |
| $2\pi$ | 0.000000 | 0.000000 | 0.000000 |

Fig. 5 displays the phase curves from each of the sequential steps of the approximation procedure of the proposed linearization method. Besides, their convergence to the exact solution of Urabe and Reiter can be followed.

The results displayed in both tables and figures demonstrate the applicability, efficiency and accuracy of the proposed linearization method above the phase curve. by comparing them to those of Urabe and Reiter, whose results were established by the Galerkin method with accuracy to 6 decimal places [53].

### 4.2. The forced van der Pol system

This section details the analysis of the harmonically forced van der Pol system, which is also a well-known system that has been studied for decades. We use it to demonstrate the efficiency of our linearization method. The governing equation of such a system may take several forms, of which the special form that can be found in [53] is given as

$$\ddot{x} - \varepsilon\left(1 - x^2\right)\dot{x} + x = \varepsilon E \sin\omega t, \qquad (4.15)$$

where $\dot{x} = \frac{dx}{dt}$. Some new parameters for (4.15) are introduced in accordance with [53], i.e., $\varepsilon = 0.1$, $\omega = 0.9$, $E = 3$. Introducing the new $\omega t = \tau$, $\frac{\varepsilon}{\omega} = \lambda$, $\frac{E}{\omega} = E_1$, $\frac{1 - \omega^2}{\varepsilon\omega} = A$ variables and parameters, (4.15) takes the form

$$x'' - \lambda\left(1 - x^2\right)x' + (1 + \lambda A)x = \lambda E_1 \sin\tau, \qquad (4.16)$$

where $x' = \frac{dx}{d\tau}$. The expression

$$f = -\lambda\left(1 - x^2\right)x' + (1 + \lambda A)x \qquad (4.17)$$
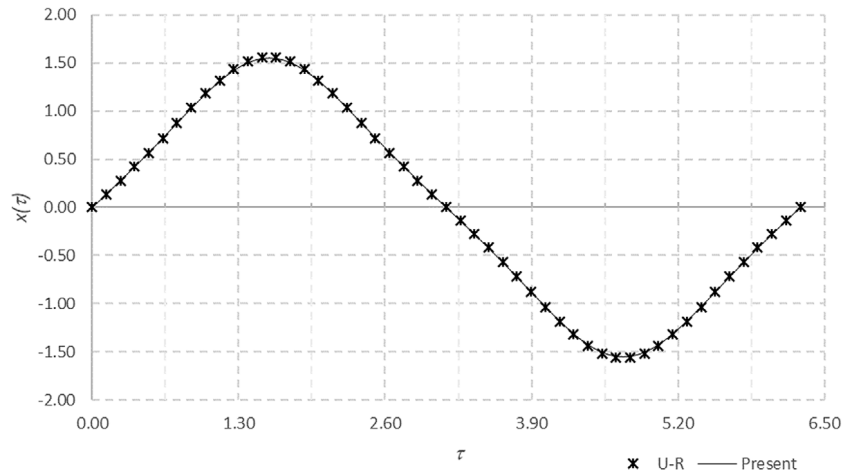
**Fig. 3.** Comparison of time-history diagram of displacement between the proposed (4.14) and the exact (4.2) solutions at $\eta = 1$.
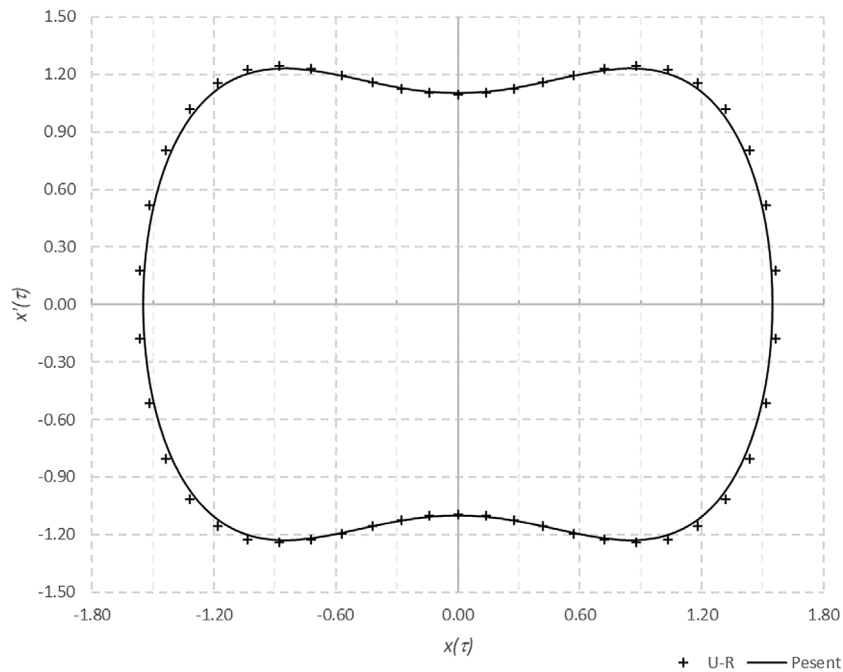


**Fig. 4.** Comparison of the phase curves belong to the proposed (4.14) and the exact (4.2) solutions for one period when $\eta = 1$.

is the nonlinear characteristic surface of (4.16) with the numerical values of $\lambda = \frac{1}{9}$, $A = \frac{19}{9}$, $E_1 = \frac{10}{3}$. Since (4.17) complies with (2.15), exploiting Eqs. (2.16), (2.6) and (4.17), we are led to the equivalent linearized equation assigned to (4.16). Without detailing the computations, the expression

$$x_1(\tau) = 1.524718 \sin \tau + 0.287549 \cos \tau \qquad (4.18)$$

is yielded. Completing (4.18) with additional higher-order harmonics, the improved 1st approximation is obtained with the form

$$x_{1,\Delta}(\tau) = x_1 + \Delta x_1 = 1.524718 \sin \tau + 0.287549 \cos \tau - 0.007266 \sin 3\tau - 0.011212 \cos 3\tau. \qquad (4.19)$$

From (4.19), having been refined with the successive procedure (3.22)–(3.25), the refined 2nd approximation is obtained:

$$x_2(\tau) = a_2 \cos \varphi + \Delta x_1 = 1.524966 \sin \tau + 0.287692 \cos \tau - 0.007266 \sin 3\tau - 0.011212 \cos 3\tau. \qquad (4.20)$$

Based on (4.20), the 2nd refined approximation, and then, by adding further terms of higher harmonics, the improved 2nd approximation solution of (4.16) can also be established, and the following expression is yielded:

$$\begin{aligned} x_{2,\Delta}(\tau) &= a_2 \cos \varphi + \Delta x_1 = 1.524966 \sin \tau \\ &+ 0.287692 \cos \tau - 0.007266 \sin 3\tau \\ &- 0.011212 \cos 3\tau - 0.000110 \sin 5\tau - 0.000153 \cos 5\tau - 0.0000011 \sin 7\tau \\ &+ 0.0000003 \cos 7\tau. \end{aligned} \qquad (4.21)$$

Let us compare our results from (4.21) to those of Urabe and Reiter [53]:

$$\begin{aligned} x(\tau) &= 1.529115 \sin \tau + 0.286713 \cos \tau + 0.007210 \sin 3\tau - 0.011371 \cos 3\tau \\ &- 0.000111 \sin 5\tau - 0.000155 \cos 5\tau - 0.0000029 \sin 7\tau \\ &+ 0.0000007 \cos 7\tau, \end{aligned} \qquad (4.22)$$
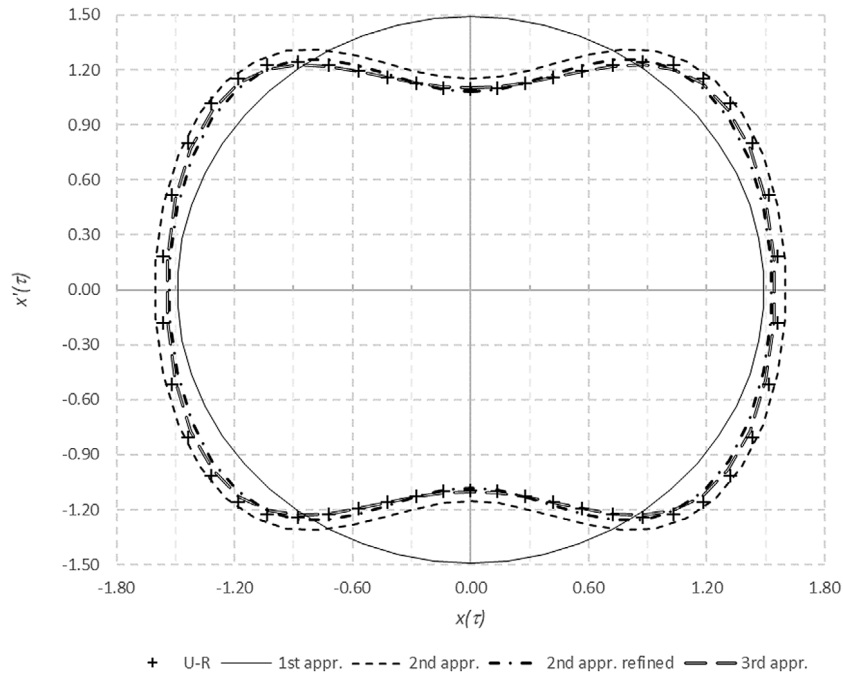
**Fig. 5.** The sequential steps of the approximation procedure.

**Table 4**
The approximation steps of the proposed method when $\lambda = \frac{1}{9}$, $A = \frac{19}{9}$, $E_1 = \frac{10}{3}$.

| The steps of approximation: | $x(\tau)$ |
|---|---|
| 1st approximation | $x_1(\tau) = 1.524718 \sin \tau + 0.287549 \cos \tau$ |
| 1st approximation — improved | $x_{1,\Delta}(\tau) = x_1 + \Delta x_1 = 1.524718 \sin \tau + 0.287549 \cos \tau - 0.007266 \sin 3\tau$ <br> $-0.011212 \cos 3\tau$ |
| 2nd approximation | $x_2(\tau) = a_2 \cos \varphi + \Delta x_1 = 1.524966 \sin \tau + 0.287692 \cos \tau - 0.007266 \sin 3\tau$ <br> $-0.011212 \cos 3\tau$ |
| 2nd approximation — improved | $x_{2,\Delta}(\tau) = a_2 \cos \varphi + \Delta x_1 = 1.524966 \sin \tau + 0.287692 \cos \tau + 0.006981 \sin 3\tau$ <br> $-0.011407 \cos 3\tau - 0.000110 \sin 5\tau - 0.000153 \cos 5\tau - 0.0000011 \sin 7\tau$ <br> $+0.0000003 \cos 7\tau$ |
| Urabe & Reiter solution [53] | $x(\tau)_{UR} = 1.529115 \sin \tau + 0.286713 \cos \tau + 0.007210 \sin 3\tau - 0.011371 \cos 3\tau$ <br> $-0.000111 \sin 5\tau - 0.000155 \cos 5\tau - 0.0000029 \sin 7\tau + 0.0000007 \cos 7\tau$ |

which is actually the expression of a limit-cycle curve established by the Galerkin method with accuracy to 6 decimal places (see Table 4, Fig. 6).

A high degree of correspondence can be observed between the exact solution by Urabe and Reiter and the solution established by the proposed linearization method above the phase curve (Fig. 6). Thus, the accuracy of our method is demonstrated also in the case of a van der Pol system.

The solution of (4.15) by the method of Galerkin was established also with another set of parameters in [53]. Now $\varepsilon = 0.1$, $\omega = 1$, $E = 1$, which yields $\lambda = 0.1$, $A = 0$, $E_1 = 1$, and thus (4.16) takes the form

$$x'' - 0,1\left(1 - x^2\right)x' + x = 0,1 \sin \tau, \qquad (4.23)$$

whose accurate solution to 6 decimal places by Urabe and Reiter can be written as

$$
\begin{aligned}
x(\tau) = {} & -0.142330 \sin \tau - 2.378786 \cos \tau + 0.041868 \sin 3\tau \\
& - 0.004647 \cos 3\tau \\
& + 0.000215 \sin 5\tau + 0.001223 \cos 5\tau - 0.0000398 \sin 7\tau \\
& + 0.0000098 \cos 7\tau.
\end{aligned} \qquad (4.24)
$$

The 1st approximation solution of (4.23) by the linearization method over the phase curve is obtained as

$$x_1(\tau) = -2.382976 \cos \tau. \qquad (4.25)$$

Adding some additional terms of higher-order harmonics to (4.25), the more accurate, "improved" 1st approximation

$$x_{1,\Delta}(\tau) = x_1 + \Delta x_1 = -2.382976 \cos \tau + 0.007256 \sin 3\tau + 0.011211 \cos 3\tau \qquad (4.26)$$

is yielded. Applying the refining procedure (3.22)–(3.25) to (4.26) and subsequently establishing additional terms of higher harmonics, the improved 2nd approximation solution of (4.23) is obtained, which can be written as follows:

$$
\begin{aligned}
x_{2,\Delta}(\tau) = {} & a_2 \cos \psi + \Delta x_1 + \Delta x_2 = -2.380544 \cos \tau + 0.042077 \sin 3\tau \\
& - 0.002904 \cos 3\tau \\
& + 0.000053 \sin 5\tau + 0.0012475 \cos 5\tau - 0.0000155 \sin 7\tau \\
& + 0.0000006 \cos 7\tau.
\end{aligned} \qquad (4.27)
$$

The comparison of (4.24) with (4.27) is illustrated in Fig. 7.

A high degree of correspondence between the exact solution (4.24) by Urabe and Reiter and the solution (4.27) established by the linearization method above the phase curve can be observed in Fig. 7.

We note, that at stronger nonlinearity, i.e. $\varepsilon \gg 0,1$, a larger number of higher-order terms need to be added to the approximation solutions than in the case of $\varepsilon = 0,1$.
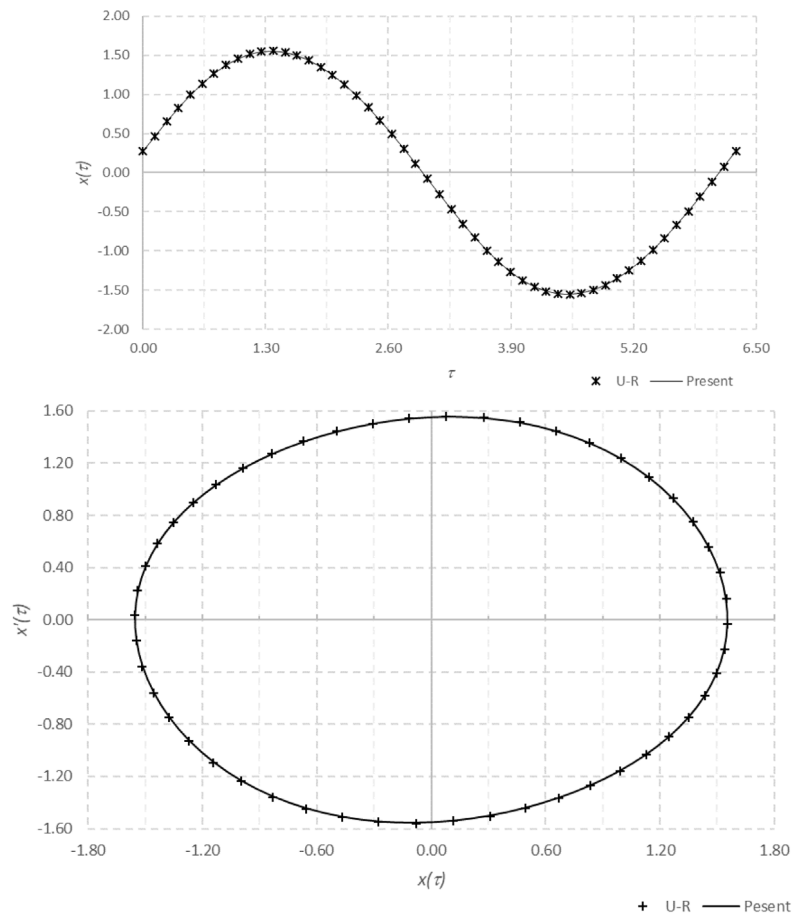
**Fig. 6.** Comparison of the time history and the phase curves belonging to the proposed (4.21) and the exact (4.22) solutions for one period when $\lambda = \frac{1}{9}$, $A = \frac{19}{9}$, $E_1 = \frac{10}{3}$.

## 5. Conclusions

This paper summarizes the theoretical background of a novel linearization method above the phase curve with the ensuing approximating and successive refining procedures. Besides the theoretical approach, through the analysis of well-known nonlinear, harmonically forced, SDOF, mass-point vibration systems, semi-analytical–numerical calculations are also applied in order to demonstrate the applicability of the proposed linearization method. The extremum principle applied for the linearization method is defined as a cyclic integral, which refers to the square differences between the nonlinear characteristic surface and the equalizing plane, and is calculated along the approximated phase curve of the original nonlinear system.

Based on the minimizing principle and assuming the first approximation solution as a simple harmonic function, the unknown coefficients of the equivalent linear equation are calculated first. This is followed by establishing the first approximation solution of the original nonlinear equation in the form of a simple harmonic expression. Then, on the basis of the first approximation, the improved expression of the first approximation is established by including additional higher-order harmonic terms. This step is followed by a successive refining procedure to obtain more accurate approximating solutions. The solutions of the proposed linearization method are compared to the results of Urabe and Reiter [53]. The analysis of the cubic TNL-system demonstrates that in the proposed linearization method, the original nonlinear equation does not require the inclusion of linear terms; hence, the method can be applied to the analytical investigation of even TNL systems.

We would like to emphasize our method's capability for geometrical interpretation, which may be important from the point of view of mechanical engineering and in education. The geometrical interpretation of the proposed linearization method can be carried out not just with the first approximation solution, but even in the case of approximating solutions that already include additional terms of higher-order harmonics.

Comparing our results obtained from the analysis of a TNL and a van der Pol system to other results from the literature, it can be claimed that the proposed linearization method, completed with additional terms of higher-order harmonics and the successive refining process, establishes approximating solutions of the original nonlinear equations accurately and efficiently.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
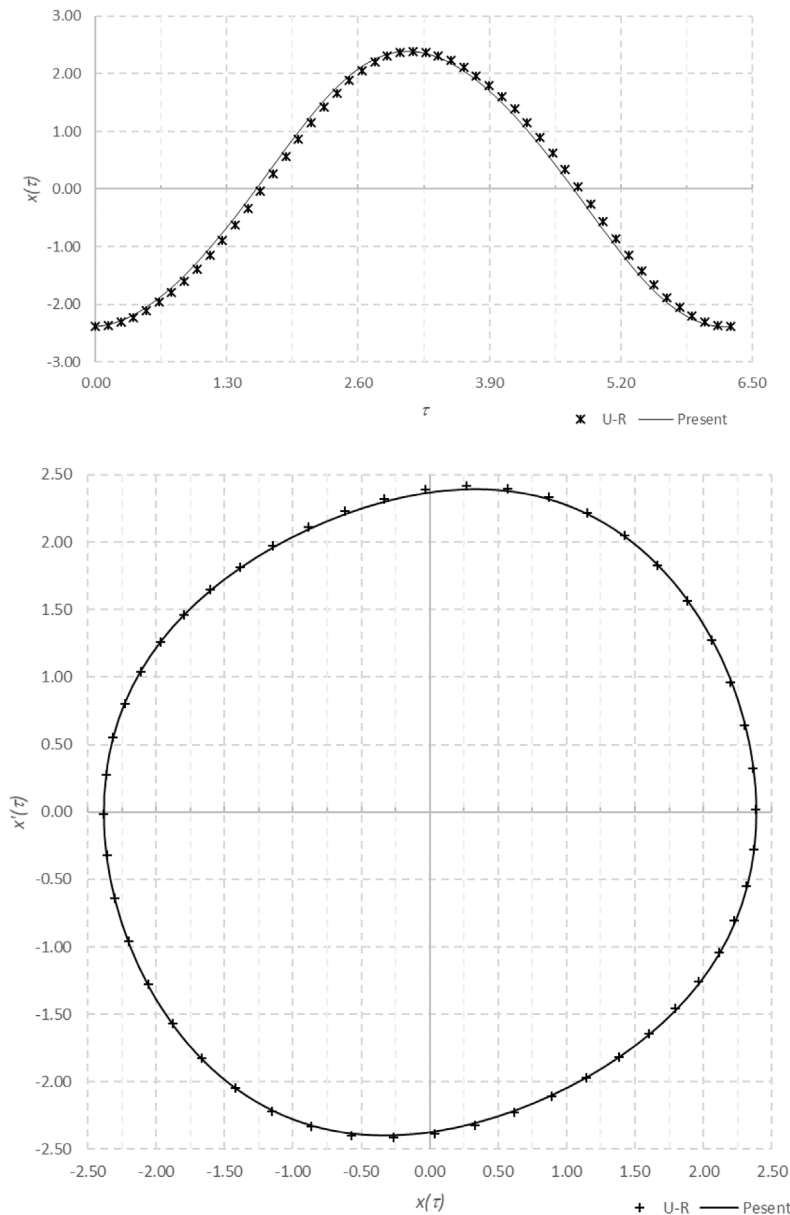
### Acknowledgments

**Fig. 7.** Comparison of the time history and the phase curves belonging to the proposed (4.27) and the exact (4.24) solutions for one period when $\lambda = 0.1$, $A = 0$, $E_1 = 1$.

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