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Extended Prelle–Singer procedure and Darboux polynomial method: An unknown interconnection

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ABSTRACT

In this work, we present a new systematic procedure to derive two integrals from a Darboux polynomial for the given second order nonlinear ordinary differential equation (ODE). We succeed it by exploring an interlink between Darboux polynomial (DP) and the extended Prelle–Singer procedure (PS) quantities, namely null forms and integrating factors. We demonstrate the interconnection with suitable examples.

1. Introduction

During the past three decades several mathematical methods have been developed to obtain the solution of nonlinear ODEs [1–11]. Among them the DP method [12,13] and the extended PS procedure [3, 14–18] got attention in recent years. In both the methods one can construct the necessary integrals, for the given ODE, algorithmically. In general, for a second order nonlinear ODE it is difficult to find the second integral. Suppose the given second order nonlinear ODE possesses an integral which is not in the form of a polynomial or rational in $\dot{x}$ then in general it is difficult to construct this integral. For example, let us consider the simple harmonic oscillator equation $\ddot{x} + x = 0$, where over dot is differentiation with respect to t. In this example, it is straightforward to determine the first integral ($I_1 = (\dot{x}^2/2) + (x^2/2)$) [15]. However, obtaining the second integral ($I_2 = t + \tan^{-1}(\dot{x}/x)$) is not an easy task in the DP method.

In this paper, we propose an alternate way to overcome this situation. We connect the known DP and its cofactor with the integrability quantifiers of the PS procedure, namely null forms and integrating factors. From the latter ones, the second integral can be derived in a systematic way [15]. By employing this interconnection one can construct both the integrals for the given ODE (if the underlying equation admits solution in terms of elementary functions) from the DP and its cofactor. The interconnection which we propose in this paper between null forms and integrating factors with the DP is new to the literature.

We organize our presentation as follows: In Section 2, we briefly recall the theory of extended PS procedure and DP method for second-order nonlinear ODEs. In Section 3, we develop the interconnection between DP and PS quantities. In Section 4, we demonstrate this interconnection with three examples. Finally, we present our conclusion in Section 5.

2. Extended PS procedure and DP method

To begin, we briefly recall the essential steps involved in the extended PS procedure. Let us consider a second-order nonlinear ODE of the form [15]

$$\ddot{x} = \phi(t, x, \dot{x}), \tag{1}$$

where ϕ is a function of t, x and $\dot{x}$.

In extended PS method one essentially seeks two sets of functions S_i and R_i, $i = 1, 2$, where S is the null form and R is the integrating factor, from which one can construct two integrals for the given equation. The required functions S_i and R_i, $i = 1, 2$, can be determined from [15]

$$D[S] = -\phi_x + S\phi_{\dot{x}} + S^2, \tag{2}$$

$$D[R] = -R(S + \phi_{\dot{x}}), \tag{3}$$

$$R_x = R_{\dot{x}}S + RS_{\dot{x}}, \tag{4}$$

where D is total derivative operator and it is given by $D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial \dot{x}}$.

The relations (2)–(4) can be derived from the expressions [15]

$$I_t = R(\phi + S\dot{x}), \tag{5}$$

$$I_x = -RS, \tag{6}$$

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$$I_{\dot{x}} = -R. \tag{7}$$

Once the functions R_i and S_i are known they can be substituted pairwise on the right hand side of Eqs. (5)–(7) and upon integration they lead to

$$\begin{aligned} I(t,x,\dot{x}) = &\int R(\phi+\dot{x}S)dt - \int\left(RS + \frac{d}{dx}\int R(\phi+\dot{x}s)dt\right)dx \\ &- \int\left\{R + \frac{d}{d\dot{x}}\left[\int R(\phi+\dot{x}S)dt \right.\right.\\ &\left.\left. - \int\left(RS+\frac{d}{dx}\int R(\phi+\dot{x}S)dt\right)dx\right]\right\}d\dot{x}. \end{aligned} \tag{8}$$

Note that for every independent set (S,R), Eq. (8) defines an integral.

Now we recall the Darboux polynomial method. Let us consider a second order ODE (1) which admits a first integral of the form $I = K(t,x,\dot{x})/G(t,x,\dot{x})$ where K and G are functions of their arguments. Differentiating this integral with respect to t and rewriting the resultant expression, we find

$$\frac{dI}{dt} = \frac{d}{dt}\left(\frac{K}{G}\right) = 0 \Rightarrow \dot{K} = b(t,x,\dot{x})K \Rightarrow D[K] = b(t,x,\dot{x})K, \tag{9}$$

where D is the total differential operator and $b(t,x,\dot{x}) = \dot{G}/G$ is the cofactor. Eq. (9) is the determining equation for the DP [12]. Solving (9), we can obtain DPs (K) and their cofactors (b).

3. Exploring the links between DP and PS method

In this section, we interconnect the null forms and integrating factors with the DP.

Proposition 1. *The DP (K) and its associated cofactor $b(t,x,\dot{x})$ can be connected through the null forms (S) and integrating factors (R) in the PS method through the expressions,*

$$D[b] = b(\phi_{\dot{x}} + 2S_1) + b^2, \tag{10}$$

$$D[R_1] = -R_1(S_1 + \phi_{\dot{x}}), \tag{11}$$

$$S_2 = S_1 + b, \quad R_2 = R_1 K. \tag{12}$$

Proof. Let $I_1 = N(t,x,\dot{x})$ and $I_2 = M(t,x,\dot{x})$ are two independent integrals for the given Eq. (1). Rewriting I_1 in terms of $\dot{x}$ and substituting it into I_2, we get $I_2 = \tilde{M}(I_1,x,t)$. Now differentiating this expression $(\tilde{M})$ with respect to x and $\dot{x}$ and comparing the obtained expressions with (6) and (7), we find

$$I_{2x} = \tilde{M}_x + \tilde{M}_{I_1} I_{1x} = -S_2R_2, \tag{13}$$

$$I_{2\dot{x}} = \tilde{M}_{I_1} I_{1\dot{x}} = -R_2. \tag{14}$$

With the help of (7), Eq. (14) can be rewritten as

$$R_2 = \tilde{M}_{I_1} R_1 = K(t,x,\dot{x})R_1, \tag{15}$$

where we have replaced $\tilde{M}_{I_1}$ as $K(t,x,\dot{x})$. Substituting Eq. (15) into Eq. (13) and simplifying the resultant expression for S_2, we find

$$S_2 = \frac{\tilde{M}_x}{\tilde{M}_{I_1} I_{1\dot{x}}} + S_1 = b(t,x,\dot{x}) + S_1, \text{ where } b(t,x,\dot{x}) = \frac{\tilde{M}_x}{\tilde{M}_{I_1} I_{1\dot{x}}}. \tag{16}$$

Differentiating the expression $R_2 = KR_1$ (vide Eq. (15)) with respect to t and substituting (3) into it we obtain the DP determining Eq. (9). On the other hand differentiating the expression $S_2 = b + S_1$ (vide Eq. (16)) with respect to t and substituting Eq. (2) into it we end up at Eq. (10).

Once the DP (K) and its cofactor (b) is known for a second order ODE then Eq. (10) provides S_1 from which R_1 can be determined through (11). The second set of (S_2,R_2) can be identified from the expression (12).

Proposition 2. *The DP $h(x,\dot{x})$ admits $\phi_{\dot{x}}$ as cofactor then the PS method quantities (S,R) can be obtained from the relations*

$$S_1 = -\frac{\phi}{\dot{x}}, \tag{17}$$

$$R_1 = \frac{\dot{x}}{h(x,\dot{x})}, \tag{18}$$

$$G = \frac{1}{R_2\dot{x}}. \tag{19}$$

Proof. Let us suppose Eq. (1) possesses a time independent integral $I_1 = N(x,\dot{x})$. In this case the null form S_1 assumes the form (17) (see Eq. (5)). Assuming the integrating factor R_1 in the form $R_1 = \frac{\dot{x}}{h(x,\dot{x})}$, where $h(x,\dot{x})$ is the DP (irreducible polynomials) with cofactor $\phi_{\dot{x}}$, that is

$$D[h(x,\dot{x})] = \phi_{\dot{x}} h(x,\dot{x}) \tag{20}$$

and substituting the expression $R_1 = \frac{\dot{x}}{h}$ in Eq. (15), we obtain

$$R_2 = K(t,x,\dot{x})R_1 = \frac{\dot{x}K(t,x,\dot{x})}{h(x,\dot{x})}. \tag{21}$$

Now defining a new function G which is of the form (19) and differentiating it with respect to t, we find

$$D[G] = G(\phi_{\dot{x}} + 2S_1 + b). \tag{22}$$

Thus the DP h is known then R_1 and S_1 can be fixed straightforwardly from (17) and (18). Substituting S_1 and b in (22) and solving the resultant equation we obtain G from which R_2 can be fixed. Finally, the function S_2 can be identified from (16).

We note that two functions G and $\hat{G}$ are solutions of Eq. (22) with the same cofactor b, then the ratio between these two functions is also an integral for the Eq. (1), that is

$$\frac{\hat{G}}{G} = \frac{\hat{R}_2}{R_2} = F(I). \tag{23}$$

In the following, we prove the above propositions with suitable examples.

4. Utility of interconnections

4.1. Example 1

To begin, we consider the nonlinear ODE discussed in [15]

$$x\ddot{x} = 3\dot{x}^2 + \frac{x\dot{x}}{t}. \tag{24}$$

The determining equation for the DP reads $K_t + \dot{x}K_x + (3\dot{x}^2 + (x\dot{x}/t))K_{\dot{x}} = bK$. Upon solving this equation, we find a particular solution for K with b, which is of the form

$$K = t^2, \; b = -\frac{2}{t}. \tag{25}$$

The DP and its cofactor (25) can be utilized to fix the null form S_1 (vide (10)) which in turn reads $S_1 = -3\dot{x}/x$. The associated integrating factor R_1 can be found from the relation (11) and it is given by $R_1 = 1/tx^3$. From (12), we obtain

$$(S_2,R_2) = \left(-\left(\frac{2}{t} + \frac{3\dot{x}}{x}\right), -\frac{t}{x^3}\right). \tag{26}$$

The functions (S_i,R_i), $i = 1,2$, also satisfy the third condition (4). In other words the obtained sets are compatible ones. Using Eq. (8), we can find the first integral I_1 in the form $I_1 = \dot{x}/tx^3$.

Substituting the second null form and the integrating factor (S_2,R_2) into Eq. (8) and integrating it, we can obtain the second integral I_2 which is of the form $I_2 = (\dot{x}t/x^3) + (1/x^2)$. Using these two integrals I_1 and I_2, we can derive the general solution of Eq. (24) in the form $x(t) = 1/\sqrt{I_2 + I_1t^2}$.

4.2. Example 2

To demonstrate Proposition 2, we consider the following second order nonlinear ODE, namely [19]

$$\ddot{x} - \frac{3}{2}\frac{\dot{x}^2}{x} + 2x^3 = 0. \tag{27}$$

The DP of this equation can be determined by solving the equation

$$h_t + \dot{x}h_x + (\frac{3}{2}\frac{\dot{x}}{x} - 2x^3)h_{\dot{x}} = \frac{3\dot{x}}{x}h. \tag{28}$$

A particular solution of (28) is given $h_1 = -x^3/2$. The null form S_1 and the integrating factor R_1 turns out to be (vide Eqs. (17) and (18))

$$S_1 = \frac{2x^3}{\dot{x}} - \frac{3\dot{x}}{2x}, \quad R_1 = \frac{-2\dot{x}}{x^3}. \tag{29}$$

Substituting the above null form S_1 in Eq. (10), and solving the resultant equation we obtain a particular solution for b which is of the form

$$b = -\frac{4x^4 + \dot{x}^2}{2t\dot{x}^2 + 2x\dot{x}}. \tag{30}$$

Substituting the expression S_1 and b in (22) and solving the resultant equation, we find a particular solution for G in the form $G = -x^3/4\dot{x}(t\dot{x} + x)$. We note that instead of finding the function G, we can determine the function K from the cofactor b. In this case, we can straightforwardly calculate R_2 using the relation (21).

From (19) we obtain

$$R_2 = -\frac{4(t\dot{x} + x)}{x^3}. \tag{31}$$

The null form S_2 can be obtained from the relation $S_2 = S_1 + b$ which in turn read

$$S_2 = \frac{4tx^4 - 3t\dot{x}^2 - 4x\dot{x}}{2tx\dot{x} + 2x^2}. \tag{32}$$

The integrals associated with the null forms and integrating factors S_i and R_i, $i = 1, 2$, read

$$I_1 = \frac{\dot{x}^2}{x^3} + 4x, \quad I_2 = \frac{8tx^4 + 2t\dot{x}^2 + 4x\dot{x}}{x^3}. \tag{33}$$

We found that the integrals I_1 and I_2 are functionally independent. From these two integrals we derive the general solution of (27) in the form

$$x(t) = \frac{16I_1}{64 + I_2^2 - 4I_1I_2t + 4I_1^2t^2}. \tag{34}$$

4.3. Example 3

For the third example, we consider a nonlinear non-polynomial oscillator equation, [20–22]

$$\ddot{x} = \frac{kx\dot{x}^2 - \omega^2 x}{(1 + kx^2)}, \tag{35}$$

where k and ω are arbitrary parameters. Eq. (35) was introduced by Mathews and Lakshmanan in 1974 [20–22]. Subsequently the classical and quantum dynamics of the oscillator (35) on the spherical configuration space was studied by Higgs and Leeman [23,24]. In recent years a considerable number of studies have been devoted on this oscillator, see for example Refs. [6,25,26]. In Refs. [6,27], the classical and quantum dynamics of the nonlinear non-polynomial oscillator (35) and its higher dimensional versions have been studied. Eq. (35) is integrable by quadrature but admits only the time-translational symmetry [28].

Eq. (35) admits a Darboux polynomial h_1 in the form [29]

$$h_1 = \frac{1}{2}k\left(1 + kx^2\right) \tag{36}$$

The associated cofactor is found to be $\phi_{\dot{x}} = \frac{2kx\dot{x}}{1+kx^2}$. Eq. (35) admits the first compatible set of null form and integrating factor (S_1, R_1) which are of the form (vide Eqs. (17) and (18))

$$S_1 = \frac{x(\omega^2 - k\dot{x}^2)}{(1 + kx^2)\dot{x}}, \quad R_1 = \frac{2k\dot{x}}{1 + kx^2}. \tag{37}$$

The associated integral reads

$$I_1 = \frac{\omega^2 - k\dot{x}^2}{1 + kx^2}. \tag{38}$$

Now we determine the function b using the expression (10) with the help of the function S_1. Our analysis yields

$$b = -\frac{(\omega^2x^2 + \dot{x}^2)(k\dot{x}^2 - \omega^2)}{\dot{x}(kt\dot{x}^3 + kt\omega^2x^2\dot{x} - k\omega^2x^3 - \omega^2x)}. \tag{39}$$

Now substituting the above expression (39) and the null form S_1 in (22), we get the following determining equation for the function G, that is

$$D[G] - \frac{2kx\dot{x}}{kx^2 + 1}G + \frac{(\omega^2x^2 + \dot{x}^2)(k\dot{x}^2 - \omega^2)}{\dot{x}(kt\dot{x}^3 + kt\omega^2x^2\dot{x} - k\omega^2x^3 - \omega^2x)}G - \frac{2x(\omega^2 - k\dot{x}^2)}{(1 + kx^2)\dot{x}}G = 0. \tag{40}$$

We find a particular solution for (40) in the form

$$G = \frac{\left(1 + kx^2\right)\left(w^2x^2 + \dot{x}^2\right)}{\dot{x}\left(-ktw^2x^2\dot{x} - kt\dot{x}^3 + kw^2x^3 + w^2x\right)}. \tag{41}$$

Using (19), we can straightforwardly fix the function R_2 and it is given by

$$R_2 = \frac{(\omega^2x + k\omega^2x^3 - kt\omega^2x^2\dot{x} - kt\dot{x}^3)}{(1 + kx^2)(\omega^2x^2 + \dot{x}^2)}. \tag{42}$$

We also find another particular solution of Eq. (40) which is of the form

$$\hat{G} = \frac{\left(w^2x^2 + \dot{x}^2\right)\sqrt{\left(w^2 - k\dot{x}^2\right)\left(kx^2 + 1\right)}}{\dot{x}\left(-ktw^2x^2\dot{x} - kt\dot{x}^3 + kw^2x^3 + w^2x\right)}. \tag{43}$$

The ratio of these two functions G and $\hat{G}$ define a integral $I = \sqrt{I_1}$ and it is confirmed here. The integrating factor $\hat{R}_2$ for the above function $\hat{G}$ turns out to be

$$\hat{R}_2 = \frac{(\omega^2x + k\omega^2x^3 - kt\omega^2x^2\dot{x} - kt\dot{x}^3)}{(1 + kx^2)(\omega^2x^2 + \dot{x}^2)\sqrt{I_1}}. \tag{44}$$

The corresponding null form S_2 can be obtained by using the relation (16) and it becomes

$$S_2 = \frac{(k\dot{x}^2 - \omega^2)(kt\omega^2x^3 + \dot{x} + kx^2\dot{x} + ktx\dot{x}^2)}{(1 + kx^2)(\omega^2x + k\omega^2x^3 - kt\omega^2x^2\dot{x} - kt\dot{x}^3)}. \tag{45}$$

This set of null form and integrating factor (S_2, R_2) does not satisfy the extra constraint which is given in Eq. (4). So the new set of null form and integrating factor $(S_2, \hat{R}_2)$ is a compatible solution for the Eqs. (2)–(4). By substituting S_2 and $\hat{R}_2$ into Eq. (8), we obtain the second integral I_2 in the form

$$I_2 = tan^{-1}(\frac{x\sqrt{I_1}}{\dot{x}}) - t\sqrt{I_1}. \tag{46}$$

From the integrals I_1 and I_2, we can write the general solution of Eq. (35) as

$$x(t) = A\sin\left(\Omega t + \delta\right), \quad \Omega = \sqrt{\frac{\omega^2}{1 + kI_1}}, \quad A = \sqrt{I_1}, \ \delta = I_2, \tag{47}$$

where I_1 and I_2 are the integration constants. We note here that the relation $\hat{I}_1 = \frac{I_1+\omega^2}{k}$ relates the above integral with the standard integral $I_1 = \frac{\dot{x}^2+k^2x^2}{1+\omega x^2}$.

5. Conclusion

In this paper, we have shown that by knowing a DP and its cofactor one can derive two sets of null forms and integrating factors. Using the latter ones one can construct the required integrals for the given second order nonlinear ODE. Through this work we have brought out an unknown interconnection that exists between DP and the extended PS procedure. The interconnection reported in this paper helps to establish the integrability of the given second order nonlinear ODE. We have also demonstrated the interconnection with suitable examples. The interconnection in the case of third order nonlinear ODEs is under investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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