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Activation strategy for relaxed asymmetric coloring games

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ABSTRACT

This paper investigates a competitive version of the coloring game on a finite graph G. An asymmetric variant of the (r, d)-relaxed coloring game is called the (r, d)-relaxed (a, b)-coloring game. In this game, two players, Alice and Bob, take turns coloring the vertices of a graph G, using colors from a set X, with $|X| = r$. On each turn Alice colors a vertices and Bob colors b vertices. A color $\alpha \in X$ is legal for an uncolored vertex u if by coloring u with color α, the subgraph induced by all the vertices colored with α has maximum degree at most d. Each player is required to color an uncolored vertex legally on each move. The game ends when there are no remaining uncolored vertices. Alice wins the game if all vertices of the graph are legally colored, Bob wins if at a certain stage there exists an uncolored vertex without a legal color. The d-relaxed (a, b)-game chromatic number of G, denoted (a, b)-$\chi_g^d(G)$, is the least r for which Alice has a winning strategy in the (r, d)-relaxed (a, b)-coloring game.

This paper extends the well-studied activation strategy of coloring games to relaxed asymmetric coloring games. The extended strategy is then applied to the (r, d)-relaxed $(a, 1)$-coloring games on planar graphs, partial k-trees and (s, t)-pseudo-partial k-trees. This paper shows that for planar graphs G, if $a \geq 2$, then $(a, 1)$-$\chi_g^d(G) \leq 6$ for all $d \geq 77$. If H is a partial k-tree, $1 \leq a < k$, then $(a, 1)$-$\chi_g^d(H) \leq k + 1$ for all $d \geq 2k + \frac{2k-1}{a}$. If H is an (s, t)-pseudo-partial k-tree, $a \geq 1$, let $\varphi(s, t, k, a) = (1 + \frac{1}{a})(k^2 + sk + tk + st + k + t + 1) + k + t$, then $(a, 1)$-$\chi_g^d(H) \leq k + 1$ for all $d \geq \varphi(s, t, k, a)$. For planar graphs G and $a \geq 1$, $(a, 1)$-$\chi_g^d(G) \leq 3$ for all $d \geq 71 + \frac{61}{a}$. These results extend the corresponding (r, d)-relaxed $(1, 1)$-coloring game results to more generalized asymmetric cases.

1. Introduction

For a graph $G = (V, E)$, let $\Pi(G)$ be the set of linear orderings on the vertex set V of G. Let $L \in \Pi(G)$. Let $V^+_{G,L}(v) = \{u : v > u \text{ in } L\}$, $V^+_{G,L}[v] = V^+_{G,L}(v) \cup \{v\}$. The orientation $G_L = (V, E_L)$ of G with respect to L is obtained by setting $E_L = \{(v, u) : \{v, u\} \in E \text{ and } v > u \text{ in } L\}$. The neighborhood of a vertex v is denoted by $N_G(v)$, the closed neighborhood $N_G(v) \cup \{v\}$ is denoted by $N_G[v]$. If $\vec{G}$ is a directed graph, then let $N^+_{\vec{G}}(v)$ denote the set of all outneighbors of v, i.e., $N^+_{\vec{G}}(v) = \{u \in V : u \leftarrow v\}$; let $N^-_{\vec{G}}(v)$ denote the set of all inneighbors of v, i.e., $N^-_{\vec{G}}(v) = \{u \in V : u \to v\}$. Also $N^+_{\vec{G}}[v] = N^+_{\vec{G}}(v) \cup \{v\}$, $N^-_{\vec{G}}[v] = N^-_{\vec{G}}(v) \cup \{v\}$. If the graph G (or $\vec{G}$, or the linear ordering L) is clear from the context, we will drop the subscripts in the notations above. Let $O(G)$ be the set of all orientations of G. For an orientation $\vec{G}$ of G, let $\Delta^+\left(\vec{G}\right) = \max_{v \in V} |N^+_{\vec{G}}(v)|$ and let $\Delta^*(G) = \min_{\vec{G} \in O(G)} \Delta^+\left(\vec{G}\right)$.

This paper investigate a variation of the coloring game that was introduced by Bodlaender in [1]. In the usual version of the coloring game on a graph G, the game is played by two players, Alice and Bob, with Alice playing first. At the start of

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the game all vertices are uncolored. A play by either player consists of coloring an uncolored vertex from a set of colors X so that no two adjacent vertices receive the same color. Alice wins if eventually the whole graph is properly colored. Bob wins if there comes a time when all the colors have been used on the neighborhood of some uncolored vertex u. The *game chromatic number* of G, denoted $\chi_g(G)$, is the least r such that Alice has a winning strategy in the coloring game on G using a set of r colors.

The *relaxed game chromatic number* was introduced by Chou, Wang and Zhu in [3], based on the concept of relaxed coloring, and has attracted some attention [3–7,15]. A *d-relaxed proper r-coloring* of a graph is an r-coloring where all of the color classes induce subgraphs of maximum degree at most d. The *d-relaxed chromatic number* of a graph G, $\chi^d(G)$, is the least r such that the graph admits a d-relaxed proper r-coloring. The (r, d)*-relaxed coloring game* is played like the usual coloring game, except that the definition of a *legal* color for an uncolored vertex is changed. Let r and d be positive integers, and let X be a set of colors with $|X| = r$. A color $\alpha \in X$ is a legal color for an uncolored vertex u if by coloring u with color α, each vertex of color α is adjacent to at most d vertices of color α. In other words, if color α is legal for an uncolored vertex u, then:

1. u is adjacent to at most d vertices that are already colored α.
2. If u is adjacent to v, and v has already been colored with α, then v is adjacent to at most $d - 1$ vertices that are already colored α.

The parameter d is called the defect. At a given time in the game, let $\mathrm{def}(v)$ be the number of neighbors of the vertex v that have the same color as v; if v is uncolored then $\mathrm{def}(v) = 0$. Each move of Alice or Bob colors an uncolored vertex with a legal color. Alice wins the game if all vertices of the graph are legally colored, otherwise Bob wins. The *d-relaxed game chromatic number* of G, denoted by $\chi_g^d(G)$, is the least r for which Alice has a winning strategy in the (r, d)-relaxed coloring game.

The relaxed coloring games on the classes of partial k-trees, planar, and outerplanar graphs have been well studied. In [4], Dunn and Kierstead showed that $\chi_g^d(G) \le k + 1$ for all $d \ge 4k - 1$, if G is a partial k-tree; $\chi_g^d(G) \le 6$ for all $d \ge 93$, if G is a planar graph; $\chi_g^d(G) \le 3$ for all $d \ge 7$, if G is an outerplanar graph. In [5], Dunn and Kierstead showed that $\chi_g^d(G) \le 3$ for all $d \ge 132$, if G is a planar graph; $\chi_g^d(G) \le 2$ for all $d \ge 30$, if G is an outerplanar graph. For outerplanar graphs G, Dunn and Kierstead [6] showed that $\chi_g^d(G) \le 2$ for all $d \ge 8$; He, Wu and Zhu [7] showed $\chi_g^d(G) \le 5$ for all $d \ge 2$; Wu and Zhu [15] showed for $0 \le d \le 4$, $\chi_g^d(G) \le 7 - d$.

A *marking game* is played by two players, Alice and Bob, with Alice playing first. At the start of the game all vertices are unmarked. A play by either player consists of marking an unmarked vertex. The game ends when all the vertices have been marked. Together the players create a linear order L on the vertices of G defined by $u <_L v$ if u is marked before v. The score of the game is s, where $s = \max_{v \in V(G)} |N^+_{G_L}[v]|$. Alice's goal is to minimize the score, while Bob's goal is to maximize the score. The *game coloring number* of G, denoted by $\mathrm{gcol}(G)$, is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then $\mathrm{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}} \mathrm{gcol}(G)$. The game coloring number was first explicitly introduced by Zhu in [21] as a tool to bound the game chromatic number. It is easy to see that for any graph G, $\chi_g(G) \le \mathrm{gcol}(G)$. The game coloring number of a graph and its extensions are also of independent interest, and have been studied extensively.

An *asymmetric* variant of the marking game is called the (a, b)-marking game. This game is played and scored like the marking game, except that on each turn Alice marks a vertices and Bob marks b vertices. (If the last vertex is marked during a player's turn, then this completes the turn.) The (a, b)-game coloring number of G, denoted by (a, b)-$\mathrm{gcol}(G)$, is the least s such that Alice has a strategy that results in a score of at most s. If $\mathbb{C}$ is a class of graphs then (a, b)-$\mathrm{gcol}(\mathbb{C}) = \max_{G \in \mathbb{C}} (a, b)$-$\mathrm{gcol}(G)$. This game was introduced by Kierstead in [9] where the (a, b)-game coloring number of the class of trees was determined for all positive integers a and b. In particular it was shown that if $a < b$ then the (a, b)-game coloring number is unbounded even for the class of trees. In [14] Kierstead and Yang showed that if a is an integer and G is a graph with $\Delta^*(G) = k \le a$, then $(a, 1)$-$\mathrm{gcol}(G) \le 2k + 2$. It is also shown that (a, b)-$\mathrm{gcol}(G)$ is bounded on the class of graphs G with $\Delta^*(G) \le k$ if and only if $k \le \frac{a}{b}$. For this reason we say that the (a, b)-marking game on G is *very asymmetric* if $\Delta^*(G) \le \frac{a}{b}$. The classes of interval, chordal, planar and outerplanar graphs were studied in [14] with respect to the (a, b)-marking game. The asymmetric marking games of the interval graphs, chordal graphs and line graphs were further explored by Kierstead and Yang in [17,19].

Similarly to the definition of (a, b)-marking game, an *asymmetric* variant of the coloring game called the (a, b)-coloring game was studied by Kierstead in [10]. The game chromatic number of G is then generalized to the (a, b)*-game chromatic number*, denoted by (a, b)-$\chi_g(G)$. In [10] it was shown that $(2, 1)$-$\chi_g(G) \le \frac{1}{2}t^2 + \frac{3}{2}t$, where t is the acyclic chromatic number of G.

Although there are relatively rich results concerning the game chromatic number and game coloring number of graphs, there are few strategies for either Alice or Bob to play the coloring game or marking game. It follows from results in [8] that there is a single strategy, *the Activation Strategy*, such that if Alice uses this strategy to play the marking game then she achieves the best known upper bounds on the game coloring numbers of the classes of forests, interval graphs, chordal graphs, partial k-trees and outerplanar graphs. For the class of planar graphs the best known upper bound on their game coloring number is obtained by using a refinement of the activation strategy [23].

In the (a, b)-marking games of a graph G, if the game is very asymmetric, i.e., if $\Delta^*(G) \le \frac{a}{b}$, then Alice can use the so-called *Harmonious Strategy* that was introduced by Kierstead and Yang [14] to achieve some good upper bounds in the game. When $\Delta^*(G) > \frac{a}{b}$, Yang and Zhu [20] extend the activation strategy to the asymmetric marking games. Using this strategy, it was proven that if G is a chordal graph with $\omega(G) = k + 1$ and $a < k$, then $(a, 1)$-gcol $(G) \le 2k + \lfloor \frac{k}{a} \rfloor + 2$. If G is an (s, t)-pseudo-partial k-tree and $1 \le a < k$, then $(a, 1)$-gcol $(G) \le 2k + s + t + \lfloor \frac{k+s}{a} \rfloor + 2$. If G is an interval graph with $\omega(G) = k + 1$ and $1 \le a < k$, then $(a, 1)$-gcol $(G) \le 2k + \lceil \frac{k}{a} \rceil + 1$. And when $k = aq$ is a multiple of a, this bound for the class of interval graphs is best possible.

Our main interest in this paper is an *asymmetric* variant of the (r, d)-relaxed coloring game called the *(r, d)-relaxed (a, b)-coloring game*. This game is played and scored like the (r, d)-relaxed coloring game, except that on each turn Alice colors a vertices and Bob colors b vertices. (If the last vertex is colored during a player's turn, then this completes the turn.) The *d-relaxed (a, b)-game chromatic number* of G, denoted by (a, b)-$\chi_g^d(G)$, is the least r for which Alice has a winning strategy in the (r, d)-relaxed (a, b)-coloring game. Similarly, the least d such that Alice has a winning strategy in the (r, d)-relaxed (a, b)-coloring game on G is called the *(a, b)-coloring r-game defect of G*, and is denoted by (a, b)-$def_g(G, r)$. If $\mathbb{C}$ is a class of graphs then (a, b)-$\chi_g^d(\mathbb{C}) = \max_{G\in\mathbb{C}} (a, b)$-$\chi_g^d(G)$, and (a, b)-$def_g(\mathbb{C}, r) = \max_{G\in\mathbb{C}} (a, b)$-$def_g(G, r)$. When $\Delta^*(G) \le \frac{a}{b}$, i.e., when the (r, d)-relaxed (a, b)-coloring game is *very asymmetric*, the following results were proven in [18] by extending the Harmonious Strategy of the marking games to the relaxed coloring games:

Theorem 1.1 ([*18*]). *Let a, k be integers and G be a graph with $\Delta^*(G) = k$. If $a \ge k$, then $(a, 1)$-$\chi_g^d(G) \le k + 1$ for all $d \ge k^2 + 2k$.*

Theorem 1.2 ([*18*]). *Let a, k be integers and G be a graph with $\Delta^*(G) = k$. If $a \ge k^3$, then $(a, 1)$-$\chi_g^d(G) \le k + 1$ for all $d \ge 2k + 1$.*

The well-studied Activation Strategy of the marking games has already been applied to the relaxed coloring games. By using the Activation Strategy, Dunn and Kierstead [4,5] showed the following results:

Theorem 1.3 ([*4*]). *For any graph $G = (V, E)$, suppose linear order L of V witnesses the admissibility of G is at most r, and $\Delta^+(G_L) = k$. Then $\chi_g^d(G) \le k + 1$ for all $d \ge (2k + 1)r + k$.*

Corollary 1.4 ([*4*]). *For any planar graph G, $\chi_g^d(G) \le 6$ for all $d \ge 93$.*

Theorem 1.5 ([*4*]). *If H is a partial k-tree, then $\chi_g^d(H) \le k + 1$ for all $d \ge 4k - 1$.*

Theorem 1.6 ([*5*]). *Suppose H is an (s, t)-pseudo-partial k-tree, $a \ge 1$. Let $\varphi(s, t, k) = 2k^2 + 2sk + 2tk + 2st + 3t + 3k + 2$. Then $\chi_g^d(H) \le k + 1$ for all $d \ge \varphi(s, t, k)$.*

Corollary 1.7 ([*5*]). *For any planar graph G, $\chi_g^d(G) \le 3$ for all $d \ge 132$.*

In this paper, we extend the Activation Strategy to relaxed asymmetric coloring games. Using this strategy, in Section 2, we prove that if a graph G has an orientation $\vec{G}$ with $\Delta^+(\vec{G}) = k > a$ and rank r (see definition in Section 2), then $(a, 1)$-$\chi_g^d(G) \le k + 1$ for all $d \ge (k + \frac{k+1}{a})r + k$. Especially for planar graphs G, if $a \ge 2$, then $(a, 1)$-$\chi_g^d(G) \le 6$ for all $d \ge 77$. In Section 3, we show that if H is a partial k-tree, $1 \le a < k$, then $(a, 1)$-$\chi_g^d(H) \le k + 1$ for all $d \ge 2k + \frac{2k-1}{a}$. In Section 4, we apply the strategy to relaxed asymmetric coloring games on (s, t)-pseudo-partial k-trees. We show that if H is an (s, t)-pseudo-partial k-tree, $a \ge 1$. Let $\varphi(s, t, k, a) = (1 + \frac{1}{a})(k^2 + sk + tk + st + k + t + 1) + k + t$. Then $(a, 1)$-$\chi_g^d(H) \le k + 1$ for all $d \ge \varphi(s, t, k, a)$. Especially for planar graphs G and $a \ge 1$, $(a, 1)$-$\chi_g^d(G) \le 3$ for all $d \ge 71 + \frac{61}{a}$. The above results extend Theorem 1.3, Corollary 1.4, Theorem 1.5, Theorem 1.6, and Corollary 1.7 to more generalized asymmetric cases respectively. We note here that the arguments used in this paper are heavily affected by the References [4,5,20].

2. Activation strategy for relaxed asymmetric coloring games

In this section we study the relaxed $(a, 1)$-coloring games of graphs G with $\Delta^*(G) = k$ and $1 \le a < k$. For $a = 1$ and the original relaxed coloring game, a well-studied strategy for Alice is the *Activation Strategy*. We will extend the Activation Strategy used by Dunn and Kierstead in [4] to the general relaxed $(a, 1)$-coloring games when $a < k$. Note that the argument of this section is analogous to that of Section 5 in [4] by Dunn and Kierstead.

Let L be a linear order on $V(G)$, and let $\vec{G}$ be an orientation of G. For each vertex v of G, let L_v be a linear order on $N^+_{\vec{G}}(v)$. Let $\Sigma = \{L_v : v \in V(G)\}$. We say that z prefers v to u if $v <_{L_z} u$. Define

$$R_{\vec{G}}(\Sigma, u) = \{v \in V(G) : \exists z, \text{ such that } u, v \in N^+_{\vec{G}}(z) \text{ and } v <_{L_z} u\} \cup \{u\}.$$

Let

$$r_{\vec{G}}(\Sigma) = \max_{u\in V(G)} \left|R_{\vec{G}}(\Sigma, u)\right|.$$

The *rank* of $\vec{G}$ is defined as

$$r_{\vec{G}} = \min_{\Sigma} r_{\vec{G}}(\Sigma) .$$

The above rank definition comes from a slight modification of the rank introduced in [20], the main difference between them is that we do not allow vertices in $R_{\vec{G}}(\Sigma, u)$ solely because they are in $N^{+}_{\vec{G}}(u)$. This definition was heavily influenced by the definition of rank [8], arrangeability [2,12], admissibility [12] and k-coloring number [13]. Comparing the two rank definitions of [8,20], the first difference is that the rank of [8] is especially applicable for the (a, b)-games with $a = 1$ and $b = 1$; while the rank of [20] can be applied to the cases with $a > 1$. The second difference is that the rank of [8] is defined through a linear order of $V(G)$; while the rank of [20] is defined through a set of local linear orders. In [12], Kierstead and Trotter showed that the arrangeability of any planar graph is at most 10. Actually, their proof in [12] also showed, for any planar G, there is an orientation $\vec{G}$ of G, such that $r_{\vec{G}} \leq 10$ and $\Delta^{+}(\vec{G}) \leq 5$. A very recent result in [11] shows that for any planar G, there is an orientation $\vec{G}$ of G, such that $r_{\vec{G}} \leq 9$ and $\Delta^{+}(\vec{G}) \leq 5$. We will prove the following theorem in this section.

Theorem 2.1. *For any graph G, if there is an orientation $\vec{G}$ of G with $\Delta^{+}\left(\vec{G}\right) = k$, $r_{\vec{G}} = r$, and $1 \leq a < k$, then $(a, 1)$-$\chi_g^d(G) \leq k + 1$ for all $d \geq (k + \frac{k+1}{a})r + k$.*

Next we describe Alice's *Activation Strategy for the relaxed $(a, 1)$-coloring games*. To unify the description we consider an equivalent version of the relaxed coloring game in which Bob plays first by coloring a new vertex x_0 with no neighbors in $V(G)$. During the game, vertices go from uncolored to colored. Let U be the set of uncolored vertices and $M = V(G) - U$ be the set of colored vertices. In the formal description of the algorithm we do not mention M. When we remove an element from U we are implicitly coloring it. Once a vertex is colored, we define $c(x)$ to be that color, where $c : M \to X$.

We need the following definition of *mother* and *father*, which is similar to the definition of mother and father that was introduced in [4]. Suppose $v \in V$. At any point in the game, we define u to be the mother of v if u is the L_v-least vertex in $N^{+}[v]$ such that $u \in U$. We denote this vertex by $m(v)$. Note that if $v \in U$, then $m(v)$ exists because v itself is a candidate. We define w to be the father of v if w is the L_v-least vertex in $N^{+}(v)$ such that either $w \in U$, or $c(w) = c(v)$ and $m(w)$ exists. We denote such a vertex by $f(v)$. We also note that these definitions are dynamic. In other words, for any $v, f(v)$ and $m(v)$ may change throughout the game, and eventually no such vertices may exist.

Initialization: $U := V(G)$; for $v \in V(G)$ do $t_v := a$; end do;

Now suppose that Bob has just colored a vertex x with color $c(x)$. Alice plays by performing the following steps.

Alice's play:

1. Step 1 (Initial Step)
   for i from 1 to a while $U \neq \emptyset$ do
   if $f(x)$ exists, and $f(x) \in U$, then $y := f(x)$;
   else if $f(x)$ exists, and $f(x)$ is colored, then $y := m(f(x))$;
   else $y := L$-min U end if;
2. Step 2 (Recursive Step)
   while $N^{+}(y) \cap U \neq \emptyset$ and $t_y > 0$ do $z := L_y$-min $N^{+}(y) \cap U$; $t_y := t_y - 1$; $y := z$ end do;
3. Step 3 (Coloring Step)
   $U := U - \{y\}$ end do;

In Step 1 Alice selects a vertex y, we say this action as *x makes a contribution to y*, or *y receives the contribution from x*. And this concept about contribution will be used similarly throughout this paper. To help clarify this concept of *contribution* and the strategy, we use it to rephrase the strategy as following.

In Step 1 (the initial step), Alice searches for $f(x)$. If $f(x)$ exists, and is uncolored, then Alice lets x make a contribution to $f(x)$, sets $y := f(x)$ and moves to Step 2 (the recursive step). If $f(x)$ exists, and is colored, then Alice lets x make a contribution to $f(x)$; in this case, $f(x)$ passes the contribution it received to $m(f(x))$ immediately. Then Alice sets $y := m(f(x))$ and moves to the recursive step. If $f(x)$ does not exist, Alice selects y to be the L-least uncolored vertex, then moves to the recursive step. In Step 2 (the recursive step), once a vertex y receives a contribution, then y passes a contribution to its uncolored outneighbors according to its preference, provided that y has made less than a contributions in total. In case y has already made a contributions or y has no uncolored outneighbors, then y will be colored when it receives another contribution. Alice repeats the above procedure a times, each time colors one vertex.

We say *Alice reaches a vertex v* if v is encountered by Alice in Step 1 or Step 2 (this is equal to say $v = f(x)$, or $v = y$ in Step 1; or $v = z$ in Step 2). The difference between this strategy and the relaxed activation strategy in [4] is that in this strategy, an uncolored vertex can be activated a times (i.e., receive a contributions) before it is colored.

In the coloring step, Alice chooses a color for y. Call a color α *eligible for y* if α has not yet been used on any outneighbor of y. We note that since $\Delta^{+}\left(\vec{G}\right) = \Delta^{*}(G) = k$, and $|X| = k + 1$, any uncolored vertex has at least one eligible color. *Alice chooses an eligible color for y that minimizes* def(y).

Note that as long as there exist uncolored vertices, the Activation Strategy on the asymmetric relaxed coloring games terminates with Alice coloring a vertex. To see this, let $t = |U| + \sum_{v \in V(G)} |t_v|$. Note that each term in the sum is always nonnegative. If Alice has not yet completed her turn then U is nonempty. So initially $t \geq 1$. At each iteration in Step 2 of the algorithm t decreases by 1 so eventually Step 2 must end at a vertex y. Then Alice colors y in Step 3. If she has not yet completed her turn then she returns to Step 1 and repeats the process.

Consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined.

Lemma 2.2. *Suppose Alice follows the Activation Strategy. Consider a time when Alice has just colored a vertex v. Then*

1. *Any uncolored vertex has received the same number of contributions as it has made.*
2. *A colored inneighbor y of u other than x has made a contributions to $f(y)$ or to $m(f(y))$.*
3. *Before a vertex is colored, it has received at most $a + 1$ contributions. And if a vertex received $a + 1$ contributions, then it is colored.*

Proof. First consider an uncolored vertex y. Suppose y has received a contribution from a vertex $v = x$ in Step 1 or $v = y'$ in Step 2. In the former case Alice progressed to Step 2 and in the latter case Alice iterated Step 2. Since y has not yet been colored, y must have contributed to one of its outneighbors in Step 2.

For observation (2), when y is colored by Alice, she is in Step 3 of her strategy after finishing the loop in Step 2. Because y has an uncolored outneighbor u, and hence y stops making contribution only if t_y is decreased to 0. Also note that the selected vertex y in each iteration of Step 2 is uncolored, therefore $f(y)$ is uncolored and receives the contribution from y, when y makes a contribution as required by Step 2. Now suppose that y was colored by Bob. Alice responds by iterating Step 1 at least a times, after which y has made a contributions to $f(y)$ or to $m(f(y))$.

Observation (3) holds because for any vertex z, initially $t_z = a$; if z receives the $(a + 1)$th contribution, z will be colored. ■

Proof of Theorem 2.1. Suppose that Alice uses the Activation Strategy on the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on graphs G with an orientation $\vec{G}$, such that $\Delta^+\left(\vec{G}\right) = k$, $r_{\vec{G}} = r$, and $1 \leq a < k$. Fix $d \geq (k + \frac{k+1}{a})r + k$.

Assume that u is an uncolored vertex. We will show that if Alice follows the Activation Strategy on graphs G, then at any time in the game, either player can legally color any uncolored vertex u with some color that is eligible for u. Thus, eventually the entire graph will be colored, and Alice will win. We do this with Lemma 2.2 and the following two lemmas.

Lemma 2.3. *Suppose Alice follows the Activation Strategy in the relaxed $(a, 1)$-coloring game. Consider a time when Alice has just colored a vertex. Then she will never reach the same vertex more than $(a + 1)k + 1$ times in response to other vertices.*

Proof. Consider a vertex v. Each time Alice reaches v, this makes a contribution to v or to $m(v) \in N^+(v)$. By Lemma 2.2 observation (3) any vertex in $N^+[v]$ can receive at most $a + 1$ contributions before it is colored. Moreover, except v may consume the last contribution by coloring itself, v passed along all the other contributions it received to some vertex (vertices) in $N^+(v)$ immediately. And once all the vertices in $N^+[v]$ are colored, then Alice will never reach v again. Therefore the number of times that Alice could reach v is at most:

$$(a + 1)\,|N^+[v]| - a \leq (a + 1)(k + 1) - a = ak + k + 1. \quad ■$$

Lemma 2.4. *Suppose Alice follows the Activation Strategy in the $(a, 1)$-coloring game. Consider a time when Alice has just colored a vertex. Suppose v is a vertex such that there exists an uncolored vertex $u \in N^+[v]$. If v is colored, let $P = \{c(v)\}$; otherwise, let P be the set of all colors. Then v has at most $(k + \frac{k+1}{a})r$ colored inneighbors that use colors from P.*

Proof. It suffices to show that v has at most $(k + \frac{k+1}{a})r - 1$ colored inneighbors other than x. This allows for the fact that if x is defined then it may be adjacent to v and otherwise it is Bob's turn and he may be about to color a vertex adjacent to v. In the former case we treat x separately because it may have not yet made a contributions its uncolored outneighbors.

Let C' be the subset of $N^-(v)$ consisting of all the vertices colored with colors from P. $C = C' - \{x\}$. For any vertex $y \in C$, let $p(y)$ be a vertex that Alice reaches when y is making or passing a contribution. We show next that v is a candidate for $p(y)$, then since $p(y)$ is chosen by Alice according to the Activation Strategy, the vertex y will witness that $p(y) \in R_{\vec{G}}(\Sigma, v)$. If we are at a time that v is uncolored, then v is a candidate for both $m(y)$ and $f(y)$. If v is colored at the time y is colored, then y is colored by using the same color as $c(v)$ (this is by the assumption of the lemma), then y must have been colored by Bob. In this case, v is a candidate for $f(y)$ (since u is an outneighbor of v, and u is uncolored by the time y is colored).

By Lemma 2.2 observation (2)y has made a contributions to $f(y)$ or to $m(f(y))$. To make these a contributions, Alice need reach vertices in $R_{\vec{G}}(\Sigma, v)$ at least a times.

By Lemma 2.3 any vertex in $R_{\vec{G}}(\Sigma, v)$ can be reached at most $ak + k + 1$ times. So we have

$$a\,|C| \leq (ak + k + 1)\,|R_{\vec{G}}(\Sigma, v)\,| \leq (ak + k + 1)r.$$

It follows that

$$|C| \leq \left(k + \frac{k + 1}{a}\right) r.$$

Finally note that if $|C| = (k + \frac{k+1}{a})r$, then all the vertices in $N^+[v]$ are colored. But this contradicts with the assumption that there exists an uncolored vertex $u \in N^+[v]$. Therefore, $|C| \le (k + \frac{k+1}{a})r - 1$. This finishes the proof of this lemma. ■

Now suppose Alice chooses to color an uncolored vertex u by using the Activation Strategy. Since $u \in N^+[u]$, by Lemma 2.4, it is possible to choose an eligible color for u such that $\mathrm{def}(u) \le (k + \frac{k+1}{a})r \le d$. If vertex v is an inneighbor of u, then $u \in N^+[v]$. By applying Lemma 2.4 again, after u is colored, $\mathrm{def}(v) \le (k + \frac{k+1}{a})r + k \le d$. Of course, the coloring of u with an eligible color will not increase the defect of any outneighbor of u. To finish the proof of Theorem 2.1, we note that Bob may borrow Alice's strategy to find a legal move for his turn. ■

By a very recent result in [11], for planar graphs G, there is an orientation $\vec{G}$ of G, such that $r_{\vec{G}} \le 9$, and $\Delta(\vec{G}) \le 5$. Applying Theorem 2.1, we have the following corollary:

Corollary 2.5. *For planar graphs G, if $1 \le a < 5$, then $(a, 1)$-$\chi_g^d(G) \le 6$ for all $d \ge 50 + \frac{54}{a}$. Especially, for $a \ge 2$, $(a, 1)$-$\chi_g^d(G) \le 6$ for all $d \ge 77$.*

3. Activation strategy for relaxed asymmetric coloring games on partial k-trees, when $1 \le a < k$

A graph $G = (V, E)$ is a *chordal graph* if every cycle of G of length ≥ 4 has a chord. An equivalent definition of a chordal graph $G = (V, E)$ is that there is a linear order say $v_1, v_2, \ldots, v_n$, on the vertex set V, such that for each i, the set $\{v_j : j < i, v_j v_i \in E\}$ induces a complete subgraph of G. We call such an order L a simplicial ordering of G. A graph H is a partial k-tree if H is a subgraph of a chordal graph G with maximum clique size $\omega(G) = k + 1$. In this section, we consider the relaxed $(a, 1)$-coloring games on the partial k-trees, where $1 \le a < k$. We will prove the following theorem in this section. Note that the argument of this section is analogous to that of Section 4 in [4].

Theorem 3.1. *If H is a partial k-tree, $1 \le a < k$, then $(a, 1)$-$\chi_g^d(H) \le k + 1$ for all $d \ge 2k + \frac{2k-1}{a}$.*

Suppose that H is a partial k-tree. Then H is a subgraph of a chordal graph G with $\omega(G) = k + 1$. By using the simplicial ordering L of the chordal graph G, we see that $\Delta^+(H_L) \le k$. Alice will play the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on H as if she were playing the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on G. Notice that at any time in the game, for any vertex v, we have $def_H(v) \le def_G(v)$. In the following parts of this section, when we use $\mathrm{def}(v)$ we mean $def_G(v)$.

Let L be a simplicial ordering of G. Then $\Delta^+(G_L) = k$. Let L_v be L restricted to $N^+_{G_L}(v)$ for all $v \in V(G)$. Then we have $r_{G_L} = k$, where r_{G_L} is the rank of G_L as defined in Section 2.

Next we show that if Alice uses the Activation Strategy as defined in Section 2, she will win the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on any chordal graph G with $\omega(G) = k + 1$ and $d \ge 2k + \frac{2k-1}{a}$. Note that this will be enough to prove Theorem 3.1.

Proof of Theorem 3.1. Fix $d \ge 2k + \frac{2k-1}{a}$. Suppose that Alice uses the Activation Strategy on the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on chordal graphs G with maximum clique size $\omega(G) = k + 1$ and $1 \le a < k$.

Consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined. Assume that u is an uncolored vertex. We will show that if Alice follows the Activation Strategy on chordal graphs, then at any time in the game, either player can legally color any uncolored vertex u with some color that is eligible for u. Thus, eventually the entire graph will be colored, and Alice will win. We show this with the following lemmas.

Lemma 3.2. *Suppose Alice follows the Activation Strategy in the relaxed $(a, 1)$-coloring game. Then any uncolored vertex u has no more than $k + \frac{k}{a} + 1$ colored inneighbors colored with colors eligible for u.*

Proof. It suffices to show that any uncolored vertex u has at most $k + \frac{k}{a}$ colored inneighbors other than x. This allows for the fact that if x is defined then it may be adjacent to u and otherwise it is Bob's turn and he may be about to color a vertex adjacent to u. In the former case we treat x separately because it may have not yet made a contributions its uncolored outneighbors.

Let C be the subset of $N^-(u) - \{x\}$ consisting of vertices colored with colors eligible for u. For any vertex $y \in C$, since u is an outneighbor of y, and u is uncolored by the time y is colored, by Lemma 2.2 observation (2) y has made a contributions to $f(y)$ or to $m(f(y))$. Since y is colored with colors eligible for u, according to the activation strategy, we concluded that all the a contributions of y come to some vertices $z \in N^+[u]$ before z is colored. Note that if a contribution goes to u, since u is uncolored, this contribution is passed on to some vertex in $N^+(u)$ immediately. By Lemma 2.2 observation (3) any vertex in $N^+(u)$ can receive at most $a + 1$ contributions before it is colored. So we have

$$a\,|C| \le (a + 1)\,|N^+(u)| \le (a + 1)\,k.$$

It follows that

$$|C| \le \left(1 + \frac{1}{a}\right) k. \quad ■$$

Lemma 3.2 shows that it is possible to choose an eligible color for u such that $\text{def}(u) \le k+\frac{k}{a}+1 \le d$. The following lemma shows that if a vertex w has been colored with α and $\text{def}(w) \ge 2k+\frac{2k-1}{a}$, then α is not an eligible color for any uncolored outneighbor of w. Of course α is not an eligible color for any inneighbor of w. This shows that coloring an uncolored vertex with an eligible color will not increase the defect of any vertex w that already has defect at least $2k+\frac{2k-1}{a}$.

Lemma 3.3. *Suppose that Alice follows the Activation Strategy. If a vertex w has been colored with α and $\text{def}(w) \ge 2k+\frac{2k-1}{a}$, then α is not an eligible color for any uncolored outneighbor of w.*

Proof. Suppose that vertex w has been colored with α, and u is an uncolored outneighbor of w for which α is an eligible color. If there are more than one outneighbor of w for which α is an eligible color, we suppose that u is the least such vertex with respect to L. We will show that $\text{def}(w) \le 2k+\frac{2k-1}{a}-1$. Let $S = N^-(u) \cap N^+(w)$, $s = |S|$, $S^- = S \cup \{w\}$, and $S^+ = S \cup \{u\}$. Since L is a simplicial ordering of G, we have $0 \le s \le k-1$. Let $T \subseteq N^-(w)$ such that T consisting of all the vertices in $N^-(w)$ which are colored with $c(w) = \alpha$.

Let

$$T^- = \{v \in T : v \text{ is colored } \alpha \text{ before } w \text{ is colored.}\}$$

and

$$T^+ = \{v \in T : v \text{ is colored } \alpha \text{ after } w \text{ is colored.}\}$$

Clearly $T = T^- \cup T^+$. Let $t = |T|$. Since α is an eligible color for u, no parents of u have been colored with α. Therefore the only vertices in $N^+(w)$ which can add to the defect of w are the vertices in S. So we have

$$\text{def}(w) \le |S| + |T| = s + t.$$

Let u' be the least element in S^-. Let $Q = N^+(u') \cup N^+[w]$ and $q = |Q|$. Notice that $u \in N^+(u') \cap N^+[w]$, we have

$$q \le |N^+(u')| + |N^+[w]| - |N^+(u') \cap N^+[w]| \le k + (k+1) - 1 = 2k.$$

Since any vertex can receive at most $a+1$ contributions before it is colored, and $u \in Q$ is uncolored yet, by Lemma 2.2 observation (3), we have the number of contributions that all the vertices in Q have received before they are colored is not more than $q * (a+1) - 1$ so far.

Let $z \in T - \{x\}$. By Lemma 2.2 observation (2), after Alice finished her moves following z's coloring with α, z has made a contributions to $f(z)$ or $m(f(z))$. When Alice tries to make a contribution from z to its parents, she will search for $f(z)$. Since w itself is a candidate, $f(z) \in N^+[w]$. If $f(z)$ is colored, it is colored with α. Since α is not used on any parent of u, thus $f(z) \in S^-$. So u is a candidate for $m(f(z))$, thus $m(f(z)) \in V^+[u]$. Since $u' \le f(z)$, and $u', f(z) \in N^+[w]$, we have $u' \in N^+[f(z)]$. Considering $m(f(z))$, $u' \in N^+[f(z)]$ and $m(f(z)) < u'$, we know that $m(f(z)) \in N^+(u')$. This shows that $m(f(z)) \in Q$. Of course if $f(z)$ is uncolored, every contribution $f(z)$ received will be counted at least once in Q.

To get an upper bound of $|T|$, we consider the worst scenario, which is $|Q| = 2k$. And notice in the worst scenario, for each vertex $z \in S^-$, until z is colored, z will receive at most $a+1$ contributions through $N^-(z)$. However, note that except z consumed the last one by coloring itself, z passed along all the other contributions to $m(z) \in N^+(z)$. Observe that u is a candidate for $m(z)$ and $u' \in N^+[z]$, by using similar arguments in the above paragraph, we concluded that $m(z) \in N^+(u') \subseteq Q$.

So we have:

$$a \cdot |T - \{x\}| \le 2k(a+1) - 1 - a|S^-|.$$

Thus we get $|T-\{x\}| \le 2k - |S^-| + \frac{2k-1}{a}$. Then we have $\text{def}(w) \le |S| + |T-\{x\}| + |\{x\}| \le |S| + 2k - |S^-| + \frac{2k-1}{a} + 1 = 2k + \frac{2k-1}{a}$.

We can improve this estimation by at least 1 by considering who colored the vertex w. If Bob colored w, first suppose $w = x$, then $\text{def}(w) \le |S| + |T - \{x\}| \le 2k + \frac{2k-1}{a} - 1$. Next we consider the case $w \ne x$. Then following Bob's coloring w, Alice has finished her responding of making a contributions for w to $f(w)$ or to $m(f(w))$ (according to the Initial Step of the strategy). All these a contributions go to Q, and are not counted in our estimation of $|T - \{x\}|$. (Note that in the above estimation of $|T - \{x\}|$, we counted the contributions from w before it is colored, but not after it is colored.) Therefore, we have overestimated $|T - \{x\}|$ by at least one.

If Alice colored w, then when w is colored, since u is an outneighbor of w and is not colored yet, she had a choice of colors when choosing α for w. Since she chose coloring w with α, it was either because w had no colored inneighbors, the choice was arbitrary; or because w had an inneighbor colored β with $\alpha \ne \beta$, and β is an eligible color for w.

First suppose that w had an inneighbor z colored β when w was colored. Now if $z = x$, then we already overestimated $|T|$ by at least one, since x is actually colored β instead of α. Now suppose $z \ne x$. Then z has made a contributions to $f(z)$ or to $m(f(z))$. When Alice tries to make a contribution from z, she will search for $f(z)$. Since w itself is a candidate, $f(z) \in N^+[w]$. If $f(z)$ is uncolored, every contribution $f(z)$ received will be counted at least once in Q. Suppose $f(z)$ is colored, it is colored with β. But this is impossible since β is an eligible color for w, and $f(z) \in N^+[w]$. So all the a contributions from z go to Q, and they are not counted in our estimation of $|T - \{x\}|$. Therefore, we have overestimated $|T - \{x\}|$ by at least one.

Next suppose that w had no colored inneighbors when w was colored α. Since u is an outneighbor of w, and u is uncolored by the time w is colored, we know that the default choice for Alice will be a vertex $u' \le_L u$. Therefore w has received $a+1$

contributions from its inneighbors. However, note that except w consumed the $(a+1)$th contribution by coloring itself, w passed along all the other a contributions to $N^+(w) \subseteq Q$. Suppose $\{v_1, \dots, v_i\}$ are the vertices in $N^-(w)$ that made the $a+1$ contributions for w. Since none of them is colored before w, we have $i \geq 2$. Now if some v_j $(1 \leq j \leq i)$ is eventually colored with a color different than α, then all the contributions that w received from v_j are over counted. Without loss of generality, we suppose that all the v_j $(1 \leq j \leq i)$ are eventually colored with α. Now, if v_j $(1 \leq j \leq i)$ are eventually colored with α, since this happens after w was colored α, it must be Bob colored v_j. Since $i \geq 2$, we suppose for all $1 \leq j \leq i-1$, $v_j \neq x$, i.e., $v_i = x$. Then following Bob's coloring v_j $(1 \leq j \leq i)$, Alice finished her responding of making a more contributions for v_j to $f(v_j)$ or to $m(f(v_j))$. And if v_j is colored with α, all these a contributions go to Q. Therefore, all the $\{v_1, \dots, v_i\}$ have made at least ia contributions to Q (a contributions before they are colored, and $(i-1)a$ more for Alice's response of Bob's coloring v_j), but they are counted only $(i-1)a$ in our estimation of $|T - \{x\}|$. Therefore, we have overestimated $|T - \{x\}|$ by at least one. This finishes the proof of this lemma. ■

To finish the proof of Theorem 3.1, we note that Bob may borrow Alice's strategy to find a legal move for his turn. ■

4. Activation strategy for relaxed asymmetric coloring games on (s, t)-pseudo-partial k-trees

The class of (s,t)-pseudo-chordal graphs and (s,t)-pseudo-partial k-trees was introduced by Zhu in [22] as a generalization of partial k-trees. For example, it was proven in [22] that planar graphs are $(3, 8)$-pseudo-partial 2-trees, although planar graphs can have arbitrarily large treewidth. Note that the argument of this section is analogous to the proof in [5] by Dunn and Kierstead.

Definition 4.1. Suppose s, t are integers such that $0 \leq s \leq t$. A connected graph $G = (V, E)$ is called an (s, t)-*pseudo-chordal graph* if there are two oriented graphs $\vec{G}_1 = (V, \vec{E}_1)$ and $\vec{G}_2 = (V, \vec{E}_2)$ such that the following is true:

1. $E_1 \cap E_2 = \emptyset$ and $E = E_1 \cup E_2$, where E_i is the set of edges obtained from $\vec{E}_i$ by omitting the orientations.
2. $\vec{G}_1$ is acyclic.
3. $\Delta^+\left(\vec{G}_2\right) \leq s$, and $\Delta\left(\vec{G}_2\right) \leq t$.
4. Let $N_1^+(x) = N_{\vec{G}_1}^+(x)$ be the set of outneighbors of x in $\vec{G}_1$. Let $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$. Then $N_1^+(x)$ induces a transitive tournament in $\vec{G}^*$.

Definition 4.2. A graph is called an (s, t)-*pseudo-partial* k-*tree* if it is a subgraph of an (s, t)-pseudo-chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree at most k.

Note that any induced subgraph of an (s, t)-pseudo-chordal graph is still an (s, t)-pseudo-chordal graph. Therefore an (s, t)-pseudo-partial k-tree can be equivalently defined as a spanning subgraph of an (s, t)-pseudo-chordal graph in which the directed graph $\vec{G}_1$ in the definition has maximum outdegree at most k. It follows from the definition that if $s = 0$ (hence $t = 0$), then a $(0, 0)$-pseudo-chordal graph is simply a chordal graph, and a $(0, 0)$-pseudo-partial k-tree is simply a partial k-tree. In this section, we will prove the following theorem for the relaxed $(a, 1)$-coloring games on the classes of (s, t)-pseudo-partial k-trees.

Theorem 4.3. *Suppose that H is an (s, t)-pseudo-partial k-tree, $a \geq 1$. Let $\varphi(s, t, k, a) = (1 + \frac{1}{a})(k^2 + sk + tk + st + k + t + 1) + k + t$. Then $(a, 1)$-$\chi_g^d(H) \leq k + 1$ for all $d \geq \varphi(s, t, k, a)$.*

Suppose that $G = (V, E)$ is an (s, t)-pseudo-chordal graph, $\vec{G}_1 = (V, \vec{E}_1)$, $\vec{G}_2 = (V, \vec{E}_2)$ and $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$ are oriented graphs, and $\Delta^+(\vec{G}_1) \leq k$, as in Definition 4.1. For convenience, for any $x \in V$, we let $N_i^+(x) = N_{\vec{G}_i}^+(x)$ $(i = 1, 2)$. Similarly, we let $N_i^-(x) = N_{\vec{G}_i}^-(x)$ $(i = 1, 2)$. We call the vertices in $N_1^+(x)$ the *major parents* of x, the vertices in $N_1^-(x)$ the *major children* of x. And we call the vertices in $N_2(x)$ the *minor relatives* of x.

Next we describe Alice's *Activation Strategy for the relaxed* $(a, 1)$-*coloring games on* (s, t)-*pseudo-partial* k-*trees*. During the game, vertices go from uncolored to colored. Let U be the set of uncolored vertices and $M = V(G) - U$ be the set of colored vertices. Once a vertex is colored, we define $c(x)$ to be that color, where $c : M \to X$.

For each vertex $v \in V$, at any point in the game, the set $M[v] = N_1^+[v] \cap U$ is defined to be the *mother-set of* v. Note that if $v \in U$, then $M[v] \neq \emptyset$ since $v \in M[v]$. Similarly, the *father-set* $F(v)$ *of* v is defined as: $F(v) = (N_1^+(v) \cap U) \cup \{w : w \in N_1^+(v),\ c(w) = c(v),\ \text{and (either } M[w] \neq \emptyset \text{ or } N_2(w) \cap U \neq \emptyset)\}$. Note that these definitions are dynamic. In other words, for any v, $M[v]$ and $F(v)$ may change throughout the game.

Let L be a linear order on $V(G) = V(\vec{G}_1)$. For each vertex $v \in V(\vec{G}_1)$, let L_v be the linear order on $N_{\vec{G}_1}^+(v)$, such that for $x, y \in N_{\vec{G}_1}^+(v)$, $x <_{L_v} y$ if and only if $(y, x) \in E(\vec{G}^*) = \vec{E}_1 \cup \vec{E}_2$. Note that the linear order L_v of $N_{\vec{G}_1}^+(v)$ is well defined, since $N_{\vec{G}_1}^+(v)$ induces a transitive tournament in $\vec{G}^*$ (this transitive tournament is a subdigraph of $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$). Then Let $\Sigma = \{L_v : v \in V(G)\}$.

At any point in the game, we define u to be the *mother of* v if u is the L_v-least vertex in $M[v]$. Note that if $v \in U$, then $m(v)$ exists because v itself is a candidate. We denote this vertex by $m(v)$. We define w to be the *father of* v if w is the L_v-least vertex in $F(v)$. We denote such a vertex by $f(v)$. At any point in the game, if $N_2(v) \cap U \neq \emptyset$, let $r(v)$ be the L-least vertex in $N_2(v) \cap U$ (the uncolored minor relatives of v). We note that these definitions are dynamic. In other words, for any $v, f(v)$, $m(v)$ and $r(v)$ may change throughout the game, and eventually no such vertices may exist. These definitions of mother and father are actually the same as the definitions of mother and father that were used in [5].

To unify the description of the strategy, we consider an equivalent version of the coloring game in which Bob plays first by coloring a new vertex x_0 with no neighbors in $V(G)$.

Initialization: $U := V(G)$; for $v \in V(G)$ do $t_v := a$; end do;

Now suppose that Bob has just colored a vertex x with color $c(x)$. Alice plays by performing the following steps.

Alice's play:

1. Step 1 (Initial Step)
   for i from 1 to a while $U \neq \emptyset$ do
   if $f(x)$ exists, and $f(x) \in U$, then $y := f(x)$;
   else if $f(x)$ exists, $f(x)$ is colored, and $m(f(x))$ exists, then $y := m(f(x))$;
   else if $f(x)$ exists, $f(x)$ is colored, and $m(f(x))$ does not exist, then $y := r(f(x))$;
   else $y := L$-min U end if;
2. Step 2 (Recursive Step)
   while $N_1^+(y) \cap U \neq \emptyset$ and $t_y > 0$ do $z := L_y$-min $N_1^+(y) \cap U$; $t_y := t_y - 1$; $y := z$ end do;
3. Step 3 (Coloring Step)
   $U := U - \{y\}$ end do;

In the coloring step, Alice chooses a color for y. Call a color α *eligible for* y if α has not yet been used on any major parent of y. We note that since $\Delta^+(\vec{G}_1) \le k$ and $|X| = k + 1$, any uncolored vertex has at least one eligible color. *Alice chooses an eligible color for y that minimizes* def(y).

Note that the strategy used here is very similar to the strategy of Section 2, besides some changes were made for the consideration of the minor relatives of the vertices in the pseudo-partial k-tree. In the following proof of Theorem 4.3, we will skip some parts that are similar to the proof of Theorem 2.1 in Section 2.

Consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined. The following analogue of Lemma 2.2 can be proved by a similar argument.

Lemma 4.4. *Suppose Alice follows the Activation Strategy. Consider a time when Alice has just colored a vertex v. Then*

1. *Any uncolored vertex has received the same number of contributions as it has made.*
2. *A colored major child y of u other than x has made a contributions to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.*
3. *Before a vertex is colored, it has received at most $a + 1$ contributions. And if a vertex received $a + 1$ contributions, then it is colored.*

Proof of Theorem 4.3. Suppose that H is an (s, t)-pseudo-partial k-tree. Then H is a spanning subgraph of an (s, t)-pseudo-chordal graph G, and k is the maximum outdegree of all the vertices of $\vec{G}_1$ in the Definition 4.1. Alice will play the $(k+1, d)$-relaxed $(a, 1)$-coloring game on H as if she were playing the $(k+1, d)$-relaxed $(a, 1)$-coloring game on G. Notice that at any time in the game, for any vertex v, we have $def_H(v) \le def_G(v)$. In the following parts of this section, when we use def(v) we mean $def_G(v)$.

Fix $d \ge \varphi(s, t, k, a)$. Suppose that Alice uses the Activation Strategy on the $(k+1, d)$-relaxed $(a, 1)$-coloring game on an (s, t)-pseudo-chordal graph $G = (V, E)$, where $\vec{G}_1 = (V, \vec{E}_1)$, $\vec{G}_2 = (V, \vec{E}_2)$ and $\vec{G}^* = (V, \vec{E}_1 \cup \vec{E}_2)$ are oriented graphs as in Definition 4.1, and $\Delta^+\left(\vec{G}_1\right) \le k$.

Assume that u is an uncolored vertex. We will show that if Alice follows the Activation Strategy on the (s, t)-pseudo-chordal graph G, then at any time in the game, either player can legally color any uncolored vertex u with some color that is eligible for u. Thus, eventually the entire graph will be colored, and Alice will win. We do this with Lemma 4.4 and the following three lemmas.

Lemma 4.5. *Suppose Alice follows the Activation Strategy on the $(k+1, d)$-relaxed $(a, 1)$-coloring game on the (s, t)-pseudo-chordal graph G. Then any uncolored vertex u is adjacent to no more than $\varphi(s, t, k, a)$ vertices colored with colors eligible for u.*

Proof. Suppose that Alice uses the Activation Strategy. Consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined. Let S be the subset of $N_1^-(u)$ consisting of vertices that are colored with colors eligible for u. Note that $N(u) = N_1^+(u) \cup N_1^-(u) \cup N_2(u)$. By definition, vertices in $N_1^+(u)$ are not colored with colors eligible for u. So any vertex adjacent to u colored with a color eligible for u belongs to $S \cup N_2(u)$. So it suffices to show that for any uncolored vertex u, S has at most $\varphi(s, t, k, a) - t - 1$ vertices other than x. This allows for the fact that if x is defined then it may be adjacent to u and otherwise it is Bob's turn and he may be about to color a vertex adjacent to u. In the former case we treat x separately because it may have not yet made a contributions. Let

$$Q = N^+_{\vec{G}^*}(u) \cup \bigcup_{v \in N_2^+(u)} (N_1^+(v) \cup (N_2(v) - \{u\})).$$

We will show that for each vertex in $y \in S - \{x\}$, y has made a contributions to some uncolored vertices in Q. For any vertex $y \in S - \{x\}$, since $u \in N_1^+(y)$, and u is uncolored by the time y is colored, by Lemma 4.4 observation (2) y has made a contributions to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.

If the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution. Since u is uncolored by the time y is colored, and $z := L_y$-min $N_1^+(y) \cap U$ by the strategy. Therefore $z \leq_{L_y} u$. By the definition of L_y, we have $z \in N^+_{\vec{G}^*}[u]$. Note that if the contribution goes to u, since u is uncolored, this contribution is passed on to some vertex in $N_1^+(u)$ immediately.

If the contribution from y goes to the father $f(y)$, and $f(y)$ is colored when y makes this contribution, then we know that colors $c(f(y)) = c(y)$ by the definition of father, and the contribution is passed on from $f(y)$ to $m(f(y))$, or to $r(f(y))$. Since the uncolored vertex u is a candidate to receive this contribution, we know that $f(y) \leq_{L_y} u$, therefore $f(y) \in N^+_{\vec{G}^*}(u)$. Since $c(y)$ is an eligible color for u and $c(f(y)) = c(y)$, we have $f(y) \notin N_1^+(u)$. Therefore $f(y) \in N_2^+(u)$. If the contribution goes to $m(f(y))$, then $m(f(y)) \in N_1^+(f(y))$. If the contribution goes to $r(f(y))$, then $r(f(y)) \in N_2(f(y))$.

Combining all the above cases, we know that $y \in S - \{x\}$ has made a contributions to some uncolored vertices in Q. By Lemma 4.4 observation (3) any vertex in Q can receive at most $a + 1$ contributions before it is colored. So we have

$$a\,|S - \{x\}| \leq (a+1)\,|Q| \leq (a+1)\,(k + s + s(k + t - 1)).$$

It follows that

$$|S| \leq \left(1 + \frac{1}{a}\right)(k + sk + st) + 1 < \varphi(s, t, k, a) - t. \quad \blacksquare$$

Lemma 4.5 shows that it is possible to choose an eligible color for u such that $\mathrm{def}(u) \leq \varphi(s, t, k, a) \leq d$. We need the next two lemmas to show that: if u is adjacent to v, α is an eligible color for u, and v has already been colored with α, then v is adjacent to at most $\varphi(s, t, k, a) - 1$ vertices that are already colored α.

Lemma 4.6. *Suppose that Alice follows the Activation Strategy. If a vertex w has been colored with α and $\mathrm{def}(w) \geq \varphi(s, t, k, a)$, then α is not an eligible color for any uncolored major parent of w.*

Proof. Suppose that Alice uses the Activation Strategy. Suppose that vertex w has been colored with α, and u is an uncolored major parent of w for which α is an eligible color. If there are more than one major parents of w for which α is an eligible color, we suppose that u is the least such vertex with respect to L_w. We will show that $\mathrm{def}(w) \leq \varphi(s, t, k, a) - 1$. As before, consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined. Let S be the subset of $N_1^-(w)$ consisting of vertices that are colored with α. Clearly

$$def(w) \leq |S| + |N_1^+(w) - \{u\}| + |N_2(w)| \leq |S| + k - 1 + t.$$

So it suffices to show that S has at most $\varphi(s, t, k, a) - k - t - 1$ vertices other than x.

Let

$$Q = N^+_{\vec{G}^*}[w] \cup \bigcup_{v \in N^+_{\vec{G}^*}(w) - \{u\}} (N_1^+(v) \cup N_2(v)).$$

We will show that for each vertex in $y \in S - \{x\}$, y has made a contributions to some uncolored vertices in Q.

Case 1: **First suppose that Alice colored** $y \in S - \{x\}$ (**with** α).

Since Alice always chooses an eligible color for y, we know that w is uncolored by the time y is colored. By Lemma 4.4 observation (2), y has made a contributions to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.

If the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution. Since w is uncolored by the time y is colored, and $z := L_y$-min $N_1^+(y) \cap U$ by the strategy. Therefore $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N^+_{\vec{G}^*}[w]$.

Suppose the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. But this never happens, since $c(y)$ is an eligible color for y.

Case 2: **Now suppose that Bob colored** $y \in S - \{x\}$ (**with** α).

Following Bob's coloring y with α, Alice iterates Step 1 (the Initial Step) a times. At each iteration of the Initial Step, Alice lets y make a contribution to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.

Subcase 2.1: **Assume w is uncolored when this contribution is made**.

Then if the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution. Since w is uncolored by the time y is colored, and $z := L_y\text{-min}\, N_1^+(y) \cap U$ by the strategy. Therefore $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N_{\vec{G}^*}^+[w]$.

If the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. Since the uncolored vertex w is a candidate to receive this contribution, we know that $f(y) \leq_{L_y} w$, therefore $f(y) \in N_{\vec{G}^*}^+(w) - \{u\}$. If the contribution goes to $m(f(y))$, then $m(f(y)) \in N_1^+(f(y))$. If the contribution goes to $r(f(y))$, then $r(f(y)) \in N_2(f(y))$.

Subcase 2.2: **Next we suppose w is colored α when a contribution is made**.

Since $c(w) = c(y)$ and $u \in N_1^+(w)$ is uncolored, w is a candidate for $f(y)$. If the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution, since w is a candidate for $f(y)$, by the strategy $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N_{\vec{G}^*}^+[w]$.

If the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. Again since w is a candidate for $f(y)$, we have $f(y) \leq_{L_y} w$, therefore $f(y) \in N_{\vec{G}^*}^+[w] - \{u\}$. If $f(y) = w$, since $u \in N_1^+(w)$ is uncolored, the contribution goes to $m(f(y)) \in N_1^+(w)$. Next suppose $f(y) \neq w$ (i.e., $f(y) \in N_{\vec{G}^*}^+(w) - \{u\}$). If the contribution goes to $m(f(y))$, then $m(f(y)) \in N_1^+(f(y))$. If the contribution goes to $r(f(y))$, then $r(f(y)) \in N_2(f(y))$.

Combining all the above cases, we know that $y \in S - \{x\}$ has made a contributions to some uncolored vertices in Q. By Lemma 4.4 observation (3) any vertex in Q can receive at most $a + 1$ contributions before it is colored. So we have

$$a\,|S - \{x\}| \leq (a+1)\,|Q| \leq (a+1)\,(k+s+1+(k+s-1)(k+t)).$$

It follows that

$$|S| \leq \left(1 + \frac{1}{a}\right)(k^2 + tk + sk + st + s - t + 1) + 1 < \varphi(s,t,k,a) - k - t - 1. \quad \blacksquare$$

Lemma 4.7. *Suppose that Alice follows the Activation Strategy. If a vertex w has been colored with α and* $\operatorname{def}(w) \geq \varphi(s,t,k,a)$, *then α is not an eligible color for any uncolored minor relative of w.*

Proof. Suppose that Alice uses the Activation Strategy. Suppose vertex w has been colored with α, and u is an uncolored minor relative of w for which α is an eligible color. We will show that $\operatorname{def}(w) \leq \varphi(s,t,k,a) - 1$. As before, consider any time when a vertex v has just been colored by Alice. If Alice has not yet completed her turn, let x be the last vertex colored by Bob. Otherwise x is undefined. Let S be the subset of $N_1^-(w)$ consisting of vertices that are colored with α. Clearly

$$def(w) \leq |S| + |N_1^+(w)| + |N_2(w) - \{u\}| \leq |S| + k + t - 1.$$

So it suffices to show that S has at most $\varphi(s,t,k,a) - k - t - 1$ vertices other than x.

Let

$$Q = N_1^+[w] \cup N_2(w) \cup \bigcup_{v \in N_{\vec{G}^*}^+(w)} (N_1^+(v) \cup N_2(v)).$$

We will show that for each vertex in $y \in S - \{x\}$, y has made a contributions to some uncolored vertices in Q.

Case 1: **First suppose that Alice colored** $y \in S - \{x\}$ (**with** α).

Since Alice always chooses an eligible color for y, w is uncolored by the time y is colored. By Lemma 4.4 observation (2), y has made a contributions to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.

If the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution. Since w is uncolored by the time y is colored, and $z := L_y\text{-min}\, N_1^+(y) \cap U$ by the strategy. Therefore $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N_{\vec{G}^*}^+[w]$.

Suppose the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. But this never happens, since $c(y)$ is an eligible color for y.

Case 2: **Now suppose that Bob colored** $y \in S - \{x\}$ (**with** α).

Following Bob's coloring y with α, Alice iterates Step 1 (the Initial Step) a times. At each iteration of the Initial Step, Alice lets y make a contribution to $f(y)$, or to $m(f(y))$, or to $r(f(y))$.

Subcase 2.1: **Assume w is uncolored when this contribution is made**.

If the contribution from y goes to $z = f(y) \in N_1^+(y)$, and z is uncolored when z receives the contribution. Since w is uncolored by the time y is colored, and $z := L_y\text{-min}\, N_1^+(y) \cap U$ by the strategy. Therefore $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N_{\vec{G}^*}^+[w]$.

Suppose the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. Since the uncolored vertex w is a candidate to receive this contribution, we know that $f(y) \leq_{L_y} w$, therefore $f(y) \in N^+_{\vec{G}^*}(w)$. If the contribution goes to $m(f(y))$, then $m(f(y)) \in N^+_1(f(y))$. If the contribution goes to $r(f(y))$, then $r(f(y)) \in N_2(f(y))$.

Subcase 2.2: **Next we suppose w is colored α when a contribution is made**.

Since $c(w) = c(y)$ and $u \in N_2(w)$ is uncolored, w is a candidate for $f(y)$. If the contribution from y goes to $z = f(y) \in N^+_1(y)$, and z is uncolored when z receives the contribution. Since w is a candidate for $f(y)$, by the strategy $z \leq_{L_y} w$. By the definition of L_y, we have $z \in N^+_{\vec{G}^*}[w]$.

If the contribution from y goes to $m(f(y))$, or to $r(f(y))$. Then the father $f(y)$ of y is colored when y makes this contribution, and colors $c(f(y)) = c(y)$ by the definition of father. Again since w is a candidate for $f(y)$, we have $f(y) \leq_{L_y} w$, therefore $f(y) \in N^+_{\vec{G}^*}[w]$. If $f(y) = w$, since $u \in N_2(w)$ is uncolored, the contribution goes to $m(f(y)) \in N^+_1(w)$ or $r(f(y)) \in N_2(w)$. Next suppose $f(y) \neq w$ (i.e., $f(y) \in N^+_{\vec{G}^*}(w)$). If the contribution goes to $m(f(y))$, then $m(f(y)) \in N^+_1(f(y))$. If the contribution goes to $r(f(y))$, then $r(f(y)) \in N_2(f(y))$.

Combining all the above cases, we know that every $y \in S - \{x\}$ has made a contributions to some uncolored vertices in Q.

Finally we note that for the vertex w, before w is colored, if any contribution goes to w, then this contribution is passed on to some vertex in $N^+_1(w) \subseteq Q$ immediately, i.e., this contribution should be counted twice to Q. Therefore if we assume w has received $a + 1$ contributions (as we will do next), then except the last contribution is consumed by w, all the other contributions should be counted twice to Q.

By Lemma 4.4 observation (3) any vertex in Q can receive at most $a + 1$ contributions before it is colored. So we have

$$a\,|S - \{x\}| \leq (a + 1)\,|Q| - a \leq (a + 1)\,(k + 1 + t + (k + s)(k + t)) - a.$$

It follows that:

$$|S| \leq \left(1 + \frac{1}{a}\right)(k^2 + tk + sk + st + k + t + 1) \leq \varphi(s, t, k, a) - k - t. \quad \blacksquare$$

Combining the above three lemmas, we see that it is possible for Alice to legally color an uncolored vertex u by choosing an eligible color for u. To finish the proof of Theorem 4.3, we note that Bob may borrow Alice's strategy to find a legal move for his turn. $\blacksquare$

Since planar graphs are (3, 8)-pseudo-partial 2-trees (refer to Zhu [22]), by Theorem 4.3 we have the following corollary.

Corollary 4.8. *For any planar graph G and $a \geq 1$, $(a, 1)$-$\chi^d_g(G) \leq 3$ for all $d \geq 71 + \frac{61}{a}$.*

When Alice has enough moves in each of her round, by applying Theorems 1.1 and 1.2 to the relaxed asymmetric coloring games on (s, t)-pseudo-partial k-trees, we have the following results:

Corollary 4.9. *Let H be an (s, t)-pseudo-partial k-tree. If $a \geq k + s$, then $(a, 1)$-$\chi^d_g(H) \leq k + s + 1$ for all $d \geq k^2 + 2ks + s^2 + 2k + 2s$. If $a \geq (k + s)^3$, then $(a, 1)$-$\chi^d_g(H) \leq k + s + 1$ for all $d \geq 2k + 2s + 1$.*

Proof. Since H is an (s, t)-pseudo-partial k-tree, we may suppose that H is a spanning subgraph of $G = (V, E)$, where G is an (s, t)-pseudo-chordal graph with $\vec{G}_1$ (as defined in Definition 4.1) has maximum outdegree at most k. Alice will play the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on H as if she were playing the $(k + 1, d)$-relaxed $(a, 1)$-coloring game on G. Notice that at any time in the game, for any vertex v, we have $def_H(v) \leq def_G(v)$. From Definition 4.1, we know $E(G) = E_1 \cup E_2$, where E_i is the set of edges obtained from $\vec{E}_i$ by omitting the orientations. And $\Delta^+\left(\vec{G}_1\right) \leq k$, $\Delta^+\left(\vec{G}_2\right) \leq s$. Therefore we have $\Delta^*(G) \leq k+s$. By applying Theorem 1.1, we have if $a \geq k+s$, then $(a, 1)$-$\chi^d_g(G) \leq k+s+1$ for all $d \geq (k+s)^2+2(k+s)$. By applying Theorem 1.2, we have if $a \geq (k + s)^3$, then $(a, 1)$-$\chi^d_g(G) \leq k + s + 1$ for all $d \geq 2(k + s) + 1$. $\blacksquare$

Let S_g be an orientable surface of genus $g \geq 1$, i.e., the sphere with g handles. We consider graphs embeddable on S_g. Let $\mathcal{G}(S_g)$ be the set of graphs of minimum degree at least 2 and embeddable on S_g. The following lemma was proven by Zhu in [22].

Lemma 4.10 (Zhu, [22]). *Given an integer $g \geq 1$, let S_g be the orientable surface of genus g. Then $\mathcal{G}(S_g)$ is a $\left(\lfloor \frac{1+\sqrt{1+48g}}{2} \rfloor, \lfloor 3 + \sqrt{1 + 48g} \rfloor\right)$-pseudo-partial 2-tree.*

By applying Theorem 4.3 and Corollary 4.9, we have the following results for the relaxed asymmetric coloring games on $\mathcal{G}(S_g)$:

Corollary 4.11. *Given an integer $g \geq 1$, let S_g be the orientable surface of genus g.*

1. *For* $a \geq 1$, *then* $(a, 1)$-$\chi_g^d\left(\mathcal{G}(S_g)\right) \leq 3$, *for all* $d \geq \varphi\left(\lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor, \lfloor 3+\sqrt{1+48g}\rfloor, 2, a\right)$.
2. *If* $a \geq \lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor + 2$, *then* $(a, 1)$-$\chi_g^d\left(\mathcal{G}(S_g)\right) \leq \lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor + 3$, *for all* $d \geq \left(\lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor + 2\right)^2 + \lfloor\sqrt{1+48g}\rfloor + 5$.
3. *If* $a \geq \left(\lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor + 2\right)^3$, *then* $(a, 1)$-$\chi_g^d\left(\mathcal{G}(S_g)\right) \leq \lfloor\frac{1+\sqrt{1+48g}}{2}\rfloor + 3$, *for all* $d \geq \lfloor\sqrt{1+48g}\rfloor + 6$.

5. Remarks

For the (r, d)-relaxed (a, b)-coloring games with $b > 1$, i.e., when Bob is allowed to color more than one vertex at a time, it can be checked that the so-called Harmonious Strategy and Enhanced Harmonious Strategy used in [18] yield the following results.

Theorem 5.1. *Let a, b, and k be positive integers, G be a graph with* $\Delta^*(G) = k \leq \frac{a}{b}$. *Then* (a, b)-$\chi_g^d(G) \leq k + 1$ *for all* $d \geq k^2 + 2k + b - 1$.

Theorem 5.2. *Let a, b, and k be positive integers, G be a graph with* $\Delta^*(G) = k$, *and* $k^3 \leq \frac{a}{b}$. *Then* (a, b)-$\chi_g^d(G) \leq k + 1$ *for all* $d \geq 2k + b$.

It can be checked that the Activation Strategy of Section 2 in this paper yields the following result for the case $b > 1$, and the main theorems of Sections 3 and 4 can be extended to the cases $b > 1$ similarly.

Theorem 5.3. *Let a, b, and k be positive integers. For any graph G, if there is an orientation* $\vec{G}$ *of G with* $\Delta^+\left(\vec{G}\right) = k$, $r_{\vec{G}} = r$, *and* $1 \leq \frac{a}{b} < k$, *then* (a, b)-$\chi_g^d(G) \leq k + 1$ *for all* $d \geq (k + \frac{k+1}{\lfloor\frac{a}{b}\rfloor})r + k + b - 1$.

For the cases $a < b$ (i.e., when for each round, Bob is allowed to color more vertices than Alice), the study on the (r, d)-relaxed (a, b)-coloring games is left open. The following problem also looks interesting to the author:

Problem 5.4. Let a, b, and k be positive integers, G be a graph with $\Delta^*(G) = k$, and $k < \frac{a}{b} < k^3$. Can the conclusion of Theorem 5.1 be improved?

Finally, it is worth mentioning that for the marking games, the study of the lower bounds of the (a, b)-game coloring number of some graphs has already attracted some recent attention, refer to [9,14,16] for examples. But the study for the lower bounds of relaxed (a, b)-game chromatic number of graphs is still left as a widely open area.

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