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AN OPERATOR EMBEDDING THEOREM FOR COMPLEXITY CLASSES OF RECURSIVE FUNCTIONS

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1. Introduction

Let $\mathcal{F}(t)$ be the set of functions computable by some machine using no more than $t(x)$ machine steps on all but finitely many arguments x. If we order the $\mathcal{F}$-classes under set inclusion as t varies over the recursive functions, then it is natural to ask how rich a structure is obtained. We show that this structure is very rich indeed. If R is any recursive countable partial order and $\underset{\sim}{F}$ is any total effective operator, then we show that there is a recursively enumerable sequence of recursive machine running times $\{\Phi_{s(k)}\}_{k\in N}$ such that if jRk, then $\mathcal{F}(\underset{\sim}{F}(\Phi_{s(j)})) \subsetneq \mathcal{F}(\Phi_{s(k)})$, and if j and k are incomparable, then $\underset{\sim}{F}(\Phi_{s(j)}) < \Phi_{s(k)}$ on infinitely many arguments, and $\underset{\sim}{F}(\Phi_{s(k)}) < \Phi_{s(j)}$ on infinitely many arguments.

An interesting feature of our proof is that we avoid appealing explicitly to the continuity of total effective operators; indeed our proof follows directly from a single appeal to the recursion theorem.

Several investigators have considered this and related problems, and in Section 4 we briefly summarize these investigations and compare them to our own.

2. Preliminaries

For notation from recursive function theory we follow Rogers [12].

By a partial order we mean a transitive asymmetric relation.

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For each $n \in N$, $\mathcal{P}_n$ stands for the partial recursive function of n-variables, and $\mathcal{R}_n$ stands for the total recursive function of n variables.

We use (a.e.) to denote "almost everywhere", which for our purposes stands for "for all but finitely many inputs". Similarly (i.o.) stands for "infinitely often".

Suppose $\{\varphi_0, \varphi_1, ...\}$ is an acceptable Gödel numbering of $\mathcal{P}_1$.

A *complexity measure* [1] $\Phi = \{\Phi_0, \Phi_1, ...\}$ is a sequence of functions in $\mathcal{P}_1$ satisfying

1. $\forall i \in N\ [\mathrm{dom}(\varphi_i) = \mathrm{dom}(\Phi_i)]$

2. $\lambda ixy\,[\Phi_i(x) = y]$ is a recursive predicate.

If we think of our Gödel numbering in the usual one-tape Turing machine formalism, then $\Phi_i(x)$ = "the number of steps in the computation of the ith Turing machine on argument x" is a complexity measure.

Henceforth let Φ be some fixed but arbitrary complexity measure. Then we define for any total function t

$$F(t) = \{i \in N \mid \varphi_i \in \mathcal{R}_1 \text{ and } \Phi_i \leqslant t \text{ (a.e.)}\},$$

and

$$\mathcal{F}(t) = \{\varphi_i \mid i \in F(t)\}.$$

That is, $F(t)$ is the set of (indices of) total machines which run in time t, and $\mathcal{F}(t)$ is the set of total functions computable within time t. $\mathcal{F}(t)$ is called a *complexity class.*

A sequence of partial functions $\Psi = \{\psi_0, \psi_1, ...\}$ is said to be an r.e. sequence of partial functions if $\lambda ix\,[\psi_i(x)] \in \mathcal{P}_2$.

The following theorem of Blum [4] shows that we can effectively uniformly enlarge complexity classes $\mathcal{F}(t)$ if t is a sufficiently well-behaved function.

Theorem (Compression Theorem). *There is a $g \in \mathcal{R}_2$ such that for every $\Phi_i \in \mathcal{R}_1$, $\mathcal{F}(\Phi_i) \subsetneqq \mathcal{F}(\lambda xg(x, \Phi_i(x))$. g is called a compression function for Φ.*

An operator is a map which takes functions to functions; we write $\underset{\sim}{F}(f)(x)$ to mean the value of the operator $\underset{\sim}{F}$ applied to the function f, evaluated at x. An operator $\underset{\sim}{F}$: $D \subseteq \mathcal{P}_1 \to \mathcal{P}_1$ is called an *effective operator* if there is an $s \in \mathcal{R}_1$ such that $\underset{\sim}{F}(\varphi_e)(x) = \varphi_{s(e)}(x)$ for all $\varphi_e \in D$.

An effective operator $\underset{\sim}{F}$ is *total effective* if for every $f \in \mathcal{R}_1$, f is in the domain of $\underset{\sim}{F}$ and $\underset{\sim}{F}(f) \in R_1$.

3. The embedding theorem

Theorem. *Let Φ be a fixed but arbitrary complexity measure. Let F be any total effective operator, and let R by any recursive countable partial order on N. Then there exists an r.e. sequence of recursive functions $p_0, p_1, ..., p_n, ...$ such that if jRk, then $\underset{\sim}{F}(p_j) < p_k$ (a.e.), and if j and k are incomparable, then $\underset{\sim}{F}(p_j) < p_k$ (i.o.) and $\underset{\sim}{F}(p_k) < p_j$ (i.o.).*

We begin with a sketch of the proof. Our result follows from a single application of the recursion theorem, which constructs the functions $\{p_k\}_{k\in N}$ simultaneously. The self-referential character of the construction implies that if p_k is total for every k, then $\{p_k\}_{k\in N}$ satisfies the theorem.

We start by equipping our construction with a special function which bounds the other functions in the partial order. Without loss of generality we take this function to be p_0. As in Lemma 3 of [9], each p_k will be identically equal to zero until p_0 has converged on all arguments less than or equal to k. This guarantees that $\{k \mid p_k \text{ non-total}\}$ is finite.

To show that p_k is total for every k we need to deal carefully with the incomparability conditions of the theorem. We list the R-incomparable pairs, and we service the kth entry only after the functions associated with the $(k-1)$st entry have converged. By keeping the incomparability conditions independent of each other in this way, a minimality argument is sufficient to show that $\{k \mid p_k \text{ non-total}\}$ is actually empty.

Proof. We assume without loss of generality that 0 is the R-greatest element, i.e., that $\forall k \neq 0$, $kR0$.

Let $a_0 = \langle i_0, k_0\rangle, a_1 = \langle i_1, k_1\rangle, \ldots, a_n = \langle i_n, k_n\rangle, \ldots$ be a recursive listing of all incomparable pairs in R such that if x and y are incomparable, then $\langle x, y\rangle$ and $\langle y, x\rangle$ both appear infinitely often in the list. As a technical convenience we define max $[\Phi] = 0$. Let $s \in \mathcal{R}_2$ be the s_1^1 function of the s-m-n theorem defined by the equation

$$\varphi_e(\langle x, y\rangle) = \varphi_{s(e,x)}(y).$$

Define $\psi \in \mathcal{P}_2$ as follows:

$$\psi(e, \langle k, x\rangle) = \begin{cases} 0 \text{ if } x < k \text{ or } \exists n < k \text{ such that } \Phi_{s(e,0)}(n) > x, & (1) \\ \max\limits_{\substack{j \leqslant x \\ jRk}} [\varphi_{s(e,j)}(x) + \underset{\sim}{F}(\varphi_{s(e,j)})(x)] + & (2)\,(\mathrm{i}) \\ [\varphi_{s(e,i_n)}(x) + \underset{\sim}{F}(\varphi_{s(e,i_n)})(x)] + 1, & (2)\,(\mathrm{ii}) \\ \text{where } n = \mu m \leqslant x\, [((m = 0)\, \mathit{and}\, (x = k_0)\, \mathit{and}\, (k = k_0))\, \mathit{or} \\ [(m > 0)\ \mathit{and}\ (k = k_m)\ \mathit{and}\ [(\forall i\, (0 \leqslant i \leqslant m)) \\ (\exists z_i \leqslant x) \text{ such that } (z_0 = k_0)\ \mathit{and} \\ (z_{i+1} = z_i + \Phi_{s(e,k_i)}(z_i))\ \mathit{and}\ (z_m = x)]]], \text{ if such an } n \\ \text{exists and (1) is not true, and} \\ \max\limits_{\substack{j \leqslant x \\ jRk}} [\varphi_{s(e,j)}(x) + \underset{\sim}{F}(\varphi_{s(e,j)})(x)] + 1 \quad \text{otherwise.} & (3) \end{cases}$$

$\psi \in \mathcal{P}_2$ since all the test computations in clauses (1) and (2) are recursive by the second complexity measure axiom. By the recursion theorem there is an e such that $\psi(e, \langle k, x\rangle) = \varphi_e(\langle k, x\rangle)$; we apply the s-1-1 version of the s-m-n theorem to obtain $\psi(e, \langle k, x\rangle) = \varphi_{s(e,k)}(x)$. To simplify our notation we now suppress mention of e and write $p_k(x) = \varphi_{s(e,k)}(x)$. Similarly we write $\Phi_{p_k}(x)$ for $\Phi_{s(e,k)}(x)$. Our definition now becomes

$$p_k(x) = \begin{cases} 0 \text{ if } x < k \text{ or } \exists n < k \text{ such that } \Phi_{p_0}(n) > x, & (1) \\ \max\limits_{\substack{j \leqslant x \\ jRk}} [p_j(x) + \underset{\sim}{F}(p_j)(x)] + & (2)\text{ (i)} \\ [p_{i_n}(x) + \underset{\sim}{F}(p_{i_n})(x)], + 1 & (2)\text{ (ii)} \\ \text{where } n = \mu m \leqslant x\, [((m = 0 \textit{ and } (k = k_0) \textit{ and } (x = k_0)) \textit{ or} & \\ [(m > 0) \textit{ and } (k = k_m) \text{ and } [(\forall i\, (0 \leqslant i \leqslant m))\, (\exists z_i \leqslant x) \text{ such} & \\ \text{that } (z_0 = k_0) \textit{ and } (z_{i+1} = z_i + \Phi_{p_{k_i}}(z_i)) \textit{ and } (z_m = x)]]], \text{ if such} & \\ \text{an } n \text{ exists and (1) is not true, and} & \\ \max\limits_{\substack{j \leqslant x \\ jRk}} [p_j(x) + \underset{\sim}{F}(p_j)(x)] + 1 \text{ otherwise.} & (3) \end{cases}$$

We first establish that at most finitely many of the functions $\{p_k\}_{k \in N}$ can be non-total. Suppose $p_k(x)$ diverges. Since p_0 is defined by (3) at all arguments, $p_0(x)$ must diverge, and so by (1) $p_j(y) \equiv 0$ for all $j > x$.

We now prove that p_k is total for all k.

Say that a_n is *serviced at* x if $p_{k_n}(x)$ is defined by (2), and if n is the least $m \leqslant x$ satisfying the body of (2) in the definition of $p_{k_n}(x)$. We allow the possibility that $p_{k_n}(x)$ may diverge. If a_n is serviced at x, (2) guarantees that $x = z_n = k_0 + \sum_{i=1}^{n-1} \Phi_{p_{k_i}}(z_i)$, and so a_n is serviced at no other argument. Moreover, if a_n is serviced at x and $p_{k_n}(x)$ diverges, then for $n' > n$, $a_{n'}$ will never be serviced, since $a_{n'}$ is serviced at y only when y bounds the computation of $\Phi_{p_{k_n}}(x)$.

Let k be an R-minimal element in the finite set $\{k' \mid p_{k'} \text{ non-total}\}$. Then if $p_k(x)$ diverges, it must do so because of (2) (ii). That is, a_n is serviced at x for some n, and p_{i_n} must be non-total.

But suppose $p_{i_n}(y)$ diverges by an instance of (2) (ii) for some y. This means that $i_n = k_j$ for some j and a_j is serviced at y. If $j < n$, then y must equal z_j, but since a_n is serviced at x, $\Phi_{p_{k_j}}(z_j) < x$ and hence $p_{k_j}(z_j)$ must converge. If $j > n$, then since a_n is serviced at x and $p_k(x)$ is assumed to diverge, a_j is never serviced. Moreover j cannot equal n, for then i_n would equal k_n. Hence p_{i_n} must be non-total because of (2) (i) or (3), and so some function $p_{i'}$ such that $i'Ri_n$ is non-total.

Let i be R-minimal among $\{i' \mid i'R\, i_n \text{ and } i' \text{ non-total}\}$. Then p_i must be non-total by an instance of (2) (ii), say at argument y. Hence $i = k_j$ for some j, and a_j must be serviced at $y = z_j = k_0 + \sum_{m=0}^{j-1} \Phi_{p_{k_m}}(z_m)$. If $j < n$, $p_{k_j}(y)$ must converge since a_n is serviced at x by assumption; and if $j = n$, then i_n and k_n are equal, a contradiction.

Furthermore if $j > n$, then a_j will never be serviced. Hence p_i is total, and we conclude that $p_k \in \mathcal{R}_1$ for every k.

If jRk, then $\underset{\sim}{F}(p_j)(z) \leqslant p_k(z)$ for all $z \geqslant m_0 = \max\,[k, j, \Phi_{p_0}(0), \Phi_{p_0}(1), \dots, \Phi_{p_0}(k-1)]$.

If j and k are incomparable, then $\langle j, k \rangle = a_{n_0}, a_{n_1}, \dots, a_{n_q}, \dots$ for some infinite sequence $n_0 < n_1 < n_2 \dots < n_q < \dots$.

For arguments $z \geqslant m_0$, $p_k(z)$ is defined by (2) or (3). Since the sequence of z_i's is strictly increasing, there is an i_0 such that for $i > i_0$, $z_i \geqslant m_0$. At those arguments z_i for $i > i_0$, $i = n_q$, $p_k(z_i)$ will be defined by clause (2) and $p_k(z_i) > F(p_j)(z_i)$. A symmetric argument shows that $p_j > F(p_k)$ (i.o.), and the theorem is proved.

Corollary. *Let Φ be a fixed but arbitrary complexity measure. Let $\underset{\sim}{F}$ be any total effective operator, and let R be any countable partial order on N. Then there exists an r.e. sequence of recursive measure functions $\Phi_{r(0)}, \Phi_{r(1)}, \ldots$ such that if jRk, then $\underset{\sim}{F}(\Phi_{r(j)}) < \Phi_{r(k)}$ (a.e.) and $\mathcal{F}(\underset{\sim}{F}(\Phi_{r(j)})) \subsetneq \mathcal{F}(\Phi_{r(k)})$, and if j and k are incomparable, then $\underset{\sim}{F}(\Phi_{r(j)}) < \Phi_{r(k)}$ (i.o.), and $\underset{\sim}{F}(\Phi_{r(k)}) < \Phi_{r(j)}$ (i.o.).*

Proof. Mostowski [11] has shown that there is a countable partial order R^* into which any countable partial order may be embedded. Moreover, Sacks [13] has shown that R^* is recursive.

We assume without loss of generality that $\underset{\sim}{F}$ is at least as large as the identity operator, and that the compression function, g, for Φ is strictly increasing in its second argument. Blum [4] has shown that there is an $h \in \mathcal{R}_2$ such that for all i it is the case that $\phi_i(x) \leqslant h(x, \Phi_i(x))$ (a.e.). We assume that h is strictly increasing in its second argument. To prove the corollary, apply the theorem to R^*, rewrite clause (2) as

$$\max_{\substack{j \leqslant x \\ jRk}} [p_j(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_j})(x)))] + [p_{i_n}(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_{i_n}})(x)))] + 1,$$

and we rewrite clause (3) as

$$\max_{\substack{j \leqslant x \\ jRk}} [p_j(x) + h(x, g(x, \underset{\sim}{F}(\Phi_{p_j})(x)))] + 1.$$

It is easy to see that the theorem goes through as before, and the monotonicity restrictions on g and h guarantee that the functions $\{\Phi_{p_k}\}_{k \in N}$ satisfy the corollary.

4. Relation to other work, and open problems

McCreight [5] is the first investigator to prove an embedding theorem for subrecursive classes. He shows that any countable partial order can be embedded in the complexity classes ordered under set inclusion. However, his theorem is weaker than our results in that the functions of his partial order are "separated" by composition with a fixed recursive function, whereas our functions are separated by a total effective operator. In [6] Enderton also proves a universal embedding theorem for subrecursive classes. His notion of a subrecursive class is quite weak, however, and his result is an immediate corollary of McCreight's theorem.

Early work on the structure of subrecursive classes was done by Feferman [7], Meyer and Ritchie [10], and Basu [3]. Feferman shows that dense chains exist for various notions of subrecursive classes. Meyer and Ritchie define what they call elementary honest classes, and they show the existence of dense chains and infinite anti-chains for such classes. Moreover, they are able to exhibit certain functions f such that dense chains of classes will exist between f and the iterate of f, $\lambda x\,[f^{(x)}(x)]$. Basu builds dense chains of subrecursive classes, where these classes are closed under the application of a fixed recursive operator.

Machtey [8] has announced universal embedding theorems for both the "honest" primitive recursive degrees and the "dishonest" primitive recursive degrees. Both of these theorems follow immediately from our results.

We also note that Alton [1, 2] has independently established our embedding theorem.

We leave open the question of the size of the functions in our embedding theorem. That is, given $\underset{\sim}{F}$, what is a reasonable upper bound on the size of p_0 in terms of $\underset{\sim}{F}$ (recall that p_0 bounds all the functions $\{p_k\}_{i \in N}$ on all arguments).

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