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Boundary elements approach for solving stochastic nonlinear problems with fractional Laplacian terms

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A R T I C L E I N F O

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A B S T R A C T

Models involving stochastic diffusion equations are utilized for describing the evolution of a number of natural phenomena and are widely discussed in the open literature. In recent years, these models have been revisited in light of experimental observations in which "anomalous" diffusion processes were identified, such as in the propagation of acoustic waves in random media. In this context, a critical characteristic of the theoretical models is the introduction of fractional derivative operators in the associated governing equations. Specifically, anomalous diffusion involves a fractional Laplacian operator replacing the classical Laplacian. Currently, solutions to equations with fractional Laplacians are available for a quite limited numbers of cases. Further, to the authors' knowledge, no solutions are available for nonlinear equations involving fractional Laplacians. This fact creates the need of developing adequate numerical methods for estimating the response of this kind of systems. This paper proposes a Boundary Element Method (BEM)-based approach to determine the response of a system governed by a nonlinear fractional diffusion equation involving a random excitation. The approach is constructed by utilizing the integral representation of the classical Poisson equation solution, in which unknown constants are determined by the BEM. Then, based on a recently proposed representation of the fractional Laplacian operator, the value of the fractional Laplacian of the response is updated progressively by matrix transformation of these constants. Numerical results pertaining to a system exposed to white noise are presented to elucidate the mechanization of the approach. Further, parameter studies are done for examining the influence of the fractional Laplacian order on the system response.

1. Introduction

The theory of fractional calculus has emerged as a potent tool for modeling and describing the evolution of a number of natural phenomena. Indeed, currently fractional models are used for studies in electrical circuits, as well as in electrochemistry, viscoelasticity etc. [1]. From a physical point of view, fractional calculus is utilized for describing phenomena associated with memory effects or with non-local effects. In the former case, the operator is applied in time domain and allows capturing the influence of the past values of the response of a system on its current state. In the latter case, the operator is applied in the space-domain and affords modeling the long-range interactions between the particles of a system. For instance, time domain fractional models are utilized in the context of viscoelasticity, where the fractional operators are employed in the constitutive equations of the viscoelastic materials [2]. The non-local models are used, for instance, in the context of heat conduction [3]; in the propagation of acoustic waves in scattered media [4]; and in the propagation of waves in periodic media [5].

This paper focuses on the problem of calculating the response of a system endowed with a non-local fractional operator. A common feature of these problems is that the fractional derivative appears as a generalization of the classical Laplacian operator. Therefore, it is denoted as a fractional Laplacian. This notation is used in relevant applications relating to the study of the anomalous diffusion, which is a phenomenon that can be observed in turbulent systems [6,7], and the study of the boundary value problem associated with the Schrödinger type equation of a fractional Laplacian [8]. Determining explicit analytical solutions of problems involving fractional operators can be a daunting task, because currently exact solutions are, in general, unavailable in the literature. Thus, numerical approaches are desirable for estimating the response of systems involving factional operators. In this regard, several research efforts have been devoted to systems endowed with fractional operators defined in the time domain. Indeed, the response of single degree of freedom systems was estimated by numerical schemes based on Newmark integration; on appropriate change of variable and discretization of the fractional derivative operator; and

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by using the Fokker–Planck–Kolmogorov equation [9–12]. However, in case of problems involving the fractional Laplacian, specific challenges related to the nonlocal properties of the operator arise. In this context, a numerical approach based on the finite difference method for the fractional Laplacian in one dimension was proposed by Huang and Oberman [13].

This paper proposes a numerical scheme based on the Boundary Elements Method (BEM) for estimating the response of a nonlinear system endowed with a fractional Laplacian operator. The BEM is an established method in the context of linear boundary value problems, and is currently adopted in practical engineering applications for studying a variety of problems spanning from sea wave propagation [14] to vibration of structural elements [15]. Its implementation in conjunction with nonlinear problems was proposed mainly by Katsikadelis and co-authors. Indeed, BEM-based algorithms were developed for solving problems pertaining to vibration of plates with variable thickness [16,17], and investigating their flutter instability [18,19]; and to large deflection of beams having variable stiffness [20]. See the monograph of Katsikadelis [21] for an overview of the implementation of the method in various problems concerned with plate response determination. Nevertheless, BEM has also emerged as a potent tool in nonlinear problems involving fractional operators. In this context, the primary implementation of the approach regards the determination of the response of viscoelastic plates based on fractional constitutive models [22], and the calculation of the response of beams and plates endowed with fractional derivative elements [23,24]. Recently, a singular boundary method was developed for solving linear problems involving the fractional Laplacian by Chen and Pang [25]. It is a meshless method which circumvents the need of meshing the boundary and the domain, and requires only collocation on scattered points of the boundary.

Based on the preceding concepts, this paper proposes the use of a BEM based numerical scheme for estimating the response of systems governed by a nonlinear fractional diffusion equation. Specifically, the approach is constructed by utilizing the integral representation of the classical Poisson equation solution, whose unknown constants are determined by the BEM. Based on the Caputo-type definition of the fractional Laplacian, the value of the fractional Laplacian of the response can then be updated progressively. Clearly, the described approach can be used for any arbitrary system with stochastic excitation. However, it is particularly apropos for Monte Carlo simulation. In this regard, numerical results pertaining to a system exposed to white noise input are presented to show the usefulness of the approach. Parameter studies are discussed for elucidating the influence of the fractional Laplacian order on the system response.

2. Preliminary remarks about the fractional Laplacian

This section introduces the main relations used in developing the proposed BEM approach. The reader interested in a detailed mathematical introduction to the fractional Laplacian operator is directed to the monograph by Samko et al. [26].

Relying on the Fourier transform theory and denoting $(-\Delta)^{\alpha/2}u(\boldsymbol{x})$ the fractional Laplacian of order α of a scalar function $u(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d$, it can be defined by the equation

$$(-\Delta)^{\alpha/2}\, u(\boldsymbol{x}) = \mathcal{F}^{-1}\, |\boldsymbol{\omega}|^{\alpha}\, \mathcal{F}u(\boldsymbol{x}), \tag{1}$$

where the Fourier transform is defined as

$$\mathcal{F}\{u(\boldsymbol{x})\} = \int_{\mathbb{R}^d} u(\boldsymbol{x}) e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}} d\boldsymbol{x}, \tag{2}$$

$\boldsymbol{\omega} \in \mathbb{R}^d$ being the Fourier variable and $\mathcal{F}^{-1}\{\cdot\}$ denoting the inverse Fourier transform

$$\mathcal{F}^{-1}\{g(\boldsymbol{\omega})\} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\boldsymbol{\omega}) e^{-i\boldsymbol{\omega}\cdot\boldsymbol{x}} d\boldsymbol{\omega}. \tag{3}$$

The construction of this operator leads to the formulation of the so-called Riesz potentials for negative values of the fractional derivative order as given by the equation

$$I^{\alpha}u(\boldsymbol{x}) = \frac{1}{\gamma_d(\alpha)} \int_{\mathbb{R}^d} \frac{u(\boldsymbol{y})\, d\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^{d-\alpha}}, \alpha \neq d, d+2, d+4, \ldots \tag{4}$$

$\gamma_d(\alpha)$ being a normalizing constant, while it leads to convolution integrals having an order of singularity higher than the dimension of the space $\mathbb{R}^d$ for positive values of the fractional derivative order. Thus, they are hypersingular integrals and the convergence of the convolution is ensured by utilizing finite differences. That is, denoting the finite differences of the function $u(\boldsymbol{x})$ by the symbol $\Delta_{\boldsymbol{y}} u(\boldsymbol{x})$, the operation $(-\Delta)^{\alpha/2}$, for $\alpha > 0$, is

$$D^{\alpha}u(\boldsymbol{x}) = \frac{1}{r_d(\alpha)} \int_{\mathbb{R}^d} \frac{\Delta_{\boldsymbol{y}} u(\boldsymbol{x})}{|\boldsymbol{y}|^{d+\alpha}} d\boldsymbol{y}, \tag{5}$$

where $r_d(\alpha)$ is a normalizing constant, which is appropriately selected to ensure consistency of Eq. (5) with the fundamental equation (1). In this manner, the fractional Laplacian operator is given by the equation

$$(-\Delta)^{\alpha/2}\, u(\boldsymbol{x}) = \mathcal{F}^{-1}\, |\boldsymbol{x}|^{\alpha}\, \mathcal{F}u(\boldsymbol{x}) = \begin{cases} I^{-\alpha}u(\boldsymbol{x}), \alpha < 0 \\ D^{\alpha}u(\boldsymbol{x}), \alpha > 0. \end{cases} \tag{6}$$

Other representations of the fractional Laplacian are available in the literature. In this work, the representation described by Chen and Holm [27] is utilized. Specifically, they proposed a Caputo-type representation of the fractional Laplacian of a bounded function $u(\boldsymbol{x})$ given by the equation

$$(-\Delta)^{\alpha/2}\, u(\boldsymbol{x}) = I_d^{2-\alpha}\left[-\Delta u(\boldsymbol{x})\right], \text{for } 1 < \alpha < 2, \tag{7}$$

with $\Delta u(x)$ denoting the classical Laplacian operator. In this context, the Riesz potential on a bounded convex domain Ω in $\mathbb{R}^d$ is defined as

$$I_d^{2-\alpha}\varphi(\boldsymbol{x}) = c(\alpha) \int_{\Omega} \frac{\varphi(\xi)}{|\boldsymbol{x}-\xi|^{d+\alpha-2}} d\xi, \tag{8}$$

where ξ is the coordinate in the domain Ω and

$$c(\alpha) = \frac{\Gamma[\frac{d-2+\alpha}{2}]}{\pi^{d/2}2^{2-\alpha}\Gamma[\frac{2-\alpha}{2}]}, \tag{9}$$

With Γ being the Gamma function [28]. This operator affords the analysis of bounded systems because it reduces the singularity of the integrand, and includes naturally the boundary conditions. In this context, under appropriate conditions of the function $u(\boldsymbol{x})$, the limit of Eq. (7), when the fractional order tends to 2, is the classical Laplace operator [29]. That is,

$$\lim_{\alpha\to 2^-} (-\Delta)^{\alpha/2}\, u(\boldsymbol{x}) = -\Delta u(\boldsymbol{x}). \tag{10}$$

Clearly, the representation is consistent with Eq. (1). That is,

$$\mathcal{F}\left\{I_d^{2-\alpha}[-\Delta u(\boldsymbol{x})]\right\} = |\boldsymbol{\omega}|^{\alpha-2}\, \mathcal{F}\{-\Delta u(\boldsymbol{x})\} = |\boldsymbol{\omega}|^{\alpha}\, \mathcal{F}\{u(\boldsymbol{x})\}. \tag{11}$$

3. Boundary element method based approach

This section describes the development of a BEM based approach for solving a class of nonlinear partial differential equations endowed with fractional Laplacian operators. The approach is described by considering a problem governed by a two-dimensional nonlinear diffusion equation. However, it can also be applied to other problems involving the fractional Laplacian operator.

Consider the two-dimensional nonlinear diffusion equation with fractional Laplacian defined over the finite domain Ω,

$$\dot{u}(x,y,t) + (-\Delta)^{\alpha/2}\, u(x,y,t) + F(u) = q(x,y,t), 1 < \alpha < 2, \tag{12}$$

where $F(u)$ is a nonlinear function of $u = u(x,y,t)$. The boundary condition associated with Eq. (12) is

$$\beta_1 u + \beta_2 \frac{\partial u}{\partial n} = \beta_3 \text{ on } \Gamma, \tag{13}$$

and the initial condition is

$$u(x, y, 0) = u_0(x, y), \tag{14}$$

where $\beta_1, \beta_2, \beta_3$ are known functions defined on the boundary Γ. It is seen that the classical diffusion equation is obtained when $\alpha = 2$, while the case $1 < \alpha < 2$ relates to the anomalous diffusion. The quantity at the right-hand side $q = q(x, y, t)$ is the load that can be either deterministic or stochastic. A BEM-based representation of u is sought by considering the solution of the classical Poisson equation

$$\Delta u = b, \tag{15}$$

in which $b = b(x, y, t)$ is an unknown load to be determined. Indeed, in this context the fundamental solution Φ associated with Eq. (15), i.e., the solution of the equation

$$\Delta\Phi = \delta(P - Q), \tag{16}$$

with $\delta(\cdot)$ denoting the delta function, is given by the equation,

$$\Phi(P, Q) = \frac{1}{2\pi}\log|P - Q|, \tag{17}$$

where $P = (x_P, y_P)$ and $Q = (x_Q, y_Q)$ are points inside the domain Ω or on the boundary Γ, and

$$|P - Q| = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}, \tag{18}$$

is their distance. Thus, an integral representation of the solution can be readily derived by applying Green's identity to the system response u and to the fundamental solution Φ,

$$\int_\Omega (\Phi\Delta u - u\Delta\Phi)\, d\Omega = \int_\Gamma \left(\Phi\frac{\partial u}{\partial n} - u\frac{\partial\Phi}{\partial n}\right) dS. \tag{19}$$

In this manner, it is seen that the solution of Eq. (12) can be represented by the integral equation

$$\rho u(P) = -\int_\Gamma \left(\Phi(P, Q)\frac{\partial u(Q)}{\partial n} - u(Q)\frac{\partial\Phi(P, Q)}{\partial n}\right) dS(Q) + \int_\Omega \Phi(P, Q)\, b(Q)\, d\Omega(Q), \tag{20}$$

where $\rho = 1$ or $1/2$ whether point P lays inside the domain Ω or on the boundary Γ. If the boundary is not smooth, value of ρ can also be determined, as described by Katsikadelis [15].

The numerical implementation of Eq. (20) is pursued by discretizing the integrals defined on the boundary and over the domain. Thus, discretizing the domain into N_1 elements, and the boundary into N_2 elements, see Fig. 1, and assuming that the load is constant over each element, the integral representation of the solution can be recast in the matrix form

$$\mathbf{u} = \mathbf{G}_d\mathbf{b} - \mathbf{L}_d\mathbf{u}_n + \mathbf{H}_d\mathbf{u}_b, \text{ in the domain,} \tag{21}$$

and

$$\mathbf{P}\mathbf{u}_b = \mathbf{G}_b\mathbf{b} - \mathbf{L}_b\mathbf{u}_n + \mathbf{H}_b\mathbf{u}_b, \text{ on the boundary.} \tag{22}$$

In these equations, $\mathbf{b}(t)$ is a vector whose elements are the values of the unknown load at nodes of the discretized domain Ω; $\mathbf{u}$ is the vector with the values of the response u at the nodes in the domain; $\mathbf{u}_b$ and $\mathbf{u}_n$ capture the response and its directional derivative at the nodes on the boundary Γ; and $\mathbf{P}$ is the matrix containing the value of coefficient ρ associated with the corresponding boundary nodes. For constant elements, the boundary is smooth at the nodal points. Hence, $\mathbf{P}$ is merely a diagonal matrix with entries equal to $1/2$. Other matrices given in Eqs. (21) and (22) are defined by the equations

$$\mathrm{G}_d(i, j) = \int_{\Omega_j} \Phi(P_i, Q)\, d\Omega(Q), P_i \in \Omega, \tag{23}$$

$$\mathrm{G}_b(i, j) = \int_{\Omega_j} \Phi(P_i, Q)\, d\Omega(Q), P_i \in \Gamma, \tag{24}$$

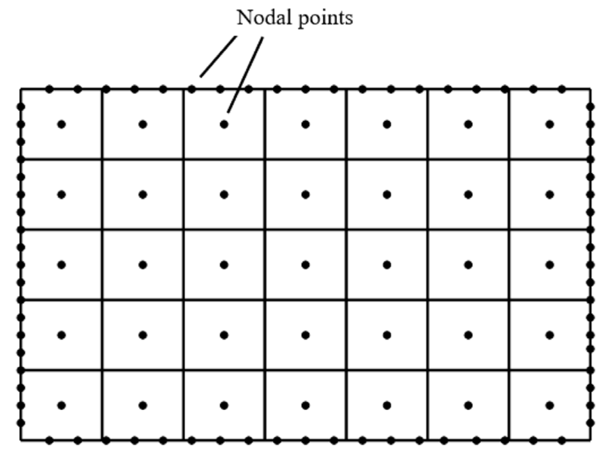

Fig. 1. Discretization of the domain into N_1 elements, and the boundary into N_2 elements.

$$\mathrm{L}_d(i, j) = \int_{\Gamma_j} \Phi(P_i, Q)\, dS(Q), P_i \in \Omega, \tag{25}$$

$$\mathrm{L}_b(i, j) = \int_{\Gamma_j} \Phi(P_i, Q)\, dS(Q), P_i \in \Gamma, \tag{26}$$

$$\mathrm{H}_d(i, j) = \int_{\Gamma_j} \frac{\partial\Phi(P_i, Q)}{\partial n} dS(Q), P_i \in \Omega, \tag{27}$$

and

$$\mathrm{H}_b(i, j) = \int_{\Gamma_j} \frac{\partial\Phi(P_i, Q)}{\partial n} dS(Q), P_i \in \Gamma, \tag{28}$$

where it is assumed that the integration is performed by keeping P_i constant, and by varying Q over the jth element of the domain or of the boundary.

Eq. (22) is used to estimate the unknown boundary quantities by introducing the boundary condition Eq. (13). In this manner, it is seen that all the boundary quantities depend upon the unknown load vector $\mathbf{b}(t)$. Thus, substituting Eqs. (22)–(28) into Eq. (21) yields the equation

$$\mathbf{u} = \mathbf{M_1}\mathbf{b} + \mathbf{e}, \tag{29}$$

where $\mathbf{M_1}$ is a known matrix and $\mathbf{e}$ is a known vector. Differentiating Eq. (29) with respect to time and taking into account that the vector $\mathbf{e}$ is a constant vector, the time variation of $\mathbf{u}$ is obtained:

$$\dot{\mathbf{u}} = \mathbf{M}_1\dot{\mathbf{b}}. \tag{30}$$

The computation of the fractional Laplacian is done by resorting to the representation given by Eq. (7). Specifically, adopting the fundamental solution associated with the Poisson equation allows collocating the load vector $\mathbf{b}(t)$ into Eq. (7). Therefore, denoting by b_j the element $\mathbf{b}$ associated with the jth domain element, it is seen that the fractional Laplacian is given by the equation

$$(-\Delta)^{\alpha/2} u(P_i) = c(\alpha)\int_\Omega \frac{-\Delta u(Q)}{|P_i - Q|^{d+\alpha-2}} d\Omega = c(\alpha)\sum_{j=1}^{N_1} b_j \int_{\Omega_j} \frac{-1}{|P_i - Q|^{d+\alpha-2}} d\Omega. \tag{31}$$

Thus, defining

$$\mathbf{M}_2(i, j) = c(\alpha)\int_{\Omega_j} \frac{1}{|P_i - Q|^{d+\alpha-2}} d\Omega, \tag{32}$$

the fractional Laplacian is calculated by the equation

$$(-\Delta)^{\alpha/2}\mathbf{u} = -\mathbf{M}_2\mathbf{b}. \tag{33}$$

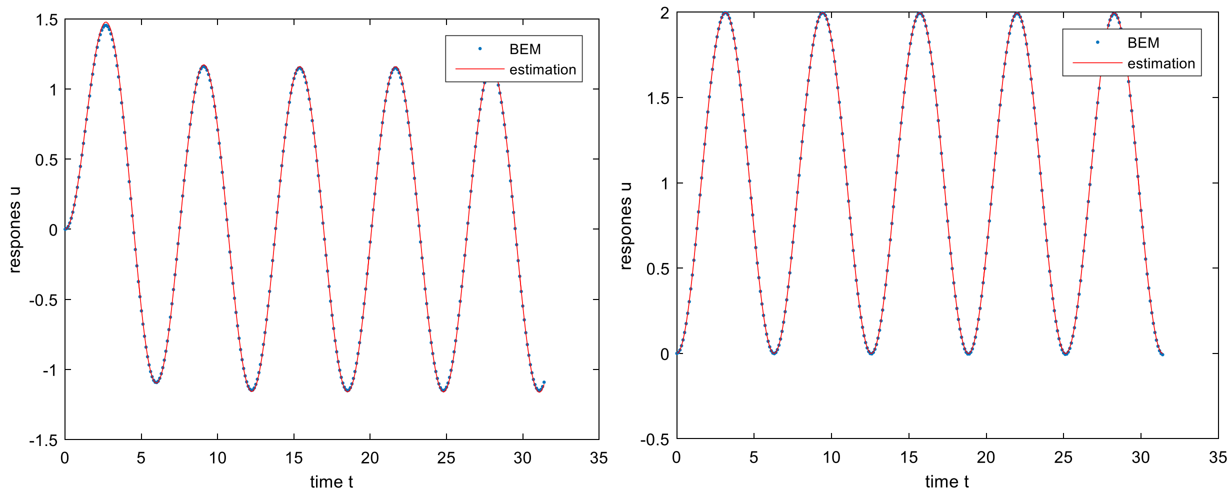

Fig. 2. Output of the BEM (blue points) vis-à-vis the estimation method of the fractional Laplacian (red line) with $\alpha = 1.95$ on a 10×5 domain (left panel), and $\alpha = 1.7$ on a 1000×500 domain (right panel).

The BEM approach is finally implemented by collocating the proposed representation into Eq. (12) for determining the values of the unknown load $\mathbf{b}(t)$. Thus, the load vector $\mathbf{b}(t)$ is governed by the nonlinear ordinary differential equation

$$\mathbf{M}_1\dot{\mathbf{b}} - \mathbf{M}_2\mathbf{b} + \mathbf{F}\left(\mathbf{b}, \mathbf{M}_1, \mathbf{e}\right) = \mathbf{q}, \tag{34}$$

where $\mathbf{F}\left(\mathbf{b}, \mathbf{M}_1, \mathbf{e}\right)$ is a nonlinear vector function collecting the nonlinear terms of the original equation. The obtained ordinary differential equation can be solved numerically by classical numerical algorithms. Once the time history of the load vector is determined by Eq. (34), the values of the unknown response vector $\mathbf{u}$ are readily obtained using Eq. (29).

Note that in this BEM procedure, $\mathbf{M}_2$ is a large non-sparse matrix, while in problems with the classical Laplacian, $\mathbf{M}_2$ is merely an identity matrix. This fact suggests that the load on the jth element of the domain influences the ith element. Thus, it introduces the nonlocal property of the fractional Laplacian. Such a contribution decreases as the distance between the elements increases. Further, note that homogeneous problems involving the classical Laplacian ($q(x, y, t) = 0$) require only a discretization of the boundary. However, in the case of the fractional Laplacian the discretization of the domain cannot be circumvented and the approach must be implemented as previously described. This issue is associated with the long-range interactions between the elements.

4. Numerical results

The approach developed in the preceding sections is applied to determining the solution of a linear fractional diffusion equation exposed to a deterministic excitation. In parallel, this problem is solved by resorting to the method based on the Riesz–Marchaud representation of the fractional Laplacian [29]. Specifically, the problem solved is obtained by setting the nonlinear parameter $k = 0$ and space-wise uniform — time-wise harmonic excitation (all values and variables are dimensionless). That is, the governing equation is

$$\dot{u} + (-\Delta)^{\alpha/2} u = p(x, y)\sin(t), \tag{35}$$

with Dirichlet boundary condition

$$u = 0 \text{ on } \Gamma, \tag{36}$$

initial condition

$$u(x, y, 0) = 0, \tag{37}$$

and $p(x, y) = 1$.

The procedure based on the Riesz–Marchaud fractional Laplacian is implemented by considering the equation,

$$(-\Delta)^{\alpha/2} u = b_{\mathrm{rm}}, \tag{38}$$

which is a fractional Poisson equation with the same boundary and initial conditions. The right hand side $b_{\mathrm{rm}}(x, y, 0)$ is constructed by adopting the approach described in the previous section. Nevertheless, in this context the fundamental solution of Eq. (38) is given by Bucur [30]. Specifically,

$$\Phi_{\mathrm{rm}}(P, Q) = \frac{\Gamma\left[(2-\alpha)/2\right]}{2^{\alpha}\pi\Gamma\left[\alpha/2\right]}\frac{1}{|P-Q|^{2-\alpha}}. \tag{39}$$

Then, the matrix equations containing the response are given as:

$$\mathbf{u} = \mathbf{G}_{d\mathrm{rm}}\mathbf{b}_{\mathrm{rm}} - \mathbf{L}_{d\mathrm{rm}}\mathbf{u}_n + \mathbf{H}_{d\mathrm{rm}}\mathbf{u}_b, \text{ in the domain,} \tag{40}$$

and

$$\mathbf{P}\mathbf{u}_b = \mathbf{G}_{b\mathrm{rm}}\mathbf{b}_{\mathrm{rm}} - \mathbf{L}_{b\mathrm{rm}}\mathbf{u}_n + \mathbf{H}_{b\mathrm{rm}}\mathbf{u}_b, \text{ on the boundary,} \tag{41}$$

where the matrices $\mathbf{G}_{d\mathrm{rm}}, \mathbf{L}_{d\mathrm{rm}}, \mathbf{H}_{d\mathrm{rm}}, \mathbf{G}_{b\mathrm{rm}}, \mathbf{L}_{b\mathrm{rm}}$ and $\mathbf{H}_{b\mathrm{rm}}$ are evaluated by Eqs. (23)–(28), with the provision that Φ is replaced by Φ_{rm}. In this manner, after all the boundary quantities are represented by $\mathbf{b}_{\mathrm{rm}}$, the ordinary differential equation

$$\mathbf{M}_{\mathrm{rm}}\dot{\mathbf{b}}_{\boldsymbol{rm}} + \mathbf{b}_{\mathrm{rm}} = \mathbf{q}, \tag{42}$$

describing the time variation of $\mathbf{b}_{\mathbf{rm}}$ is derived, where $\mathbf{M}_{\mathrm{rm}}$ is a known matrix, and $\mathbf{q}$ is a known vector. After Eq. (42) is solved numerically, the response can be estimated as

$$\mathbf{u} = \mathbf{M}_{\mathrm{rm}}\mathbf{b}_{\mathrm{rm}} + \mathbf{e}_{\mathrm{rm}}, \tag{43}$$

where $\mathbf{e}_{\mathrm{rm}}$ is a known vector.

Fig. 2 shows numerical results obtained by the proposed BEM approach and the estimation method, under different values of the fractional Laplacian order and of the domain size.

It is seen that, when the fractional order is close to 2, or when the domain is large enough, the two methods are in quite good agreement. This result is consistent with the properties of the fractional Laplacian representations. That is, when the order is close to 2, they converge to the classical Laplace operator. Further, a conjecture is that, in an unbounded domain, different representations are expected to be equal. It is worth mentioning that, the proposed BEM approach is directly derived by Green's identity and the Caputo-type representation of the fractional Laplacian. On the other hand, in the estimation method based on the Riesz–Marchaud representation, a similar integral transformation is only assumed to be valid [29].

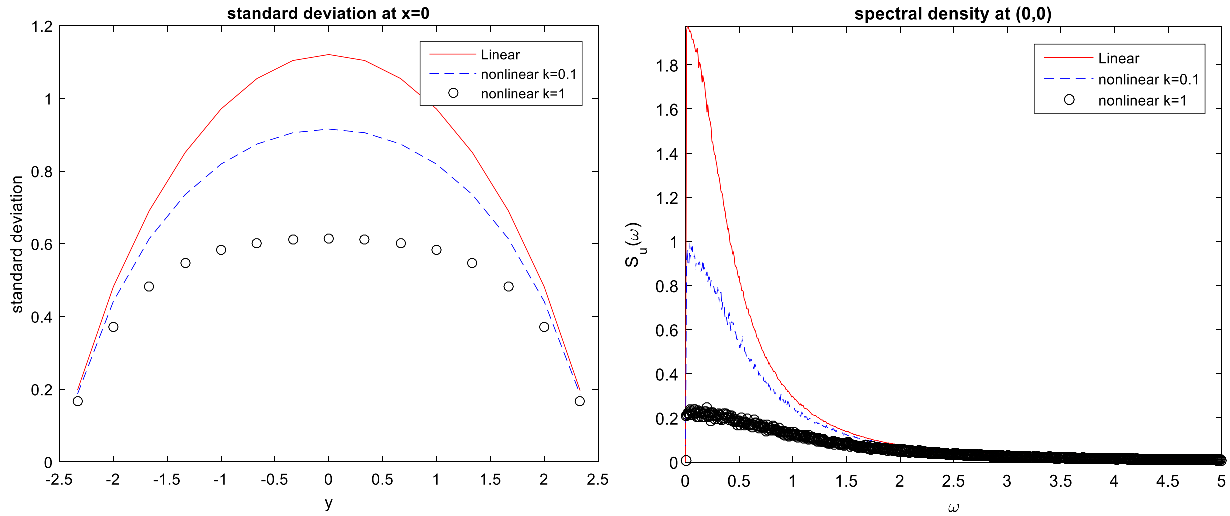

Fig. 3. Standard deviation of the response u along the line $x = 0$ and the power spectral density of u at the point (0,0) with fractional order 1.9. The values of coefficient of the nonlinear term are: $k = 0$ linear (continuous line); $k = 0.1$ (dashed line); $k = 1$ (circles).

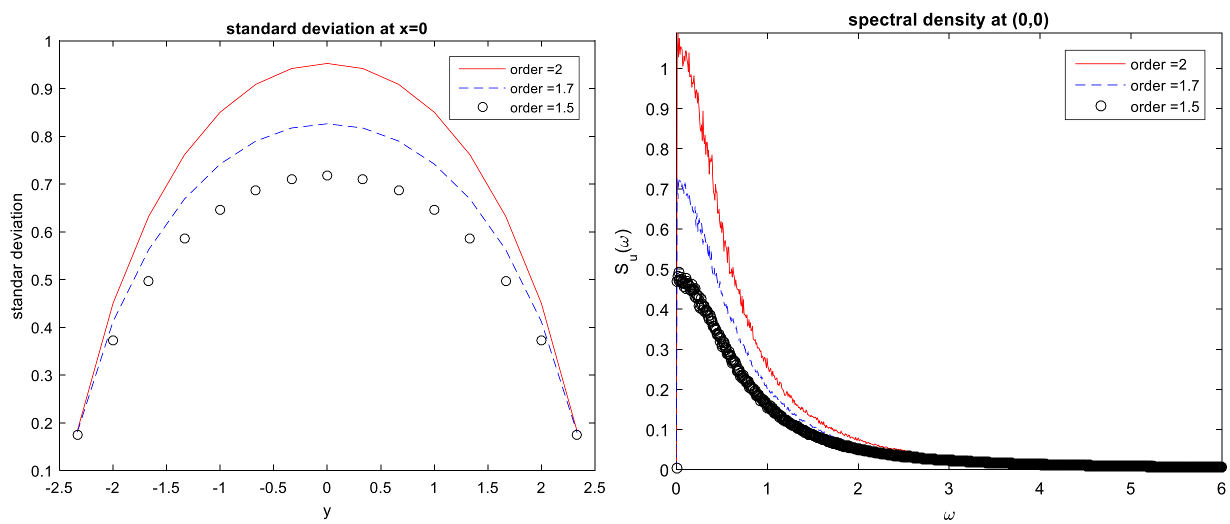

Fig. 4. Standard deviation of the temperature u along the line $x = 0$ and the power spectral density of u at the point (0,0) with coefficient of the nonlinear term $k = 0.1$. The values of fractional orders are: $\alpha = 2$, classical Laplacian (continuous line); $\alpha = 1.7$ (dashed line); $\alpha = 1.5$ (circles).

Next, to demonstrate the effectiveness of the approach, the proposed approach is implemented for determining the response of a nonlinear fractional diffusion equation defined over a rectangular domain. The system has a cubic nonlinearity proportional to a constant parameter k. The rectangular domain occupies the region $\Omega = (-5 \le x \le 5, -2.5 \le y \le 2.5)$, (all values and variables are dimensionless), while the governing equation is

$$\dot{u} + (-\Delta)^{\alpha/2} u + ku^3 = p(x, y) f(t), \tag{44}$$

with mixed boundary conditions

$$\frac{\partial u}{\partial n}(-5, y, t) = 0 = \frac{\partial u}{\partial n}(5, y, t), \tag{45}$$

$$u(x, -2.5, t) = 0 = u(x, 2.5, t), \tag{46}$$

and initial condition

$$u(x, y, t = 0) = 0. \tag{47}$$

Note that the excitation of system (44) is of a separable type. The time-dependent part $f(t)$ of the load is broad-band noise with a given power spectral density, while the deterministic part is $p(x, y) = 1$, so that the load is uniformly distributed over the domain.

To generate spectrum-compatible stochastic excitations, the spectral method [29,31] is utilized. Specifically, given the power spectral density $S(\omega) = 0.5$ for $0 < \omega \le 20\pi$ and $S(\omega = 0) = 0$, the load $f(t)$ is represented by the truncated infinite series,

$$f(t) = \sqrt{2} \sum_{n=0}^{N-1} A_n \cos\left(\omega_n t + \phi_n\right), \tag{48}$$

where

$$A_n = \sqrt{2S\left(\omega_n\right)\Delta\omega}, (n = 0, 1, 2, \ldots, N-1), \tag{49}$$

$$\omega_n = n\Delta\omega, \Delta\omega = \frac{\omega_c}{N}, \tag{50}$$

and

$$A_0 = 0 \text{ or } S\left(\omega_0 = 0\right) = 0. \tag{51}$$

In Eq. (48), ϕ_n are phase angles uniformly distributed over the interval $(0, 2\pi)$; and ω_c is the cut-off frequency of the target spectrum.

Having realizations of the time-dependent part of the excitation, the BEM approach is implemented by discretizing the domain into 225 rectangular elements, while the boundary is discretized into 400 elements. This procedure leads to the problem of estimating properly the singular integrals arising by the application of Eq. (32), since the calculation of the diagonal elements of the matrix $\mathbf{M_2}$ involves points P_i located in the domain Ω_i. In this context, the method of Chen and Pang [25] is applied for the evaluation of these entries. Further, the fourth Runge–Kutta method is utilized to solve the system of nonlinear ordinary differential equations (34).

Figs. 3 and 4 show results derived by post-processing the Monte Carlo data. The numerical results relate to various values of the nonlinear parameter k, and of the fractional Laplacian order α. The left panel of each figure shows values of response standard deviation along the line $x = 0$. The right panel is the power spectral density function of the response calculated at the center of the domain (0, 0). The figures show that decreasing the order of the derivative induces a reduction of the response standard deviation, which is also observed in frequency domain via a reduction of the maximum spectrum level.

5. Concluding remarks

A Boundary Element Method-based approach has been developed for estimating the response of a nonlinear diffusion equation incorporating a fractional Laplacian term with deterministic or stochastic excitation. The BEM is implemented by a surrogate equation having an unknown time dependent load vector determined by establishing a system of nonlinear ordinary differential equations by relying on the original fractional diffusion equation. The approach has been implemented under the assumption that the fractional Laplacian can be represented by the Caputo-type formula. The critical feature of the approach is that the influence of the fractional Laplacian operator manifests itself by a non-diagonal matrix, which renders the nonlocal property of the fractional Laplacian operator. Numerical examples have been considered by estimating the response of a system defined over a rectangular domain excited by a random excitation. The numerical examples have shown that the fractional Laplacian is associated with a reduction of the response statistics as the fractional derivative order is reduced. Although the approach has been applied to a two-dimensional system, it can be applied in three-dimensional cases, as well.

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