Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 221 (2009) 1069–1114
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

A determinant formula for a holonomic q-difference system associated with Jackson integrals of type BC_n

Kazuhiko Aomoto [a], Masahiko Ito [b,*]

[a] *Department of Mathematics, Kyoto Sangyo University, Kyoto 603-8555, Japan*
[b] *Department of Physics and Mathematics, Aoyama Gakuin University, Kanagawa 229-8558, Japan*

Received 12 August 2006; accepted 3 February 2009
Available online 24 February 2009
Communicated by the Managing Editors of AIM

Abstract

A Jackson integral of type BC_n is a multisum generalization of the very-well-poised-balanced ${}_{2r}\psi_{2r}$ basic hypergeometric series. We state an explicit product formula for the determinant of a matrix with entries given by the BC_n type Jackson integrals. In order to show this, we treat the determinant as a solution of a holonomic q-difference equation. In particular we give the q-difference equation explicitly as a two-term recurrence relation, which the determinant satisfies, by introducing a set of new symmetric polynomials via the symplectic Schur functions.

MSC: primary 33D15, 33D67; secondary 39A13

Keywords: Jackson integral of type BC_n; Symplectic Schur functions; Vandermonde type determinant

1. Introduction

Throughout this paper, we assume $0 < q < 1$ and denote the q-*shifted factorial* for all integers N by $(x)_\infty := \prod_{i=0}^{\infty}(1 - q^i x)$ and $(x)_N := (x)_\infty/(q^N x)_\infty$.

As we see in [1,10,28], etc., there are a lot of summation and transformation formulae for basic hypergeometric series. BC_n type Jackson integrals are a multisum generalization of the basic hypergeometric series in a class of what is called *very-well-poised-balanced* ${}_{2r}\psi_{2r}$. A key

* Corresponding author.
E-mail addresses: aokazu@cc.kyoto-su.ac.jp (K. Aomoto), mito@gem.aoyama.ac.jp (M. Ito).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.02.003

reason to consider the BC_n type Jackson integrals, which permit the Weyl group symmetry, is to give an explanation and an extension of these hypergeometric series from the view points of the Weyl group symmetry and the q-difference equations satisfied by the BC_n type Jackson integrals. For example, the formula called Slater's transformation for a very-well-poised-balanced ${}_{2r}\psi_{2r}$ series [10,28] can be regarded as a connection formula for the solutions of q-difference equations of the Jackson integral of type BC_1. (See [20] for the exact correspondence between them. See also [17] for a connection formula for the BC_n case.)

In [4,5], finiteness of the cohomologies associated with the Jackson integrals of type BC_n depending on $(2s+2)+l$ parameters $a_1, a_2, \ldots, a_{2s+2}$ and $t_1, t_2, \ldots, t_l$ has been discussed. In this paper, we restrict ourselves to the case where $l = 1$. Then the dimension of its nth cohomology is equal to $\kappa := \binom{s+n-1}{n}$. This means that we can regard the Jackson integrals as a solution of a holonomic system of q-difference equations of rank κ.

For a point $\xi \in (\mathbb{C}^*)^n$ and a function $\varphi(z)$ on $(\mathbb{C}^*)^n$ which is holomorphic and invariant under the Weyl group action, we consider the following function $\langle\varphi, \xi\rangle$ defined as

$$\langle\varphi, \xi\rangle := \int_{\Lambda_\xi} \varphi(z)\Phi(z)\Delta(z)\varpi_q \quad \text{where } \varpi_q = \frac{d_q z_1}{z_1} \wedge \ldots \wedge \frac{d_q z_n}{z_n},$$

where the integral is taken in the Jackson integral sense, i.e., sum over the lattice $\Lambda_\xi \simeq \mathbb{Z}^n$ (see Section 3.1 for the definition of the Jackson integral). Here $\Phi(z)$ is the q-multiplicative function of $z = (z_1, z_2, \ldots, z_n) \in (\mathbb{C}^*)^n$ defined by

$$\Phi(z) := \prod_{i=1}^{n} \prod_{m=1}^{2s+2} z_i^{1/2-\alpha_m} \frac{(qa_m^{-1}z_i)_\infty}{(a_m z_i)_\infty} \prod_{1\leqslant j<k\leqslant n} z_j^{1-2\tau} \frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty} \frac{(qt^{-1}z_j z_k)_\infty}{(tz_j z_k)_\infty}, \tag{1.1}$$

where $q^{\alpha_m} = a_m$ and $q^\tau = t$. The function $\Phi(z)$ is a similar one in BC_n case to what is called a 'phase function' in the context [29]. The function $\Delta(z)$ is the Weyl denominator of type C_n defined by

$$\Delta(z) := \prod_{i=1}^{n} \frac{1 - z_i^2}{z_i} \prod_{1\leqslant j<k\leqslant n} \frac{(1 - z_j/z_k)(1 - z_j z_k)}{z_j}.$$

We call the sum $\langle\varphi, \xi\rangle$ the *Jackson integral of type BC_n*. When we regard the sum $\langle\varphi, z\rangle$ as a function of $z \in (\mathbb{C}^*)^n$, we can construct a unique holomorphic function on $(\mathbb{C}^*)^n$ from $\langle\varphi, z\rangle$ by *regularization* of the Jackson integral of type BC_n (see Section 3 for its definition). We denote the holomorphic function by $\langle\langle\varphi, z\rangle\rangle$ and call it the *regularized Jackson integral* of type BC_n.

We define some terminology to state the main results. Let B be the set defined by

$$B = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n;\ s-1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\},$$

which consists of κ elements. Corresponding to the set B, we can take the symplectic Schur functions $\chi_\lambda(z)$ for the partitions $\lambda \in B$. (See Section 2 for the definition of $\chi_\lambda(z)$.) We can construct a holonomic system of q-difference equations of rank κ with the basis $\{\chi_\lambda(z);\ \lambda \in B\}$, which is stated in [4,5] as follows:

Proposition 1.1. *Let T_u be the q-shift operator on the parameter $u \to qu$. If all parameters $a_1, a_2, \dots, a_{2s+2}$ and t are generic, then there exist invertible matrices Y_{a_i} $(i = 1, 2, \dots, 2s+2)$ and Y_t whose entries $y^{(a_i)}_{\lambda\nu}$, $y^{(t)}_{\lambda\nu}$ are rational functions of $a_1, a_2, \dots, a_{2s+2}$ and t, such that*

$$T_{a_i}\langle \chi_\lambda, z\rangle = \sum_{\nu\in B} y^{(a_i)}_{\lambda\nu}\langle \chi_\nu, z\rangle, \qquad T_t\langle \chi_\lambda, z\rangle = \sum_{\nu\in B} y^{(t)}_{\lambda\nu}\langle \chi_\nu, z\rangle$$

where λ runs over the set B.

In this paper, by a suitable change of bases, the matrix Y_{a_i} is transformed into a triangular matrix whose diagonal entries are determined explicitly, and as a consequence we obtain the following:

Theorem 1.2. *The determinant of the matrix Y_{a_i} is evaluated as*

$$\det Y_{a_i} = (-a_i)^{-s\binom{s+n-1}{s}} \prod_{k=1}^{n}\left[\frac{\prod_{j=1}^{2s+2}(1-t^{n-k}a_ia_j)}{(1-t^{n-k}a_i^2)(1-t^{n+k-2}a_1a_2\dots a_{2s+2})}\right]^{\binom{s+k-2}{k-1}}.$$

Theorem 1.2 has the following application. Let Z be the set defined by

$$Z = \{\mu = (\mu_1, \mu_2, \dots, \mu_s) \in \mathbb{Z}^s;\ \mu_1 + \dots + \mu_s = n,\ \mu_1 \geqslant 0, \dots, \mu_s \geqslant 0\},$$

which consists of κ elements. Corresponding to the set Z, we can define the special points $\zeta_{(\mu)} \in (\mathbb{C}^*)^n$ for the s-tuples $\mu \in Z$. (See Section 3.2 for the definition of $\zeta_{(\mu)}$.) The sum $\langle \chi_\lambda, \zeta_{(\mu)}\rangle$ (or $\langle\langle \chi_\lambda, \zeta_{(\mu)}\rangle\rangle$), where $\lambda \in B$, $\mu \in Z$, can be explained as the pairing between the nth cohomology and nth homology associated with $\Phi(z)$. The difference forms $\chi_\lambda(z)\Delta(z)\varpi_q$ and the lattices $\Lambda_{\zeta_{(\mu)}}$ containing the points $\zeta_{(\mu)}$ give the bases for the cohomology and the homology, respectively (see [2,3]). In particular, in order to establish the non-degeneracy of the pairing, we need the non-vanishing of the determinant of the matrix whose entries are $\langle\langle \chi_\lambda, \zeta_{(\mu)}\rangle\rangle$, $\lambda \in B$, $\mu \in Z$. More explicitly we can use Theorem 1.2 to show not only the determinant does not vanish but is expressible as follows:

Theorem 1.3. *Let $\theta(x)$ be the function defined by $\theta(x) = (x)_\infty(q/x)_\infty$. The $\kappa \times \kappa$ determinant with (λ, μ) entry $\langle\langle \chi_\lambda, \zeta_{(\mu)}\rangle\rangle$ is evaluated as*

$$\begin{aligned}&\{(1-q)(q)_\infty\}^{n\binom{s+n-1}{n}} \prod_{k=1}^{n}\left[\frac{(qt^{-(n-k+1)})^s_\infty}{(qt^{-1})^s_\infty}\frac{\prod_{1\leqslant i<j\leqslant 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1})_\infty}\right]^{\binom{s+k-2}{k-1}}\\ &\times\prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\frac{\theta(t^{2r-(n-k)}a_ia_j^{-1})\theta(t^{n-k}a_ia_j)}{t^ra_i}\right]^{\binom{s+k-3}{k-1}},\end{aligned} \tag{1.2}$$

where the rows $\lambda \in B$ and the columns $\mu \in Z$ of the matrix $(\langle\langle \chi_\lambda, \zeta_{(\mu)}\rangle\rangle)_{\lambda,\mu}$ are arranged in the orders $<$ and $\prec_Z$, respectively.

(The symbol $<$ denotes the reverse lexicographic ordering on the set B which is stated in Section 2.2. For the definition of the ordering $\prec_Z$ on Z, see Section 3.2.) Theorem 1.3 is similar to Theorem 5.9–5.11 in [29] that established formulae for the determinants of matrices formed by q-hypergeometric integrals of type A_n (q-analogues of the Selberg integral).

Note that in the case $s=1$ the matrix size of $(\langle\langle\chi_\lambda,\zeta_{(\mu)}\rangle\rangle)_{\lambda,\mu}$ reduces to 1 and Theorem 1.2 becomes exactly the same as the following formula first proved by van Diejen [8]:

$$\langle\langle 1,\xi\rangle\rangle=(1-q)^n(q)_\infty^n\prod_{i=1}^{n}\frac{(qt^{-i})_\infty}{(qt^{-1})_\infty}\frac{\prod_{1\leqslant j<k\leqslant 4}(qt^{-(i-1)}a_j^{-1}a_k^{-1})_\infty}{(qt^{-(n+i-2)}a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})_\infty},\tag{1.3}$$

which is equivalent to the q-Macdonald–Morris identity of type $(C_n^\vee,C_n)$ studied by Gustafson [11]. (See also [14,15,21,23,25].) In this case the last factor including θ functions in (1.2) disappears.

We mention that the determinant of the matrix Y_t in Proposition 1.1 is eventually obtained from Theorem 1.3 as follows:

Corollary 1.4. *The determinant of the matrix Y_t is written as*

$$\det Y_t=t^{-n(n-1)\binom{s+n-1}{n}}\prod_{k=1}^{n}\left[\left(\frac{(t^{n-k+1})_{n-k+1}}{1-t}\right)^s\frac{\prod_{1\leqslant i<j\leqslant 2s+2}(t^{n-k}a_ia_j)_{n-k}}{(t^{n+k-2}a_1a_2\ldots a_{2s+2})_{n+k-2}}\right]^{\binom{s+k-2}{k-1}}.$$

Since $\langle\langle\chi_\lambda,\zeta_{(\mu)}\rangle\rangle\to\chi_\lambda(\zeta_{(\mu)})$ if we take the limit $q\to 0$, a Vandermonde type determinant whose entries are symplectic Schur functions can be deduced from Theorem 1.3 as follows:

Corollary 1.5. *The $\kappa\times\kappa$ determinant with (λ,μ) entry $\chi_\lambda(\zeta_{(\mu)})$ is evaluated as*

$$\det\bigl(\chi_\lambda(\zeta_{(\mu)})\bigr)_{\lambda,\mu}=\prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\frac{(1-t^{2r-(n-k)}a_ia_j^{-1})(1-t^{n-k}a_ia_j)}{t^ra_i}\right]^{\binom{s+k-3}{k-1}}.\tag{1.4}$$

(See [12] for another simple proof for Corollary 1.5, which does not go through Theorem 1.3.) Note that, in Section 7, in order to prove Theorem 1.3 we state the following Vandermonde type determinant of matrix formed by ordinary Schur functions $S_\lambda(z)$:

Proposition 1.6. *The $\kappa\times\kappa$ determinant with (λ,μ) entry $S_\lambda(\zeta_{(\mu)})$ is evaluated as*

$$\det\bigl(S_\lambda(\zeta_{(\mu)})\bigr)_{\lambda,\mu}=\prod_{k=1}^{n}\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\bigl(t^ra_i-t^{n-k-r}a_j\bigr)^{\binom{s+k-3}{k-1}}.\tag{1.5}$$

Eq. (1.5) exactly coincides with the ordinary Vandermonde determinant if $n=1$. Though Proposition 1.6 is a direct consequence from the principal term of asymptotic behavior of the formula (1.4) as $a_i\to+\infty$ $(1\leqslant i\leqslant s)$, in fact Proposition 1.6 can be used in the proof of Theorem 1.3. We give a proof of Proposition 1.6 in Appendix D (see [12] or [29, Eq. (A.14)] for other proofs). Note in passing that other Vandermonde type determinant formulae similar to (1.4) and (1.5) can be found in [18,19].

Theorem 1.3 follows from Theorem 1.2 and the asymptotic behavior of the determinant $J := \det(\langle\chi_\lambda, \zeta_{(\mu)}\rangle)_{\lambda,\mu}$. Here we outline its proof. It is sufficient to obtain an explicit form of J because the evaluation of the determinant of the matrix $(\langle\langle\chi_\lambda, \zeta_{(\mu)}\rangle\rangle)_{\lambda,\mu}$ is deduced from that of J. We evaluate J in two steps. The first is to establish the recurrence relation for J as follows:

$$\frac{T_{a_i} J}{J} = (-a_i)^{-s\binom{s+n-1}{s}} \prod_{k=1}^{n} \left[\frac{\prod_{j=1}^{2s+2}(1 - t^{n-k} a_i a_j)}{(1 - t^{n-k} a_i^2)(1 - t^{n+k-2} a_1 a_2 \dots a_{2s+2})} \right]^{\binom{s+k-2}{k-1}}, \tag{1.6}$$

which is independent of choice of the points $\zeta_{(\mu)}$ and is a direct consequence of Theorem 1.2. By repeated use of (1.6), J can be evaluated up to some factors. The next step is to determine the indefinite factors which depend on the points $\zeta_{(\mu)}$. The factors can be calculated using the asymptotic behavior of J as the parameters tend to infinity in the following direction:

$$T^N : \begin{cases} a_i \to a_i q^{(s+1)N} & \text{if } 1 \leqslant i \leqslant s, \\ a_j \to a_j q^{-sN} & \text{if } s+1 \leqslant j \leqslant 2s+2 \end{cases}$$

with $N \to +\infty$. The explicit expression of the principal term of the asymptotic behavior of $T^N J$ is given in Proposition 7.2. Imposing this asymptotic behavior as a boundary condition of the recurrence relation for J completes the proof of Theorem 1.2.

This paper is organized as follows. In Section 2, we define the symplectic Schur functions $\chi_\lambda(z)$ and introduce two kinds of orderings on the set B. In Section 3, we give the definition of the Jackson integral of type BC_n, its truncation and regularization. The truncated Jackson integral is defined by introducing the special points $\zeta_{(\mu)}$, $\mu \in Z$. We also state the main theorems in this section. In Section 4, we construct new polynomials $e_\lambda(z)$ via the symplectic Schur functions $\chi_\lambda(z)$ and state their vanishing properties. In Section 5, using a property of Jackson integrals (Proposition 3.3) we show homogeneous linear relations among $\langle e_\lambda, z\rangle$. The explicit expression of the coefficients of $\langle e_\lambda, z\rangle$ in these relations is important for computing the determinant of the matrix Y_{a_i}. We establish Theorem 1.2 in Section 6. In the holonomic system of q-difference equations stated in Proposition 1.1, by the change of basis from $\{\chi_\lambda(z);\ \lambda \in B\}$ to $\{e_\lambda(z);\ \lambda \in B\}$, the matrix Y_{a_i} is transformed into a triangular matrix. From the diagonal entries of the triangular matrix we obtain the explicit expression of the determinant of Y_{a_i}. In Section 7 we state the asymptotic behavior of the determinant of matrix formed by truncated Jackson integrals. Finally we establish our main theorem in Section 8.

We should mention the polynomials $e_\lambda(z)$ defined in Section 4. The polynomials $e_\lambda(z)$ are constructed from the *'elementary' symmetric polynomials* $e_i(z)$, $i = 0, 1, \dots, n$, introduced in [15] to study the structure of the BC_n type Jackson integral in the simplest case $s = 1$. In another context [7,9], van Diejen defined a set of polynomials to describe the Pieri-type formula for the Macdonald–Koornwinder polynomials. Despite the polynomials $e_i(z)$ and van Diejen's polynomials $\hat{E}_i(z)$ in [7, Eq. (6.10), p. 254] differ in appearance, both can be rewritten as

$$e_i(z) = \hat{E}_i(z) = \sum_{1 \leqslant j_1 < \dots < j_i \leqslant n} \prod_{k=1}^{i} e\big(z_{j_k}; a t^{j_k - k}\big) \quad \text{for } z = (z_1, z_2, \dots, z_n), \tag{1.7}$$

where $e(x; y) := x + x^{-1} - (y + y^{-1})$ and we fix $a = a_1$, which is one of parameters $a_1, \dots, a_{2s+2}$ in (1.1). This confirms that the polynomials $e_i(z)$ coincide with the polynomials $\hat{E}_i(z)$. Moreover it is remarkable that the polynomials $e_i(z)$ also coincide with Okounkov's *interpolation Macdonald polynomials* I_μ in [27, Eq. (1.2), p. 294] where $\mu = (1^i)$ under a suitable change of parameters. See also [26]. In a recent work [22], Komori, Noumi and Shiraishi encountered the polynomials (1.7) in a context very different from ours. This provides a further interpretation of their origin.

As another application, the polynomials $e_\lambda(z)$ are useful for evaluating a similar determinant formula for BC_n type Jackson integrals that are obtained as a generalization of Gustafson's multiple ${}_6\psi_6$ summation formula. For further details, see [6,16,19].

2. Symplectic Schur functions

Let W be the Weyl group of type C_n, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathcal{S}_n$ where $\mathcal{S}_n$ is the symmetric group on $\{1, 2, \ldots, n\}$. W is generated by the following reflections of the coordinates $z = (z_1, z_2, \ldots, z_n) \in (\mathbb{C}^*)^n$:

$$\begin{aligned} \tau_1\colon&\quad z_1 \leftrightarrow z_1^{-1},\\ \sigma_i\colon&\quad z_1 \leftrightarrow z_i \quad \text{for } i = 2, 3, \ldots, n. \end{aligned} \tag{2.1}$$

For a function $f(z)$ on $(\mathbb{C}^*)^n$, we define action of the Weyl group W on $f(z)$ by

$$wf(z) := f\bigl(w^{-1}(z)\bigr) \quad \text{for } w \in W.$$

We say that a function $f(z)$ on $(\mathbb{C}^*)^n$ is W-*symmetric* or W-*skew-symmetric* if $wf(z) = f(z)$ or $wf(z) = (\operatorname{sgn} w) f(z)$ for all $w \in W$, respectively.

We denote by $\mathcal{A}f(z)$ the alternating sum over W defined by

$$\mathcal{A}f(z) := \sum_{w \in W} (\operatorname{sgn} w) wf(z),$$

which is W-skew-symmetric. Let $\mathcal{P}$ be the set of partitions defined by

$$\mathcal{P} := \bigl\{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n;\ \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\bigr\}.$$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}$, we set $A_\lambda(z) := \mathcal{A}(z_1^{\lambda_1} z_2^{\lambda_2} \ldots z_n^{\lambda_n})$. The following holds for $\rho := (n, n-1, \ldots, 2, 1) \in \mathcal{P}$:

$$A_\rho(z) = \prod_{i=1}^{n} \bigl(z_i - z_i^{-1}\bigr) \prod_{1 \leqslant j < k \leqslant n} \frac{(z_k - z_j)(1 - z_j z_k)}{z_j z_k}, \tag{2.2}$$

which is called Weyl's denominator formula. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}$, the following (Laurent) polynomial is said to be the *symplectic Schur function*:

$$\chi_\lambda(z) := \frac{A_{\lambda+\rho}(z)}{A_\rho(z)}$$

which occurs in Weyl's character formula. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P}$, if we denote by m_i the multiplicity of i in λ, i.e., $m_i = \#\{j\colon \lambda_j = i\}$, it is convenient to use the notation $\lambda = (1^{m_1} 2^{m_2} \ldots r^{m_r} \ldots)$ and $\chi_\lambda(z) = \chi_{(1^{m_1} 2^{m_2} \ldots r^{m_r} \ldots)}(z)$, for example, $\chi_{(2,1,1,0)}(z_1, z_2, z_3, z_4) = \chi_{(1^2 2)}(z_1, z_2, z_3, z_4)$.

2.1. The sets B and L

Let s be an arbitrary positive integer. Throughout the paper the number s is fixed. Let B and L be subsets of $\mathcal{P}$ defined by the following:

$$B := \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P};\ s-1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\},$$
$$L := \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathcal{P};\ s \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}.$$

The definition of the sets B and L differ only by the upper bound for λ_1. We divide B and L into $n+1$ parts as follows:

$$B = \bigcup_{i=0}^{n} B_i, \qquad L = \bigcup_{i=0}^{n} L_i.$$

Here $B_0 = L_0 = \{(0)\}$, where $(0) = (0, 0, \ldots, 0) \in \mathcal{P}$, and

$$B_i = \{(\lambda_1, \lambda_2, \ldots, \lambda_i, 0, 0, \ldots, 0) \in \mathcal{P};\ s-1 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_i > 0\},$$
$$L_i = \{(\lambda_1, \lambda_2, \ldots, \lambda_i, 0, 0, \ldots, 0) \in \mathcal{P};\ s \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_i > 0\}.$$

By definition, it follows that

$$|B| = \binom{s+n-1}{n}, \qquad |B_i| = \binom{s+i-2}{i}, \qquad |L| = \binom{s+n}{n}, \qquad |L_i| = \binom{s+i-1}{i}.$$

This indicates

$$\binom{s+n-1}{n} = \sum_{i=0}^{n} \binom{s+i-2}{i}, \qquad \binom{s+n}{n} = \sum_{i=0}^{n} \binom{s+i-1}{i}, \tag{2.3}$$

which are consequences of Pascal's triangle.

2.2. Orderings on B

We define the reverse lexicographic ordering $<$ on $\mathcal{P}$. For $\lambda, \mu \in \mathcal{P}$, we denote $\lambda < \mu$ if the following holds for some $k \in \{1, 2, \ldots, n\}$:

$$\lambda_1 = \mu_1, \quad \lambda_2 = \mu_2, \quad \ldots, \quad \lambda_{k-1} = \mu_{k-1} \quad \text{and} \quad \lambda_k < \mu_k.$$

In this paper, we consider two orderings on B. One is the reverse lexicographic ordering $<$ restricted on B. The other is defined as follows. For $\lambda, \mu \in B$, we denote $\lambda \prec \mu$ if a pair of $\lambda \in B_i$ and $\mu \in B_j$ satisfies either (1) $i > j$ or (2) $i = j$ and $\lambda < \mu$.

3. Definitions and the main results

Throughout the paper we assume $0 < q < 1$ and define the *q-shifted factorial* for all integers N by $(x)_\infty := \prod_{i=0}^{\infty}(1 - q^i x)$ and $(x)_N := (x)_\infty/(q^N x)_\infty$. For the fixed positive integer s we denote the number $\binom{s+n-1}{n}$ by κ.

3.1. BC_n type Jackson integral

For an arbitrary $z=(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$, we define a q-shift $z\to q^\nu z$ by a lattice point $\nu=(\nu_1,\nu_2,\dots,\nu_n)\in\mathbb{Z}^n$, where

$$q^\nu z:=\left(q^{\nu_1}z_1,q^{\nu_2}z_2,\dots,q^{\nu_n}z_n\right)\in(\mathbb{C}^*)^n.$$

The set $\Lambda_z:=\{q^\nu z\in(\mathbb{C}^*)^n;\ \nu\in\mathbb{Z}^n\}$ forms an orbit of a lattice subgroup of $(\mathbb{C}^*)^n$.

Definition 3.1. For a point $\xi\in(\mathbb{C}^*)^n$ and a function $f(z)$ on $(\mathbb{C}^*)^n$, we define the sum over the lattice Λ_ξ by

$$\int_{\Lambda_\xi} f(z)\,\varpi_q:=(1-q)^n\sum_{\nu\in\mathbb{Z}^n}f\left(q^\nu\xi\right)\quad\text{where }\varpi_q=\frac{d_qz_1}{z_1}\wedge\dots\wedge\frac{d_qz_n}{z_n}. \tag{3.1}$$

If this integral converges, we call it the *Jackson integral*.

Let $\Phi(z)$ be the q-multiplicative function of $z=(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$ defined by

$$\Phi(z):=\prod_{i=1}^{n}\prod_{m=1}^{2s+2}z_i^{1/2-\alpha_m}\frac{(qa_m^{-1}z_i)_\infty}{(a_mz_i)_\infty}\prod_{1\leqslant j<k\leqslant n}z_j^{1-2\tau}\frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty}\frac{(qt^{-1}z_jz_k)_\infty}{(tz_jz_k)_\infty},$$

where $q^{\alpha_m}=a_m$ and $q^\tau=t$. By definition, the following holds for $\Phi(z)$:

Lemma 3.2. *If we set $U_w(z):=w\Phi(z)/\Phi(z)$ for $w\in W$, then $U_w(z)$ is invariant under the q-shift $z\to q^\nu z$ for $\nu\in\mathbb{Z}^n$.*

Let T_{z_1} be the q-shift of variable z_1 such that $T_{z_1}:z_1\to qz_1$. Set

$$\nabla\varphi(z):=\varphi(z)-\frac{T_{z_1}\Phi(z)}{\Phi(z)}\,T_{z_1}\varphi(z), \tag{3.2}$$

where ratio $T_{z_1}\Phi(z)/\Phi(z)$ is written as the rational function

$$\frac{T_{z_1}\Phi(z)}{\Phi(z)}=q^{s+n}\prod_{m=1}^{2s+2}\frac{(1-a_mz_1)}{(a_m-qz_1)}\prod_{j=2}^{n}\frac{(1-tz_1/z_j)(1-tz_1z_j)}{(t-qz_1/z_j)(t-qz_1z_j)}. \tag{3.3}$$

The following is a key lemma which will be used in Section 5:

Proposition 3.3. *Let $\varphi(z)$ be an arbitrary function such that $\int_{\Lambda_\xi}\varphi(z)\Phi(z)\varpi_q$ converges. The following holds for $\varphi(z)$:*

$$\int_{\Lambda_\xi}\Phi(z)\nabla\varphi(z)\varpi_q=0. \tag{3.4}$$

Moreover,

$$\int_{\Lambda_\xi} \Phi(z)\,\mathcal{A}\nabla\varphi(z)\varpi_q = 0. \tag{3.5}$$

Proof. See Appendix A. □

We set

$$\Delta(z) := \prod_{i=1}^{n} \frac{1-z_i^2}{z_i} \prod_{1\leqslant j<k\leqslant n} \frac{(1-z_j/z_k)(1-z_jz_k)}{z_j}.$$

Using Weyl's denominator formula (2.2), $\Delta(z)$ can be written as

$$\Delta(z) = (-1)^n A_\rho(z) \quad \text{where } \rho = (n, n-1, \ldots, 2, 1) \in \mathcal{P}. \tag{3.6}$$

Definition 3.4. For a point $\xi \in (\mathbb{C}^*)^n$ and an arbitrary W-symmetric function $\varphi(z)$, the *BC_n type Jackson integral of $\varphi(z)$ over the lattice Λ_ξ* is defined by

$$\int_{\Lambda_\xi} \varphi(z)\Phi(z)\Delta(z)\varpi_q. \tag{3.7}$$

We will denote this by $\langle\varphi, \xi\rangle$, or simply $\langle\varphi\rangle$ if the point ξ is fixed.

By definition the sum $\langle\varphi, z\rangle$ is invariant under the q-shift $z \to q^\nu z$ for $\nu \in \mathbb{Z}^n$. From $w\varphi(z) = \varphi(z)$, $w\Delta(z) = (\operatorname{sgn} w)\Delta(z)$ and Lemma 3.2, it follows that

$$w\langle\varphi, z\rangle = (\operatorname{sgn} w)U_w(z)\langle\varphi, z\rangle \quad \text{for } w \in W. \tag{3.8}$$

We assume the following conditions for $a_1, a_2, \ldots, a_{2s+2}, t$ and ξ:

$$\left|a_1a_2\ldots a_{2s+2}t^{n+i-2}\right| > q^s \quad \text{for } i = 1, 2, \ldots, n, \tag{3.9}$$

and

$$\begin{cases} a_m\xi_i \notin \{q^l;\ l \in \mathbb{Z}\} & \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant m \leqslant 2s+2, \\ t\xi_j/\xi_k, t\xi_j\xi_k \notin \{q^l;\ l \in \mathbb{Z}\} & \text{for } 1 \leqslant j < k \leqslant n. \end{cases}$$

Then the convergence of $\langle 1, \xi\rangle$ can be confirmed in the same way as [13, Theorem 4, p. 158]. Throughout the paper we also assume the following condition

$(\mathcal{C})$ all the parameters $a_1, a_2, \ldots, a_{2s+2}$ and t are generic.

Let T_u be the q-shift operator on a parameter u, i.e., $T_u : u \to qu$. Let $\vec{\chi}$ and $\langle\vec{\chi}\rangle$ be the vectors defined by

$$\vec{\chi} := \big(\chi_\lambda(z)\big)_{\lambda\in B} \quad \text{and} \quad \langle\vec{\chi}\rangle := \big(\langle\chi_\lambda, \xi\rangle\big)_{\lambda\in B},$$

where the indices $\lambda \in B$ are arranged in increasing order of $<$.

Proposition 3.5. *Under the condition* ($\mathcal{C}$), *there exist invertible* $\kappa \times \kappa$ *matrices* Y_{a_i} ($i = 1, 2, \ldots, 2s+2$) *and* Y_t *whose entries are rational functions of* $a_1, a_2, \ldots, a_{2s+2}$ *and* t, *such that*

$$T_{a_i}\langle\vec{\chi}\rangle = \langle\vec{\chi}\rangle Y_{a_i}, \qquad T_t\langle\vec{\chi}\rangle = \langle\vec{\chi}\rangle Y_t.$$

Proof. See [5]. □

We now state one of the main theorems, which is the same as Theorem 1.2.

Theorem 3.6. *The determinant of the matrix* Y_{a_i} *is evaluated as*

$$\det Y_{a_i} = (-a_i)^{-s\binom{s+n-1}{s}} \prod_{k=1}^{n} \left[\frac{\prod_{j=1}^{2s+2}(1 - t^{n-k}a_i a_j)}{(1 - t^{n-k}a_i^2)(1 - t^{n+k-2}\prod_{m=1}^{2s+2} a_m)} \right]^{\binom{s+k-2}{k-1}}.$$

We shall prove Theorem 3.6 in Section 6.

3.2. Truncation

Let Z be the set of all s-tuples defined by

$$Z := \{(\mu_1, \mu_2, \ldots, \mu_s) \in \mathbb{Z}^s;\ \mu_1 + \cdots + \mu_s = n,\ \mu_1 \geqslant 0, \ldots, \mu_s \geqslant 0\},$$

which consists of κ elements. For s-tuples $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_s) \in Z$, we define the ordering $\mu \prec_Z \nu$ on Z if there exists i such that $\mu_1 = \nu_1, \mu_2 = \nu_2, \ldots, \mu_{i-1} = \nu_{i-1}$, $\mu_i < \nu_i$. Corresponding to the s-tuple $\mu = (\mu_1, \mu_2, \ldots, \mu_s) \in Z$, we take the point

$$\zeta_{(\mu)} = (\zeta_{(\mu)1}, \zeta_{(\mu)2}, \ldots, \zeta_{(\mu)n}) \in (\mathbb{C}^*)^n \tag{3.10}$$

satisfying

$$\begin{cases} \zeta_{(\mu)i} = a_i & \text{if } i \in \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\}, \\ \zeta_{(\mu)j}/\zeta_{(\mu)j+1} = t & \text{if } j \notin \{\mu_1, \mu_1 + \mu_2, \ldots, \mu_1 + \mu_2 + \cdots + \mu_s\} \end{cases}$$

or equivalently

$$\zeta_{(\mu)i} = \begin{cases} a_1 t^{\mu_1 - i} & \text{if } 1 \leqslant i \leqslant \mu_1, \\ a_2 t^{\mu_1 + \mu_2 - i} & \text{if } \mu_1 + 1 \leqslant i \leqslant \mu_1 + \mu_2, \\ \vdots & \\ a_s t^{n-i} & \text{if } \mu_1 + \cdots + \mu_{s-1} + 1 \leqslant i \leqslant n. \end{cases}$$

For example, when $s = 4$ and $n = 9$, for the 4-tuple $\mu = (4, 0, 2, 3) \in Z$ the point $\zeta_{(\mu)}$ is written as

$$\zeta_{(\mu)} = \big(\underbrace{a_1t^3, a_1t^2, a_1t, a_1}_{4}, \underbrace{a_3t, a_3}_{2}, \underbrace{a_4t^2, a_4t, a_4}_{3}\big).$$

For the point $\zeta_{(\mu)} \in (\mathbb{C}^*)^n$, we denote by $\Lambda^+_{\zeta_{(\mu)}}$ the fan with the vertex $\zeta_{(\mu)}$ such that

$$\Lambda^+_{\zeta_{(\mu)}} := \{q^\nu \zeta_{(\mu)} \in (\mathbb{C}^*)^n;\ \nu \in D_\mu\}$$

where D_μ is the fan in $\mathbb{Z}^n$ defined by

$$D_\mu := \left\{(\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}^n;\ \begin{array}{ll} \nu_i > 0 & \text{if } i = \sum_{k=1}^j \mu_k\ (1 \leqslant j \leqslant s), \\ \nu_i - \nu_{i+1} > 0 & \text{if } i \text{ is otherwise} \end{array}\right\}.$$

We call the Jackson integral $\langle \varphi, \zeta_{(\mu)} \rangle$ summed over the lattice $\Lambda_{\zeta_{(\mu)}}$ *truncated*. Since $\Phi(z) = 0$ if $z \in \Lambda_{\zeta_{(\mu)}} - \Lambda^+_{\zeta_{(\mu)}}$, the truncated Jackson integral $\langle \varphi, \zeta_{(\mu)} \rangle$ is summed only over the fan $\Lambda^+_{\zeta_{(\mu)}}$. We will discuss in Section 7 the asymptotic behavior of the truncated Jackson integrals.

3.3. Regularization

Let $\Theta(z)$ be the function on $(\mathbb{C}^*)^n$ defined by

$$\Theta(z) := \prod_{i=1}^n \frac{z_i^s \theta(z_i^2)}{\prod_{m=1}^{2s+2} z_i^{\alpha_m} \theta(a_m z_i)} \prod_{1 \leqslant j<k \leqslant n} \frac{\theta(z_j/z_k)\theta(z_j z_k)}{z_j^{2\tau} \theta(t z_j/z_k)\theta(t z_j z_k)} \tag{3.11}$$

where $\theta(x)$ denotes the function $(x)_\infty (q/x)_\infty$. By definition we see

$$w\Theta(z) = (\operatorname{sgn} w) U_w(z) \Theta(z) \quad \text{for } w \in W. \tag{3.12}$$

Proposition 3.7. *Under the condition* $(\mathcal{C})$, *if* $\varphi(z)$ *is* W*-symmetric and holomorphic on* $(\mathbb{C}^*)^n$, *then there exists a holomorphic function* $f(z)$ *on* $(\mathbb{C}^*)^n$ *such that* $\langle \varphi, z \rangle = f(z)\Theta(z)$.

Proof. See Appendix B. □

Definition 3.8. If $\varphi(z)$ is W-symmetric and holomorphic on $(\mathbb{C}^*)^n$, we call the holomorphic function $\langle \varphi, z \rangle / \Theta(z)$ the *regularized Jackson integral* and denote it by $\langle\!\langle \varphi, z \rangle\!\rangle$.

From (3.8) and (3.12), the regularized Jackson integral $\langle\!\langle \varphi, z \rangle\!\rangle$ is also W-symmetric.

Remark. In particular, if $s = 1$, the function $\Theta(z)$ is periodic under the q-shift $z \to q^\nu z$ for $\nu \in \mathbb{Z}^n$. This implies that the function $f(z)$ in Proposition 3.7 becomes a constant independent of z, and $\langle\!\langle 1, z \rangle\!\rangle$ coincides with the right-hand side of (1.3). See [8,11,14,15] for the constant in the case where $s = 1$.

We now state the other main theorem for the BC_n type regularized Jackson integral, which is the same as Theorem 1.3.

Theorem 3.9. *The $\kappa \times \kappa$ determinant with (λ, μ) entry $\langle\langle \chi_\lambda, \zeta_{(\mu)} \rangle\rangle$ is evaluated as*

$$\big\{(1-q)(q)_\infty\big\}^{n\binom{s+n-1}{n}} \prod_{k=1}^{n} \left[\frac{(qt^{-(n-k+1)})_\infty^s}{(qt^{-1})_\infty^s} \frac{\prod_{1\leqslant i<j\leqslant 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1})_\infty} \right]^{\binom{s+k-2}{k-1}}$$
$$\times \prod_{k=1}^{n} \left[\prod_{r=0}^{n-k} \prod_{1\leqslant i<j\leqslant s} \frac{\theta(t^{2r-(n-k)}a_i a_j^{-1})\theta(t^{n-k}a_i a_j)}{t^r a_i} \right]^{\binom{s+k-3}{k-1}},$$

where the rows $\lambda \in B$ and the columns $\mu \in Z$ of the matrix $(\langle\langle \chi_\lambda, \zeta_{(\mu)} \rangle\rangle)_{\lambda,\mu}$ are arranged in the orders $<$ and $\prec_Z$, respectively.

We will prove Theorem 3.9 in Section 8.

4. The polynomials $e_\lambda(z)$

In this section, we give the definition of polynomials $e_\lambda(z)$ and state some properties of $e_\lambda(z)$.

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^n$, we denote by z^λ the monomial $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_n^{\lambda_n}$. For $\lambda \in \mathcal{P}$ the *orbit-sums* $m_\lambda(z)$ are defined by

$$m_\lambda(z) := \sum_{\mu \in W\lambda} z^\mu$$

where $W\lambda := \{w\lambda;\ w \in W\}$ is the W-orbit of λ. The symplectic Schur functions $\chi_\lambda(z)$ are expanded in the orbit-sums $m_\mu(z)$ as follows:

$$\chi_\lambda(z) = m_\lambda(z) + \sum_{\mu<\lambda} K_{\lambda\mu} m_\mu(z) \tag{4.1}$$

where the $K_{\lambda\mu}$ are integers (see [24]). Using $\chi_\lambda(z)$, we first define polynomials $e_i(z)$ of degree i, $0 \leqslant i \leqslant n$, as follows:

$$e_i(z) := \sum_{j=0}^{i} (-1)^j \chi_{(1^{i-j})}(\underbrace{z_1, z_2, \dots, z_n}_{n}) \chi_{(j)}\big(\underbrace{a_1, a_1 t, \dots, a_1 t^{n-i}}_{n-i+1}\big), \tag{4.2}$$

which we call the *ith 'elementary' symmetric polynomial* as was noted in [15]. From (4.1), we have

$$e_i(z) = m_{(1^i)}(z) + \text{lower order terms w.r.t. } <. \tag{4.3}$$

Lemma 4.1. *If $1 \leqslant j \leqslant i \leqslant n$, then*

$$e_i\big(z_1, z_2, \dots, z_{j-1}, a_1 t^{n-j}, a_1 t^{n-j-1}, \dots, a_1 t, a_1\big) = 0.$$

Proof. This is a direct consequence of the relation

$$\sum_{j=0}^{i}(-1)^j \chi_{(1^{i-j})}(z_1, z_2, \dots, z_n)\chi_{(j)}(z_1, z_2, \dots, z_{n-i+1}) = 0$$

for $i = 1, 2, \dots, n$, which was proved in [15]. □

Lemma 4.2. *The product expression of the nth 'elementary' symmetric polynomial $e_n(z)$ is the following:*

$$e_n(z) = \prod_{i=1}^{n} \frac{(a_1 - z_i)(1 - a_1 z_i)}{a_1 z_i} = \frac{T_{a_1}\Phi(z)}{\Phi(z)}. \tag{4.4}$$

Proof. See [15]. □

We now define $e_\lambda(z)$ for an arbitrary $\lambda \in \mathcal{P}$ as follows:

$$e_\lambda(z) := \prod_{i=1}^{n} e_i(z)^{\lambda_i - \lambda_{i+1}}, \tag{4.5}$$

where we regard $\lambda_{n+1} = 0$ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}$. Since

$$m_\lambda(z) m_\mu(z) = m_{\lambda+\mu}(z) + \text{lower order terms w.r.t.} <$$

and $\lambda = \sum_{i=1}^{n}(\lambda_i - \lambda_{i+1})(1^i)$, Eq. (4.3) implies that $e_\lambda(z)$ is expanded in the orbit-sums $m_\mu(z)$ as

$$e_\lambda(z) = m_\lambda(z) + \sum_{\mu<\lambda} M_{\lambda\mu} m_\mu(z), \tag{4.6}$$

so that $e_\lambda(z)$ is also expanded in $\chi_\mu(z)$ as

$$e_\lambda(z) = \chi_\lambda(z) + \sum_{\mu<\lambda} E_{\lambda\mu} \chi_\mu(z). \tag{4.7}$$

Lemma 4.3. *If $\lambda \in L_i$ and $1 \leqslant j \leqslant i \leqslant n$, then*

$$e_\lambda\big(z_1, z_2, \dots, z_{j-1}, a_1 t^{n-j}, a_1 t^{n-j-1}, \dots, a_1 t, a_1\big) = 0.$$

Proof. Since $\lambda_i \neq 0$ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_i, 0, \dots, 0) \in L_i$, using Lemma 4.1, if $1 \leqslant j \leqslant i \leqslant n$, then $e_i(z_1, z_2, \dots, z_{j-1}, a_1 t^{n-j}, a_1 t^{n-j-1}, \dots, a_1 t, a_1)^{\lambda_i} = 0$. By definition, $e_\lambda(z)$ is expressed as

$$e_\lambda(z) = e_1(z)^{\lambda_1-\lambda_2} e_2(z)^{\lambda_2-\lambda_3} \dots e_i(z)^{\lambda_i}$$

and has the factor $e_i(z)^{\lambda_i}$. Thus we have Lemma 4.3. □

Let x be a real number satisfying $x > 0$. For $i = 1, 2, \dots, n+1$, we set

$$\zeta_i = (\zeta_{i1}, \zeta_{i2}, \dots, \zeta_{in}) \in (\mathbb{C}^*)^n, \tag{4.8}$$

and

$$\zeta_{ij} := \begin{cases} x^{k_{ij}} & \text{if } 1 \leqslant j < i, \\ t^{n-j} a_1 & \text{if } i \leqslant j \leqslant n, \end{cases}$$

where $k_{ij} > 0$ is to be suitably chosen, for example $k_{ij} := (s+1)^{i-j-1}$, to satisfy Lemma 4.5 and $k_{ij} > k_{i\ell}$ if $j < \ell$.

Corollary 4.4. *If $\lambda \in L_i$ and $1 \leqslant j \leqslant i \leqslant n$, then $e_\lambda(\zeta_j) = 0$.*

Proof. It is straightforward from the definition (4.8) of ζ_j and Lemma 4.3. □

Lemma 4.5. *If $\lambda \in L_{i-1}$, then*

$$\lim_{x \to 0} \big[z^\lambda e_\mu(z)\big]_{z=\zeta_i} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda > \mu. \end{cases}$$

Proof. By the definition (4.8) of ζ_j, if $\lambda \in L_{i-1}$, then we have

$$\lim_{x \to 0} \big[z^\lambda m_\mu(z)\big]_{z=\zeta_i} = \begin{cases} 1 & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda > \mu. \end{cases}$$

Using the expression (4.6) of $e_\mu(z)$ proves Lemma 4.5. □

5. Homogeneous linear relations for $\langle e_\lambda \rangle$

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots, 0) \in B_\ell$ and $0 \leqslant i \leqslant n - \ell$, we set

$$s^i \lambda := (\underbrace{s, s, \dots, s}_{i}, \lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots, 0) \in L_{i+\ell},$$

so that s^i maps B_ℓ into L for $i + \ell \leqslant n$. In this section our aim is to prove the following:

Proposition 5.1. *If $\lambda \in B_\ell$, then*

$$\langle e_{s^{n-\ell}\lambda} \rangle - K_\ell \langle e_\lambda \rangle \in \bigoplus_{\substack{\mu \prec \lambda \\ \mu \in B}} \mathbb{C}\langle e_\mu \rangle, \tag{5.1}$$

where the coefficient K_ℓ is written as

$$K_\ell = \prod_{i=1}^{n-\ell} \big(-a_1^{-s}\big) \frac{t^{i+\ell-1}}{t^{s(n-i-\ell)}} \frac{\prod_{k=2}^{2s+2}(1 - t^{n-i-\ell} a_1 a_k)}{(1 - t^{n+i+\ell-2} \prod_{m=1}^{2s+2} a_m)}. \tag{5.2}$$

Proposition 5.1 is a key of the proof of Theorem 3.6 and is used in Section 6. The rest of this section is devoted to proving Proposition 5.1.

To specify the number n of variables $z_1, z_2, \dots, z_n$, we simply use $e_\lambda^{(n)}(z)$ and $A^{(n)}(z)$ instead of the polynomials $e_\lambda(z)$ and Weyl's denominator $A_\rho(z)$, respectively.

For the point $z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n$ we set

$$\widehat{z}_k := (z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n) \in (\mathbb{C}^*)^{n-1}. \tag{5.3}$$

For $\widehat{z}_k$ $(k = 1, 2, \dots, n)$ we have the following, immediately from Corollary 4.4 and Lemma 4.5.

Lemma 5.2. *The following holds for $\mu \in L_{i-1}$ and the point $\zeta_j \in (\mathbb{C}^*)^n$ defined in* (4.8)*:*

$$\big[e_\mu^{(n-1)}(\widehat{z}_k)\big]_{z=\zeta_j} = 0 \quad \text{if } 1 \leqslant k \leqslant j < i,$$

and

$$\big[e_\mu^{(n-1)}(\widehat{z}_k)\big]_{z=\zeta_i} = 0 \quad \text{if } 1 \leqslant k < i.$$

Moreover,

$$\lim_{x\to 0}\big[z^\mu e_\nu^{(n-1)}(\widehat{z}_i)\big]_{z=\zeta_i} = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu < \mu. \end{cases}$$

Let $f(z)$ and $g(z)$ be functions defined as follows:

$$f(z) := \prod_{m=1}^{2s+2}(a_m - z_1)\prod_{j=2}^{n}(t - z_1/z_j)(t - z_1 z_j),$$
$$g(z) := \prod_{m=1}^{2s+2}(1 - a_m z_1)\prod_{j=2}^{n}(1 - t z_1/z_j)(1 - t z_1 z_j).$$

We set

$$\begin{cases} f_1(z) := f(z), & g_1(z) := g(z), \\ f_i(z) := \sigma_i f(z), & g_i(z) := \sigma_i g(z) \quad \text{for } i = 2, 3, \dots, n, \end{cases} \tag{5.4}$$

where σ_i is defined in (2.1). By definition of τ_1 in (2.1), we have

$$\tau_1\left(\frac{f_1(z)}{z_1^{n+s}}\right) = \frac{g_1(z)}{z_1^{n+s}}. \tag{5.5}$$

Lemma 5.3. *For the point $\zeta_j \in (\mathbb{C}^*)^n$ defined in* (4.8) *the following holds for $f_k(z)$, $g_k(z)$:*

$$f_k(\zeta_j) = 0 \quad \text{if } 1 \leqslant j \leqslant k \leqslant n,$$

while,

$$g_k(\zeta_j)=0 \quad \textit{if } 1\leqslant j<k\leqslant n.$$

Moreover,

$$\lim_{x\to 0}\left[z_1z_2\dots z_{i-1}\frac{g_i(z)}{z_i^{n+s}}\right]_{z=\zeta_i} = (-t)^{i-1}\frac{\prod_{k=2}^{2s+2}(1-a_ka_1t^{n-i})}{(1-t)(t^{n-i}a_1)^{n-i+s+1}}\prod_{j=0}^{n-i}\big(1-t^{j+1}\big)\big(1-t^{n-i+j}a_1^2\big), \tag{5.6}$$

and if $i\geqslant k$, then

$$\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\dots\frac{z_{k-1}}{z_k}\big(f_k(z)-g_k(z)\big)\right]_{z=\zeta_{i+1}} = (-1)^k\left(t^{k-1}-t^{2n-k-1}\prod_{m=1}^{2s+2}a_m\right). \tag{5.7}$$

Proof. From (5.4), $f_k(z)$ has the factor $(t-z_k/z_{k+1})$ if $1\leqslant k\leqslant n-1$, and f_n has the factor (a_1-z_n). When $z=\zeta_j$, from the definition (4.8) of ζ_j, it follows that $t-z_k/z_{k+1}=0$ if $j\leqslant k\leqslant n-1$ and $a_1-z_n=0$ if $j\leqslant n$. Thus $f_k(\zeta_j)=0$ if $j\leqslant k\leqslant n$. From (5.4) it follows that $g_k(z)$ has the factor $(1-tz_k/z_{k-1})$ so that $g_k(\zeta_j)=0$ if $j+1\leqslant k\leqslant n$.

Next we prove formulae (5.6) and (5.7). If we put $z=\zeta_i$ (see (4.8)), then we have

$$\begin{aligned}
&\left[z_1z_2\dots z_{i-1}\frac{g_i(z)}{z_i^{n+s}}\right]_{z=\zeta_i}\\
&\quad=\frac{(1-a_1^2t^{n-i})\prod_{k=2}^{2s+2}(1-a_ka_1t^{n-i})}{(t^{n-i}a_1)^{n+s}}\\
&\qquad\times\big(x^{k_{i1}}-t^{n-i+1}a_1\big)\big(x^{k_{i2}}-t^{n-i+1}a_1\big)\dots\big(x^{k_{i,i-1}}-t^{n-i+1}a_1\big)\\
&\qquad\times\big(1-x^{k_{i1}}t^{n-i+1}a_1\big)\big(1-x^{k_{i2}}t^{n-i+1}a_1\big)\dots\big(1-x^{k_{i,i-1}}t^{n-i+1}a_1\big)\\
&\qquad\times\big(1-t^2\big)\big(1-t^3\big)\dots\big(1-t^{n-i+1}\big)\big(1-t^{2(n-i)}a_1^2\big)\big(1-t^{2(n-i)-1}a_1^2\big)\dots\big(1-t^{n-i+1}a_1^2\big),
\end{aligned}$$

so that we obtain (5.6) taking the limit $x\to 0$. Suppose $k\leqslant i$. Then, putting $z=\zeta_{i+1}$ (see (4.8)), we have the following:

$$\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\dots\frac{z_{k-1}}{z_k}f_k(z)\right]_{z=\zeta_{i+1}} = (-1)^{k-1}t^{2n-k-1}\prod_{m=1}^{2s+2}a_m,$$

$$\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\dots\frac{z_{k-1}}{z_k}g_k(z)\right]_{z=\zeta_{i+1}} = (-t)^{k-1},$$

which completes the proof. □

Let $\bar\varphi_\lambda(z)$, $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in L$, be the function defined by

$$\bar\varphi_\lambda(z):=\frac{\mathcal{A}\nabla\varphi_\lambda(z)}{2}$$

where ∇ is defined in (3.2) and

$$\varphi_\lambda(z):=\frac{f(z)}{z_1^{\lambda_1+n}}\,z_2^{n-1}z_3^{n-2}\dots z_n\,e^{(n-1)}_{(\lambda_2,\dots,\lambda_n)}(z_2,z_3,\dots,z_n). \tag{5.8}$$

Lemma 5.4. *For $\lambda\in B_\ell$ and $1\leqslant i\leqslant n-\ell$, the function $\bar\varphi_{s^i\lambda}(z)$ is expressed as*

$$\bar\varphi_{s^i\lambda}(z)=\sum_{k=1}^{n}(-1)^{k+1}\frac{f_k(z)-g_k(z)}{z_k^{s+n}}e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)A^{(n-1)}(\widehat{z}_k), \tag{5.9}$$

where $\widehat{z}_k$ is the point defined by (5.3)*. On the other hand, $\bar\varphi_{s^i\lambda}(z)$ is expanded in the functions $e^{(n)}_\mu(z)A^{(n)}(z)$, $\mu\leqslant s^i\lambda$, as follows:*

$$\bar\varphi_{s^i\lambda}(z)=\sum_{\mu\leqslant s^i\lambda}c_{s^i\lambda,\mu}e^{(n)}_\mu(z)A^{(n)}(z). \tag{5.10}$$

Proof. For $\lambda\in B_\ell$ and $1\leqslant i\leqslant n-\ell$, (5.8) implies that

$$\varphi_{s^i\lambda}(z)=\frac{f(z)}{z_1^{s+n}}\,z_2^{n-1}z_3^{n-2}\dots z_n\,e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_1). \tag{5.11}$$

From the definition (3.2) of ∇ and (3.3) we have

$$\nabla\varphi_{s^i\lambda}(z)=\frac{f(z)-g(z)}{z_1^{n+s}}z_2^{n-1}z_3^{n-2}\dots z_n\,e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_1).$$

Then from (2.1) and (5.5) it follows that

$$\begin{aligned}\bar\varphi_{s^i\lambda}(z)&=\mathcal{A}\nabla\varphi_{s^i\lambda}(z)/2\\&=\frac{f_1(z)-g_1(z)}{z_1^{n+s}}e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_1)A^{(n-1)}(\widehat{z}_1)\\&\quad+\sum_{k=2}^{n}(\operatorname{sgn}\sigma_k)\sigma_k\left[\frac{f_1(z)-g_1(z)}{z_1^{n+s}}e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_1)A^{(n-1)}(\widehat{z}_1)\right].\end{aligned} \tag{5.12}$$

Thus, we obtain the expression (5.9) by substituting (5.4) and the following for (5.12):

$$\operatorname{sgn}\sigma_k=-1,\qquad \sigma_k e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_1)=e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k),\qquad \sigma_k A^{(n-1)}(\widehat{z}_1)=(-1)^k A^{(n-1)}(\widehat{z}_k).$$

Next, from the degrees of the monomials in the expansion of (5.11) we eventually obtain the expression (5.10). □

Lemma 5.5. *Suppose that $\lambda \in B_\ell$ and $\mu \in L_{j-1}$. If $c_{s^i\lambda,\nu} = 0$ for $\nu \in L_{j-1}$ satisfying $\mu < \nu$, then*

$$\lim_{x\to 0}\left[\left(z^\mu \prod_{m=1}^{j-1} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_j} = (-1)^{j-1} c_{s^i\lambda,\mu} A^{(n-j+1)}\big(t^{n-j}a_1,\ldots,a_1\big). \quad (5.13)$$

Proof. Using (5.10) and Corollary 4.4, we have

$$\overline{\varphi}_{s^i\lambda}(\zeta_j) = \sum_{\nu \leqslant s^i\lambda} c_{s^i\lambda,\nu} e_\nu^{(n)}(\zeta_j) A^{(n)}(\zeta_j) = \sum_{k=1}^{j-1}\sum_{\nu\in L_k} c_{s^i\lambda,\nu} e_\nu^{(n)}(\zeta_j) A^{(n)}(\zeta_j).$$

If $c_{s^i\lambda,\nu} = 0$ for $\nu \in L_{j-1}$ satisfying $\mu < \nu$, then $\overline{\varphi}_{s^i\lambda}(\zeta_j)$ is written as

$$\overline{\varphi}_{s^i\lambda}(\zeta_j) = \sum_{\nu \leqslant \mu} c_{s^i\lambda,\nu} e_\nu^{(n)}(\zeta_j) A^{(n)}(\zeta_j),$$

so that

$$\left[\left(z^\mu \prod_{m=1}^{j-1} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_j} = \sum_{\nu\leqslant\mu} c_{s^i\lambda,\nu}\big[\big(z^\mu e_\nu^{(n)}(z)\big)\big(z_1^n z_2^{n-1}\ldots z_{j-1}^{n-j+2} A^{(n)}(z)\big)\big]_{z=\zeta_j}. \quad (5.14)$$

From Lemma 4.5, we have

$$\lim_{x\to 0}\big[z^\mu e_\nu^{(n)}(z)\big]_{z=\zeta_j} = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu < \mu. \end{cases} \quad (5.15)$$

From Weyl's denominator formula (2.2) and the expression (4.8) of ζ_j, it follows that

$$\lim_{x\to 0}\big[\big(z_1^n z_2^{n-1}\ldots z_{j-1}^{n-j+2} A^{(n)}(z)\big)\big]_{z=\zeta_j} = (-1)^{j-1} A^{(n-j+1)}\big(t^{n-j}a_1,\ldots,a_1\big). \quad (5.16)$$

Taking the limit $x \to 0$ in both sides of (5.14) and using (5.15) and (5.16), we obtain (5.13). □

Lemma 5.6. *For $\lambda \in B_\ell$ the coefficient $c_{s^i\lambda,\mu}$ in* (5.10) *vanishes if $\mu \in L_{j-1}$ where $j = 1, 2, \ldots, i+\ell-1$.*

Proof. From Lemma 5.5, in order to prove $c_{s^i\lambda,\mu} = 0$ for $\mu \in L_{j-1}$, it is sufficient to show that

$$\lim_{x\to 0}\left[\left(z^\mu \prod_{m=1}^{j-1} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_j} = 0 \quad \text{if } 1 \leqslant j \leqslant i+\ell-1. \quad (5.17)$$

We now suppose $1 \leqslant j \leqslant i+\ell-1$. By Lemma 5.3, if $j < k \leqslant n$, then $f_k(\zeta_j) = g_k(\zeta_j) = 0$. Moreover, by Lemma 5.2, if $k \leqslant j < i+\ell$, then $[e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)]_{z=\zeta_j} = 0$. Since the summand of

$\overline{\varphi}_{s^i\lambda}(z)$ in (5.9) has the factors $f_k(z) - g_k(z)$ and $e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)$, if we put $z = \zeta_j$, then $\overline{\varphi}_{s^i\lambda}(\zeta_j) = 0$, which proves (5.17). □

Lemma 5.7. *For $\lambda \in B_\ell$ the coefficient $c_{s^i\lambda,\mu}$ in* (5.10) *vanishes if*

$$\mu \in L_{i+\ell-1} \quad \textit{and} \quad s^{i-1}\lambda < \mu.$$

Moreover, the coefficient $c_{s^i\lambda,s^{i-1}\lambda}$ is evaluated as

$$c_{s^i\lambda,s^{i-1}\lambda} = \frac{(1-t^{n-i-\ell+1})t^{i+\ell-1}}{(1-t)t^{s(n-i-\ell)}} \frac{\prod_{k=2}^{2s+2}(1-t^{n-i-\ell}a_1a_k)}{a_1^s}. \tag{5.18}$$

Proof. By Lemma 5.3, $f_k(\zeta_{i+\ell}) = g_k(\zeta_{i+\ell}) = 0$ if $i+\ell < k \leqslant n$, and $f_{i+\ell}(\zeta_{i+\ell}) = 0$. Moreover, by Lemma 5.2, $[e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)]_{z=\zeta_{i+\ell}} = 0$ if $k < i+\ell$. Since the summand of $\overline{\varphi}_{s^i\lambda}(z)$ in (5.9) has the factors $f_k(z) - g_k(z)$ and $e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)$, if we put $z = \zeta_{i+\ell}$, then

$$\overline{\varphi}_{s^i\lambda}(\zeta_{i+\ell}) = \left[(-1)^{i+\ell}\frac{g_{i+\ell}(z)}{z_{i+\ell}^{n+s}} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+\ell}) A^{(n-1)}(\widehat{z}_{i+\ell})\right]_{z=\zeta_{i+\ell}}. \tag{5.19}$$

Thus, we have

$$\begin{aligned}
&\left[\left(z^\mu \prod_{m=1}^{i+\ell-1} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell}} \\
&\quad = (-1)^{i+\ell}\left[\left(z_1z_2\ldots z_{i+\ell-1}\frac{g_{i+\ell}(z)}{z_{i+\ell}^{n+s}}\right)\left(z^\mu e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+\ell})\right)\right. \\
&\qquad \left.\times \left(z_1^{n-1}z_2^{n-2}\ldots z_{i+\ell-1}^{n-i-\ell+1} A^{(n-1)}(\widehat{z}_{i+\ell})\right)\right]_{z=\zeta_{i+\ell}}.
\end{aligned} \tag{5.20}$$

From Lemma 5.2, when $\mu \in L_{i+\ell-1}$, we have

$$\lim_{x\to 0}\left[z^\mu e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+\ell})\right]_{z=\zeta_{i+\ell}} = \begin{cases} 1 & \text{if } s^{i-1}\lambda = \mu \\ 0 & \text{if } s^{i-1}\lambda < \mu. \end{cases} \tag{5.21}$$

Using (4.8) and Weyl's denominator formula (2.2), we also have

$$\begin{aligned}
&\lim_{x\to 0}\left[z_1^{n-1}z_2^{n-2}\ldots z_{i+\ell-1}^{n-i-\ell+1} A^{(n-1)}(\widehat{z}_{i+\ell})\right]_{z=\zeta_{i+\ell}} \\
&\quad = (-1)^{i+\ell-1} A^{(n-i-\ell)}\left(t^{n-i-\ell-1}a_1, \ldots, a_1\right).
\end{aligned} \tag{5.22}$$

From (5.20), (5.21) and (5.22), if $\mu > s^{i-1}\lambda$, then

$$\lim_{x\to 0}\left[\left(z^\mu \prod_{m=1}^{i+\ell-1} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell}} = 0.$$

By virtue of Lemma 5.5, we therefore obtain

$$c_{s^i\lambda,\mu}=0 \quad \text{if} \quad \mu\in L_{i+\ell-1} \text{ and } s^{i-1}\lambda<\mu. \tag{5.23}$$

Next we evaluate the coefficient $c_{s^i\lambda,s^{i-1}\lambda}$. By Lemma 5.5 and (5.23), we have

$$\begin{aligned}&\lim_{x\to 0}\Biggl[\Biggl(z^{s^{i-1}\lambda}\prod_{m=1}^{i+\ell-1}z_m^{n-m+1}\Biggr)\overline{\varphi}_{s^i\lambda}(z)\Biggr]_{z=\zeta_{i+\ell}}\\&\quad=(-1)^{i+\ell-1}c_{s^i\lambda,s^{i-1}\lambda}A^{(n-i-\ell+1)}\bigl(t^{n-i-\ell}a_1,\dots,a_1\bigr).\end{aligned} \tag{5.24}$$

On the other hand, using (5.21) and (5.22) and putting $\mu=s^{i-1}\lambda$ in (5.20), it follows that

$$\begin{aligned}&\lim_{x\to 0}\Biggl[\Biggl(z^{s^{i-1}\lambda}\prod_{m=1}^{i+\ell-1}z_m^{n-m+1}\Biggr)\overline{\varphi}_{s^i\lambda}(z)\Biggr]_{z=\zeta_{i+\ell}}\\&\quad=-\lim_{x\to 0}\biggl[z_1z_2\dots z_{i+\ell-1}\frac{g_{i+\ell}(z)}{z_{i+\ell}^{n+s}}\biggr]_{z=\zeta_{i+\ell}}A^{(n-i-\ell)}\bigl(t^{n-i-\ell-1}a_1,\dots,a_1\bigr).\end{aligned} \tag{5.25}$$

Comparing (5.24) and (5.25), we have

$$c_{s^i\lambda,s^{i-1}\lambda}=(-1)^{i+\ell}\lim_{x\to 0}\biggl[z_1z_2\dots z_{i+\ell-1}\frac{g_{i+\ell}(z)}{z_{i+\ell}^{n+s}}\biggr]_{z=\zeta_{i+\ell}}\frac{A^{(n-i-\ell)}(t^{n-i-\ell-1}a_1,\dots,a_1)}{A^{(n-i-\ell+1)}(t^{n-i-\ell}a_1,\dots,a_1)}. \tag{5.26}$$

From Weyl's denominator formula (2.2), it follows that

$$\frac{A^{(j+1)}(z_1,z_2,\dots,z_{j+1})}{A^{(j)}(z_2,\dots,z_{j+1})}=-\frac{1-z_1^2}{z_1}\prod_{k=2}^{j+1}\frac{(1-z_1/z_k)(1-z_1z_k)}{z_1},$$

so that

$$\frac{A^{(n-i-\ell+1)}(t^{n-i-\ell}a_1,\dots,a_1)}{A^{(n-i-\ell)}(t^{n-i-\ell-1}a_1,\dots,a_1)}=\frac{-1}{(1-t^{n-i-\ell+1})}\prod_{j=0}^{n-i-\ell}\bigl(1-t^{j+1}\bigr)\frac{(1-t^{n-i-\ell+j}a_1^2)}{t^{n-i-\ell}a_1}. \tag{5.27}$$

From (5.6), (5.26) and (5.27), we obtain (5.18). □

Lemma 5.8. *For $\lambda\in B_\ell$, the coefficient $c_{s^i\lambda,s^i\lambda}$ in* (5.10) *is evaluated as*

$$c_{s^i\lambda,s^i\lambda}=\frac{1-t^i}{1-t}\Biggl(1-t^{2n-i-1}\prod_{m=1}^{2s+2}a_m\Biggr).$$

Proof. Using Lemma 5.3, $f_k(\zeta_{i+\ell+1}) = g_k(\zeta_{i+\ell+1}) = 0$ if $i+\ell+2 \leqslant k \leqslant n$. Since the summand of $\overline{\varphi}_{s^i\lambda}(z)$ in (5.9) has the factors $f_k(z) - g_k(z)$, if we put $z = \zeta_{i+\ell+1}$, then

$$\overline{\varphi}_{s^i\lambda}(\zeta_{i+\ell+1}) = \left[\sum_{k=1}^{i+\ell+1} (-1)^{k+1} \frac{f_k(z) - g_k(z)}{z_k^{n+1}} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k) A^{(n-1)}(\widehat{z}_k)\right]_{z=\zeta_{i+\ell+1}}.$$

Thus, it follows that

$$\left[\left(z^{s^i\lambda} \prod_{m=1}^{i+\ell} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell+1}} = S_1(\zeta_{i+\ell+1}) + S_2(\zeta_{i+\ell+1}) + S_3(\zeta_{i+\ell+1})$$

where $S_1(z)$, $S_2(z)$ and $S_3(z)$ are functions defined by the following:

$$\begin{aligned} S_1(z) := {} & \sum_{k=1}^{i} (-1)^{k+1} \frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\big(f_k(z) - g_k(z)\big) \\ & \times \big(\underbrace{z_1^s z_2^s \ldots z_{k-1}^s z_{k+1}^s \ldots z_i^s}_{i-1} z_{i+1}^{\lambda_1} \ldots z_{i+\ell}^{\lambda_\ell} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)\big) \\ & \times \big(z_1^{n-1} z_2^{n-2} \ldots z_{k-1}^{n-k+1} z_{k+1}^{n-k} \ldots z_{i+\ell}^{n-i-\ell+1} A^{(n-1)}(\widehat{z}_k)\big), \end{aligned} \tag{5.28}$$

$$\begin{aligned} S_2(z) := {} & \sum_{k=1}^{\ell} (-1)^{i+k+1} \frac{z_1}{z_{i+k}}\frac{z_2}{z_{i+k}}\cdots\frac{z_{i+k-1}}{z_{i+k}}\big(f_{i+k}(z) - g_{i+k}(z)\big)\big(z^{s^i\lambda} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+k})\big) \\ & \times \big(z_1^{n-1} z_2^{n-2} \ldots z_{i+k-1}^{n-i-k+1} z_{i+k+1}^{n-i-k} \ldots z_{i+\ell}^{n-i-\ell+1} A^{(n-1)}(\widehat{z}_{i+k})\big), \end{aligned} \tag{5.29}$$

$$\begin{aligned} S_3(z) := {} & (-1)^{i+\ell}\left(z_1 z_2 \ldots z_{i+\ell} \frac{f_{i+\ell+1}(z) - g_{i+\ell+1}(z)}{z_{i+\ell+1}^{n+s}}\right)\big(z^{s^i\lambda} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+\ell+1})\big) \\ & \times \big(z_1^{n-1} z_2^{n-2} \ldots z_{i+\ell}^{n-i-\ell} A^{(n-1)}(\widehat{z}_{i+\ell+1})\big). \end{aligned} \tag{5.30}$$

We now prove that $\lim_{x\to 0} S_2(\zeta_{i+\ell+1}) = \lim_{x\to 0} S_3(\zeta_{i+\ell+1}) = 0$. We show $\lim_{x\to 0} S_3(\zeta_{i+\ell+1}) = 0$ first. Since $f_{i+\ell+1}(\zeta_{i+\ell+1}) = 0$ by Lemma 5.3, it follows that

$$\left[\left(z_1 z_2 \ldots z_{i+\ell} \frac{f_{i+\ell+1}(z) - g_{i+\ell+1}(z)}{z_{i+\ell+1}^{n+s}}\right)\right]_{z=\zeta_{i+\ell+1}} = -\left[z_1 z_2 \ldots z_{i+\ell} \frac{g_{i+\ell+1}(z)}{z_{i+\ell+1}^{n+s}}\right]_{z=\zeta_{i+\ell+1}}. \tag{5.31}$$

From (5.6) in Lemma 5.3, the above factor in $S_3(\zeta_{i+\ell+1})$ is a constant if we take the limit $x \to 0$. Since $s^i\lambda > s^{i-1}\lambda$, we have

$$\lim_{x\to 0}\big[z^{s^i\lambda} e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+\ell+1})\big]_{z=\zeta_{i+\ell+1}} = 0. \tag{5.32}$$

Moreover, if $x \to 0$, the other factors in $S_3(\zeta_{i+\ell+1})$ are the following:

$$\begin{aligned} & \lim_{x\to 0}\big[z_1^{n-1} z_2^{n-2} \ldots z_{i+\ell}^{n-i-\ell} A^{(n-1)}(\widehat{z}_{i+\ell+1})\big]_{z=\zeta_{i+\ell+1}} \\ & \quad = (-1)^{i+\ell} A^{(n-i-\ell-1)}\big(t^{n-i-\ell-2}a_1, \ldots, a_1\big). \end{aligned} \tag{5.33}$$

Combining (5.30)–(5.33), we obtain $\lim_{x\to 0} S_3(\zeta_{i+\ell+1}) = 0$.

Next we show $\lim_{x\to 0} S_2(\zeta_{i+\ell+1}) = 0$. From (5.7) in Lemma 5.3, the factor

$$\left[\frac{z_1}{z_{i+k}}\frac{z_2}{z_{i+k}}\cdots\frac{z_{i+k-1}}{z_{i+k}}\big(f_{i+k}(z)-g_{i+k}(z)\big)\right]_{z=\zeta_{i+\ell+1}} \tag{5.34}$$

in $S_2(\zeta_{i+\ell+1})$ is a constant if we take the limit $x \to 0$. Since $s^i\lambda > s^{i-1}\lambda$, we have

$$\lim_{x\to 0}\big[z^{s^i\lambda}\, e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_{i+k})\big]_{z=\zeta_{i+\ell+1}} = 0 \tag{5.35}$$

for $1 \leqslant k \leqslant \ell$. If $x \to 0$, the other factors in $S_2(\zeta_{i+\ell+1})$ are the following:

$$\begin{aligned}&\lim_{x\to 0}\big[z_1^{n-1}z_2^{n-2}\cdots z_{i+k-1}^{n-i-k+1}z_{i+k+1}^{n-i-k}\cdots z_{i+\ell}^{n-i-\ell+1}\,A^{(n-1)}(\widehat{z}_{i+k})\big]_{z=\zeta_{i+\ell+1}}\\ &\quad= (-1)^{i+\ell-1}A^{(n-i-\ell)}\big(t^{n-i-\ell-1}a_1,\dots,a_1\big).\end{aligned} \tag{5.36}$$

Combining (5.29), (5.34), (5.35) and (5.36), we obtain $\lim_{x\to 0} S_2(\zeta_{i+\ell+1}) = 0$. Therefore,

$$\lim_{x\to 0}\left[\left(z^{s^i\lambda}\prod_{m=1}^{i+\ell} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell+1}} = \lim_{x\to 0} S_1(\zeta_{i+\ell+1}). \tag{5.37}$$

Finally, we evaluate $\lim_{x\to 0} S_1(\zeta_{i+\ell+1})$. From (4.8) and definition (4.2) of $e^{(n)}_\mu(z)$, if $k \leqslant i$, we have

$$\lim_{x\to 0}\big[\underbrace{z_1^s z_2^s\cdots z_{k-1}^s z_{k+1}^s\cdots z_i^s}_{i-1}\underbrace{z_{i+1}^{\lambda_1}\cdots z_{i+\ell}^{\lambda_\ell}}_{\ell}\, e^{(n-1)}_{s^{i-1}\lambda}(\widehat{z}_k)\big]_{z=\zeta_{i+\ell+1}} = 1. \tag{5.38}$$

We also have

$$\begin{aligned}&\lim_{x\to 0}\big[z_1^{n-1}z_2^{n-2}\cdots z_{k-1}^{n-k+1}z_{k+1}^{n-k}\cdots z_{i+\ell}^{n-i-\ell+1}\,A^{(n-1)}(\widehat{z}_k)\big]_{z=\zeta_{i+\ell+1}}\\ &\quad= (-1)^{i+\ell-1}A^{(n-i-\ell)}\big(t^{n-i-\ell-1}a_1,\dots,a_1\big)\end{aligned} \tag{5.39}$$

by using (4.8) and Weyl's denominator formula (2.2). Using (5.28), (5.37), (5.38) and (5.39), we therefore obtain

$$\begin{aligned}&\lim_{x\to 0}\left[\left(z^{s^i\lambda}\prod_{m=1}^{i+\ell} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell+1}}\\ &\quad= \lim_{x\to 0} S_1(\zeta_{i+\ell+1})\\ &\quad= (-1)^{i+\ell-1}A^{(n-i-\ell)}\big(t^{n-i-\ell-1}a_1,\dots,a_1\big)\\ &\qquad\times\sum_{k=1}^{i}(-1)^{k+1}\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\big(f_k(z)-g_k(z)\big)\right]_{z=\zeta_{i+\ell+1}}.\end{aligned} \tag{5.40}$$

On the other hand, by Lemma 5.5, it follows that

$$\lim_{x\to 0}\left[\left(z^{s^i\lambda}\prod_{m=1}^{i+\ell} z_m^{n-m+1}\right)\overline{\varphi}_{s^i\lambda}(z)\right]_{z=\zeta_{i+\ell+1}} = (-1)^{i+\ell} c_{s^i\lambda,s^i\lambda} A^{(n-i-\ell)}\big(t^{n-i-\ell+1}a_1,\dots,a_1\big). \tag{5.41}$$

Comparing (5.40) with (5.41), and using (5.7) in Lemma 5.3, we obtain

$$\begin{aligned} c_{s^i\lambda,s^i\lambda} &= -\sum_{k=1}^{i}(-1)^{k+1}\lim_{x\to 0}\left[\frac{z_1}{z_k}\frac{z_2}{z_k}\cdots\frac{z_{k-1}}{z_k}\big(f_k(z)-g_k(z)\big)\right]_{z=\zeta_{i+\ell+1}} \\ &= \sum_{k=1}^{i}\left(t^{k-1}-t^{2n-k-1}\prod_{m=1}^{2s+2}a_m\right) = \frac{1-t^i}{1-t}\left(1-t^{2n-i-1}\prod_{m=1}^{2s+2}a_m\right), \end{aligned}$$

which completes the proof. □

For $\lambda\in B_\ell$, $\ell=0,1,2,\dots,n$, we set

$$\partial F^\lambda := \big\{\lambda, s\lambda, s^2\lambda,\dots,s^{n-\ell}\lambda\big\} \quad\text{and}\quad F^\lambda := \bigcup_{\substack{\mu\prec\lambda\\ \mu\in B}}\partial F^\mu. \tag{5.42}$$

Then we have

$$F^\lambda\subset F^\mu \quad \text{if } \lambda\prec\mu.$$

Moreover, if we set

$$F_i^\lambda := F^\lambda\cap\left(\bigcup_{i\leqslant j\leqslant n} L_j\right),$$

then

$$F^\lambda = F_0^\lambda = F_1^\lambda = \cdots = F_\ell^\lambda \supset F_{\ell+1}^\lambda \supset F_{\ell+2}^\lambda \supset\cdots\supset F_n^\lambda. \tag{5.43}$$

Lemma 5.9. *Let $\lambda\in B_\ell$. There exists a relation between $e_{s^i\lambda}$ and $e_{s^{i-1}\lambda}$ such that*

$$\langle e_{s^i\lambda}\rangle - C_{i\ell}\,\langle e_{s^{i-1}\lambda}\rangle \in \bigoplus_{\mu\in F_{i+\ell-1}^\lambda}\mathbb{C}\langle e_\mu\rangle \tag{5.44}$$

where the coefficient $C_{i\ell}$ is evaluated as

$$C_{i\ell} = -\frac{(1-t^{n-i-\ell+1})t^{i+\ell-1}}{(1-t^i)t^{s(n-i-\ell)}a_1^s}\,\frac{\prod_{k=2}^{2s+2}(1-t^{n-i-\ell}a_1a_k)}{(1-t^{2n-i-1}\prod_{m=1}^{2s+2}a_m)}. \tag{5.45}$$

Proof. From Lemmas 5.6 and 5.7, for $\lambda \in B_\ell$, the function $\overline{\varphi}_{s^i\lambda}(z)$ is expanded as

$$\overline{\varphi}_{s^i\lambda}(z) = c_{s^i\lambda,s^i\lambda} e_{s^i\lambda}(z) A^{(n)}(z) + c_{s^i\lambda,s^{i-1}\lambda} e_{s^{i-1}\lambda}(z) A^{(n)}(z) + \sum_{\mu \in F^\lambda_{i+\ell-1}} c_{s^i\lambda,\mu} e_\mu(z) A^{(n)}(z).$$

Since $A_\rho(z) = A^{(n)}(z)$, from (3.6) and (3.7), it follows that

$$\langle e_\mu \rangle = (-1)^n \int_{\Lambda_\xi} e_\mu(z) \Phi(z) A^{(n)}(z) \varpi_q. \tag{5.46}$$

Since $\int_{\Lambda_\xi} \overline{\varphi}_{s^i\lambda}(z)\Phi(z)\varpi_q = 0$ by Proposition 3.3, using (5.46), we have

$$c_{s^i\lambda,s^i\lambda}\langle e_{s^i\lambda}\rangle + c_{s^i\lambda,s^{i-1}\lambda}\langle e_{s^{i-1}\lambda}\rangle + \sum_{\mu \in F^\lambda_{i+\ell-1}} c_{s^i\lambda,\mu}\langle e_\mu\rangle = 0.$$

If we set

$$C_{i\ell} := -\frac{c_{s^i\lambda,s^{i-1}\lambda}}{c_{s^i\lambda,s^i\lambda}},$$

we obtain the expression (5.44). The evaluation of the constant $C_{i\ell}$ is given by Lemmas 5.7 and 5.8. □

Lemma 5.10. *If $\lambda \in B_\ell$, then*

$$\langle e_{s^i\lambda}\rangle - \tilde{C}_{i\ell}\langle e_\lambda\rangle \in \bigoplus_{\mu \in F^\lambda} \mathbb{C}\langle e_\mu\rangle \quad \textit{where } \tilde{C}_{i\ell} := \prod_{k=1}^{i} C_{k\ell}.$$

Proof. This is straightforward from repeated use of Lemma 5.9 and (5.43). □

Lemma 5.11. *The following holds for $\lambda \in B_\ell$ and $1 \leqslant i \leqslant n-\ell$:*

$$\langle e_{s^i\lambda}\rangle - \tilde{C}_{i\ell}\langle e_\lambda\rangle \in \bigoplus_{\substack{\mu \prec \lambda \\ \mu \in B}} \mathbb{C}\langle e_\mu\rangle. \tag{5.47}$$

Proof. We show (5.47) by induction on $\lambda \in B$ with ordering $\prec$. For $\eta \prec \lambda$, we assume that

$$\langle e_{s^j\eta}\rangle - \tilde{C}_{j\ell'}\langle e_\eta\rangle \in \bigoplus_{\substack{\nu \prec \eta \\ \nu \in B}} \mathbb{C}\langle e_\nu\rangle \quad \text{if } \eta \in B_{\ell'}. \tag{5.48}$$

By definition (5.42) of F^λ, if $\mu \in F^\lambda$, there exists $\eta \in B$ such that $\eta \prec \lambda$ and $\mu \in \partial F^\eta$. Then μ is written as $\mu = s^j\eta$. By the inductive hypothesis (5.48), we have

$$\langle e_\mu\rangle - \tilde{C}_{j\ell'}\langle e_\eta\rangle \in \bigoplus_{\substack{\nu \prec \eta \\ \nu \in B}} \mathbb{C}\langle e_\nu\rangle \quad \text{if } \eta \in B_{\ell'}.$$

Hence

$$\langle e_\mu\rangle \in \bigoplus_{\substack{\nu\preceq\eta\\ \nu\in B}} \mathbb{C}\langle e_\nu\rangle \subset \bigoplus_{\substack{\nu\prec\lambda\\ \nu\in B}} \mathbb{C}\langle e_\nu\rangle. \tag{5.49}$$

Combining Lemma 5.10 and (5.49), we have (5.47). □

Proof of Proposition 5.1. In Lemma 5.11, we consider the case where $i = n-\ell$. We put $K_\ell = \tilde{C}_{n-\ell,\ell}$. Since $\tilde{C}_{n-\ell,\ell} = \prod_{i=1}^{n-\ell} C_{i\ell}$, using (5.45), we have the expression

$$K_\ell = \prod_{i=1}^{n-\ell}\left(-\frac{(1-t^{n-i-\ell+1})t^{i+\ell-1}}{(1-t^i)t^{s(n-i-\ell)}a_1^s}\,\frac{\prod_{k=2}^{2s+2}(1-t^{n-i-\ell}a_1a_k)}{(1-t^{2n-i-1}\prod_{m=1}^{2s+2}a_m)}\right).$$

Since

$$\prod_{i=1}^{n-\ell}\frac{(1-t^{n-i-\ell+1})}{(1-t^i)} = 1 \quad \text{and} \quad \prod_{i=1}^{n-\ell}\left(1-t^{2n-i-1}\prod_{m=1}^{2s+2}a_m\right) = \prod_{j=1}^{n-\ell}\left(1-t^{n+j+\ell-2}\prod_{m=1}^{2s+2}a_m\right),$$

we obtain (5.2). □

6. Proof of Theorem 3.6

Since the parameters $a_1, a_2, \ldots, a_{2s+2}$ can be replaced symmetrically in Theorem 3.6, it is sufficient to prove it only for $\det Y_{a_1}$:

Theorem 6.1. *The determinant of the matrix Y_{a_1} is evaluated as*

$$\det Y_{a_1} = (-a_1)^{-s\binom{s+n-1}{s}} \prod_{j=1}^{n}\left[\frac{\prod_{k=2}^{2s+2}(1-t^{n-j}a_1a_k)}{(1-t^{n+j-2}\prod_{m=1}^{2s+2}a_m)}\right]^{\binom{s+j-2}{j-1}}.$$

Proof. If we set the following vectors which consist of $e_\lambda(z)$, $\lambda\in B$, with ordering $<$:

$$\vec{e}_< := \big(e_\lambda(z)\big)_{0\leqslant\lambda\leqslant(s-1)^n} \quad \text{and} \quad \langle\vec{e}_<\rangle := \big(\langle e_\lambda\rangle\big)_{0\leqslant\lambda\leqslant(s-1)^n},$$

then using the expression (4.7) we have

$$\vec{e}_< = \vec{\chi}E \quad \text{and} \quad \det E = 1 \tag{6.1}$$

where E forms an upper triangular matrix whose elements are $E_{\lambda\mu}$. Let $T_{a_1}^{-1}$ be the operator which represents the q-shift $a_1\to q^{-1}a_1$. From (6.1), setting

$$\vec{e}_<^{\,-} := \big(T_{a_1}^{-1}e_\lambda(z)\big)_{0\leqslant\lambda\leqslant(s-1)^n}, \qquad \langle\vec{e}_<^{\,-}\rangle := \big(\langle T_{a_1}^{-1}e_\lambda\rangle\big)_{0\leqslant\lambda\leqslant(s-1)^n},$$

it follows that

$$\vec{e}_<^{\,-} = \vec{\chi}\big(T_{a_1}^{-1}E\big),$$

so that

$$T_{a_1}\langle\vec{e}^{\,-}_{<}\rangle = \big(T_{a_1}\langle\vec{\chi}\rangle\big)E. \tag{6.2}$$

On the other hand, since

$$\begin{aligned}T_{a_1}\langle\varphi\rangle &= \int_{\Lambda_\xi} T_{a_1}\varphi(z)T_{a_1}\Phi(z)\Delta(z)\varpi_q = \int_{\Lambda_\xi} \frac{T_{a_1}\Phi(z)}{\Phi(z)}T_{a_1}\varphi(z)\Phi(z)\Delta(z)\varpi_q \\ &= \int_{\Lambda_\xi} e_n(z)T_{a_1}\varphi(z)\Phi(z)\Delta(z)\varpi_q \quad \big(\text{by using (4.4)}\big) \\ &= \langle e_nT_{a_1}\varphi\rangle,\end{aligned}$$

from the definition (4.5) of $e_\lambda(z)$ we have

$$T_{a_1}\langle\vec{e}^{\,-}_{<}\rangle = \big(\langle e_ne_\lambda\rangle\big)_{0\leqslant\lambda\leqslant(s-1)^n} = \big(\langle e_{(1^n)+\lambda}\rangle\big)_{0\leqslant\lambda\leqslant(s-1)^n} = \big(\langle e_\lambda\rangle\big)_{\lambda\in L_n}. \tag{6.3}$$

Set the following vectors which consist of $e_\lambda(z)$, $\lambda\in B$, with ordering $\prec$:

$$\vec{e}_\prec := \big(e_\lambda(z)\big)_{1^n\preceq\lambda\preceq 0} \quad\text{and}\quad \langle\vec{e}_\prec\rangle := \big(\langle e_\lambda\rangle\big)_{1^n\preceq\lambda\preceq 0}.$$

As a consequence of Proposition 5.1, it follows that

$$\big(\langle e_\lambda\rangle\big)_{\lambda\in L_n} = \langle\vec{e}_\prec\rangle Y \tag{6.4}$$

where Y is an upper triangular matrix of size κ. The diagonal entries of the matrix Y consist of $K_{n-\ell}$ with multiplicities $\binom{s+n-\ell-2}{n-\ell}$, $\ell=0,1,\dots,n$, where K_ℓ is defined in (5.2). By definition, the following holds between $\vec{e}_<$ and $\vec{e}_\prec$:

$$\vec{e}_\prec = \vec{e}_< P \tag{6.5}$$

where P is a matrix of size κ, which represents permutation of $e_\lambda(z)$'s. Combining (6.2)–(6.5), we have a system of q-difference equations as follows:

$$T_{a_1}\langle\vec{\chi}\rangle = \langle\vec{\chi}\rangle Y_{a_1}$$

where Y_{a_1} is the matrix of size κ satisfying

$$Y_{a_1} = EPYE^{-1},$$

so that

$$\det Y_{a_1} = \det P\,\det Y. \tag{6.6}$$

From (6.6), in order to prove Theorem 6.1, it is sufficient to show the following two propositions. □

Proposition 6.2. *The determinant of the matrix P is given by*

$$\det P = (-1)^{(s-1)\binom{s+n-1}{s}}.$$

Proof. See Appendix C. □

Proposition 6.3. *The determinant of the matrix Y is evaluated as*

$$\det Y = \left(-a_1^{-s}\right)^{\binom{s+n-1}{s}} \prod_{j=1}^{n} \left[\frac{\prod_{k=2}^{2s+2}(1-t^{n-j}a_1a_k)}{(1-t^{n+j-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+j-2}{j-1}}. \tag{6.7}$$

Proof. From the diagonal entries of the triangular matrix Y and (5.2) in Proposition 5.1, it follows that

$$\det Y = \prod_{\ell=0}^{n} K_\ell^{\binom{s+\ell-2}{\ell}} = \prod_{\ell=0}^{n}\prod_{i=1}^{n-\ell} \left[\left(-a_1^{-s}\right) \frac{t^{i+\ell-1}}{t^{s(n-i-\ell)}} \frac{\prod_{k=2}^{2s+2}(1-t^{n-i-\ell}a_1a_k)}{(1-t^{n+i+\ell-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+\ell-2}{\ell}}$$
$$= \prod_{(i,\ell)\in I} \left[\left(-a_1^{-s}\right) \frac{t^{i+\ell-1}}{t^{s(n-i-\ell)}} \frac{\prod_{k=2}^{2s+2}(1-t^{n-i-\ell}a_1a_k)}{(1-t^{n+i+\ell-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+\ell-2}{\ell}}$$

where the set I is defined as

$$I := \left\{(i,\ell)\in \mathbb{Z}^2;\ 1\leqslant i\leqslant n-\ell,\ 0\leqslant \ell\leqslant n\right\}.$$

We put $j=i+\ell$. If (i,ℓ) runs over I, then (j,ℓ) runs over the set $I' := \{(j,\ell)\in\mathbb{Z}^2;\ 1\leqslant j\leqslant n, 0\leqslant \ell\leqslant j-1\}$. Using (2.3) we have the following:

$$\det Y = \prod_{(j,\ell)\in I'} \left[\left(-a_1^{-s}\right) \frac{t^{j-1}}{t^{s(n-j)}} \frac{\prod_{k=2}^{2s+2}(1-t^{n-j}a_1a_k)}{(1-t^{n+j-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+\ell-2}{\ell}}$$
$$= \prod_{j=1}^{n}\prod_{\ell=0}^{j-1} \left[\left(-a_1^{-s}\right) \frac{t^{j-1}}{t^{s(n-j)}} \frac{\prod_{k=2}^{2s+2}(1-t^{n-j}a_1a_k)}{(1-t^{n+j-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+\ell-2}{\ell}}$$
$$= \left(-a_1^{-s}\right)^{\sum_{j=1}^{n}\binom{s+j-2}{j-1}} t^{\sum_{j=1}^{n}(j-1-s(n-j))\binom{s+j-2}{j-1}} \prod_{j=1}^{n} \left[\frac{\prod_{k=2}^{2s+2}(1-t^{n-j}a_1a_k)}{(1-t^{n+j-2}\prod_{m=1}^{2s+2}a_m)} \right]^{\binom{s+j-2}{j-1}}.$$

To prove (6.7) it is sufficient to show that

$$\sum_{j=1}^{n} \binom{s+j-2}{j-1} = \binom{s+n-1}{n-1} = \binom{s+n-1}{s} \tag{6.8}$$

and

$$\sum_{j=1}^{n}\bigl(j-1-s(n-j)\bigr)\binom{s+j-2}{j-1}=0. \tag{6.9}$$

For (6.8), it is obvious from (2.3). It is also easy to deduce (6.9) from (6.8) by induction on n and is left to the reader. □

7. Asymptotic behavior of the truncated Jackson integrals

Let $A'_{(i_1,i_2,\dots,i_n)}(z)$ be the function of $z=(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$ defined in the form of the following determinant:

$$A'_{(i_1,i_2,\dots,i_n)}(z):=\det\bigl(z_j^{i_k}\bigr)_{1\leqslant j,k\leqslant n}=\begin{vmatrix} z_1^{i_1} & z_2^{i_1} & \cdots & z_n^{i_1}\\ z_1^{i_2} & z_2^{i_2} & \cdots & z_n^{i_2}\\ \vdots & \vdots & \ddots & \vdots\\ z_1^{i_n} & z_2^{i_n} & \cdots & z_n^{i_n}\end{vmatrix},$$

where $(i_1,i_2,\dots,i_n)\in\mathbb{Z}^n$. For $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_n)\in\mathcal{P}$, we define the *Schur function* $S_\lambda(z)$ as follows:

$$S_\lambda(z):=\frac{A'_{\lambda+\rho'}(z)}{A'_{\rho'}(z)}=\frac{A'_{(\lambda_1+n-1,\lambda_2+n-2,\dots,\lambda_{n-1}+1,\lambda_n+0)}(z)}{A'_{(n-1,n-2,\dots,1,0)}(z)}$$

where $\rho'=(n-1,n-2,\dots,1,0)$. We see

$$\chi_\lambda(z)\sim S_\lambda\bigl(z^{-1}\bigr)\quad \text{if } (z_1,z_2,\dots,z_n)\to(0,0,\dots,0) \tag{7.1}$$

where $z^{-1}:=(z_1^{-1},z_2^{-1},\dots,z_n^{-1})$ for $z=(z_1,z_2,\dots,z_n)\in(\mathbb{C}^*)^n$.

For $S_\lambda(\zeta_{(\mu)})$ where $\zeta_{(\mu)}$ is defined in (3.10), we state a Vandermonde type determinant of matrix formed by $S_\lambda(\zeta_{(\mu)})$, which is the same as Proposition 1.6.

Proposition 7.1. *The $\kappa\times\kappa$ determinant with (λ,μ) entry $S_\lambda(\zeta_{(\mu)})$ is evaluated as*

$$\det\bigl(S_\lambda(\zeta_{(\mu)})\bigr)_{\lambda,\mu}=\prod_{k=1}^{n}\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\bigl(t^r a_i-t^{n-k-r}a_j\bigr)^{\binom{s+k-3}{k-1}} \tag{7.2}$$

where the rows $\lambda\in B$ and the columns $\mu\in Z$ of the matrix $(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ are arranged in the orders $<$ and $\prec_Z$, respectively.

Proof. See Appendix D. □

We now give the asymptotic behavior of $J := \det(\langle \chi_\lambda, \zeta_{(\mu)}\rangle)_{\lambda,\mu}$ in a specific direction as indicated in Proposition 7.2. Let T^N be the q-shift operator of the parameters a_k for the specific direction defined by

$$T^N : \begin{cases} a_i \to a_i q^{(s+1)N} & \text{if } 1 \leqslant i \leqslant s, \\ a_j \to a_j q^{-sN} & \text{if } s+1 \leqslant j \leqslant 2s+2, \end{cases}$$

which keeps the parameters within the domain (3.9) of convergence for sufficiently large N. We divide $\Phi(z)\Delta(z)$ into the following three parts:

$$\Phi(z)\Delta(z) = I_1(z)I_2(z)I_3(z) \tag{7.3}$$

where

$$I_1(z) = \prod_{i=1}^n z_i^{s-\alpha_1-\cdots-\alpha_{2s+2}-2(n-i)\tau}, \tag{7.4}$$

$$I_2(z) = \prod_{i=1}^n \prod_{\ell=1}^s \big(qa_\ell^{-1} z_i\big)_\infty \prod_{1\leqslant j<k\leqslant n} (1 - z_j/z_k) \frac{(qt^{-1}z_j/z_k)_\infty}{(tz_j/z_k)_\infty}, \tag{7.5}$$

and

$$I_3(z) = \prod_{i=1}^n \left[\frac{1-z_i^2}{\prod_{\ell=1}^s (a_\ell z_i)_\infty} \prod_{\ell=s+1}^{2s+2} \frac{(qa_\ell^{-1}z_i)_\infty}{(a_\ell z_i)_\infty} \right] \prod_{1\leqslant j<k\leqslant n} (1 - z_j z_k) \frac{(qt^{-1}z_j z_k)_\infty}{(tz_j z_k)_\infty}.$$

Proposition 7.2. *The asymptotic behavior of $T^N J$ as $N \to +\infty$ is expressed as*

$$T^N J \sim F(N)(1-q)^{n\binom{s+n-1}{n}} \det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu} \prod_{\mu\in Z} I_1(\zeta_{(\mu)}) I_2(\zeta_{(\mu)}) \tag{7.6}$$

where the factor $F(N)$ depending on N is

$$F(N) = q^{\binom{s+1}{2}\binom{s+n-1}{n-1}(s+1+2sN)N} t^{-s(s^2+s+1)\binom{s+n-1}{n-2}N} \times \left[\frac{1}{(a_1a_2\dots a_s)^s\,(a_{s+1}a_{s+2}\dots a_{2s+2})^{s+1}} \right]^{s\binom{s+n-1}{s}N}. \tag{7.7}$$

The rest of this section is devoted to proving Proposition 7.2.

Lemma 7.3. *The following holds for Z:*

$$\prod_{\mu\in Z} \zeta_{(\mu)1}\zeta_{(\mu)2}\dots\zeta_{(\mu)n} = (a_1a_2\dots a_s)^{\binom{s+n-1}{n-1}} t^{s\binom{s+n-1}{n-2}}.$$

Proof. By definition (3.10) of $\zeta_{(\mu)}$, for s-tuple $\mu=(\mu_1,\mu_2,\dots,\mu_s)$ we have

$$\zeta_{(\mu)1}\zeta_{(\mu)2}\cdots\zeta_{(\mu)n}=\prod_{k=1}^{s}a_k^{\mu_k}t^{\mu_k(\mu_k-1)/2},$$

so that

$$\prod_{\mu\in Z}\zeta_{(\mu)1}\zeta_{(\mu)2}\cdots\zeta_{(\mu)n}=\prod_{\mu\in Z}\prod_{k=1}^{s}a_k^{\mu_k}t^{\mu_k(\mu_k-1)/2}=\prod_{k=1}^{s}\prod_{i=0}^{n}a_k^{(n-i)\binom{s+i-2}{i}}t^{\frac{(n-i)(n-i-1)}{2}\binom{s+i-2}{i}}$$

because the number of s-tuples μ satisfying $\mu_k=n-i$ is $\binom{s+i-2}{i}$ for $i=0,1,\dots,n$. Thus we obtain Lemma 7.3 using the following lemma with $j=1$ and 2. □

Lemma 7.4. *The following holds for $j=0,1,2,\dots,n$:*

$$\sum_{i=0}^{n}\binom{n-i}{j}\binom{s+i-2}{i}=\binom{s+n-1}{n-j}. \tag{7.8}$$

Proof. This identity can be proved by induction on n. □

Lemma 7.5. *The asymptotic behavior of $T^N\chi_\lambda(\zeta_{(\mu)})$ as $N\to+\infty$ is expressed as*

$$T^N\chi_\lambda(\zeta_{(\mu)})\sim T^N S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\quad(N\to+\infty) \tag{7.9}$$

for $\zeta_{(\mu)}^{-1}=(\zeta_{(\mu)1}^{-1},\zeta_{(\mu)2}^{-1},\dots,\zeta_{(\mu)n}^{-1})$. Moreover,

$$\begin{aligned}T^N\det\big(\chi_\lambda(\zeta_{(\mu)})\big)_{\lambda,\mu}&\sim T^N\det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu}\quad(N\to+\infty)\\&=q^{-(s+1)\binom{s}{2}\binom{s+n-1}{s}N}\times\det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu},\end{aligned} \tag{7.10}$$

where the explicit form of $\det(S_\lambda(\zeta_{(\mu)}^{-1}))_{\lambda,\mu}$ *is*

$$\det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu}=\prod_{k=1}^{n}\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\left(\frac{1-t^{2r-(n-k)}a_i/a_j}{t^r a_i}\right)^{\binom{s+k-3}{k-1}}. \tag{7.11}$$

Proof. From (7.1) we have (7.9). Proposition 7.1 implies (7.11), so that we have

$$\begin{aligned}T^N\det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu}&=\prod_{k=1}^{n}\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\left(\frac{1-t^{2r-(n-k)}a_i/a_j}{q^{(s+1)N}t^r a_i}\right)^{\binom{s+k-3}{k-1}}\\&=q^{-(s+1)\binom{s}{2}\binom{s+n-1}{s}N}\times\det\big(S_\lambda\big(\zeta_{(\mu)}^{-1}\big)\big)_{\lambda,\mu}.\end{aligned}$$ □

Proof of Proposition 7.2. From (7.3), $T^N\langle\chi_\lambda,\zeta_{(\mu)}\rangle$ is expressed as

$$T^N\langle\chi_\lambda,\zeta_{(\mu)}\rangle=(1-q)^n\sum_{\nu\in D_\mu}T^N\chi_\lambda\big(q^\nu\zeta_{(\mu)}\big)T^NI_1\big(q^\nu\zeta_{(\mu)}\big)\,T^NI_2\big(q^\nu\zeta_{(\mu)}\big)T^NI_3\big(q^\nu\zeta_{(\mu)}\big) \tag{7.12}$$

where

$$T^NI_1\big(q^\nu\zeta_{(\mu)}\big)=\prod_{i=1}^n\big(\zeta_{(\mu)i}q^{\nu_i+(s+1)N}\big)^{s-\alpha_1-\cdots-\alpha_{2s+2}-2(n-i)\tau+sN}, \tag{7.13}$$

$$\begin{aligned}T^NI_2\big(q^\nu\zeta_{(\mu)}\big)&=\prod_{i=1}^n\prod_{\ell=1}^s\big(q^{1+\nu_i}a_\ell^{-1}\zeta_{(\mu)i}\big)_\infty\\&\quad\times\prod_{1\leqslant j<k\leqslant n}\big(1-q^{\nu_j-\nu_k}\zeta_{(\mu)j}/\zeta_{(\mu)k}\big)\frac{(q^{1+\nu_j-\nu_k}t^{-1}\zeta_{(\mu)j}/\zeta_{(\mu)k})_\infty}{(q^{\nu_j-\nu_k}t\,\zeta_{(\mu)j}/\zeta_{(\mu)k})_\infty}\end{aligned} \tag{7.14}$$

and

$$\begin{aligned}&T^NI_3\big(q^\nu\zeta_{(\mu)}\big)\\&\quad=\prod_{i=1}^n\left[\frac{1-(\zeta_{(\mu)i})^2q^{\nu_i+2(s+1)N}}{\prod_{m=1}^s(a_m\zeta_{(\mu)i}\,q^{\nu_i+2(s+1)N})_\infty}\prod_{m=s+1}^{2s+2}\frac{(\zeta_{(\mu)i}\,a_m^{-1}q^{1+\nu_i+(2s+1)N})_\infty}{(\zeta_{(\mu)i}\,a_mq^{\nu_i+N})_\infty}\right]\\&\quad\quad\times\prod_{1\leqslant j<k\leqslant n}\left[\big(1-\zeta_{(\mu)j}\zeta_{(\mu)k}q^{\nu_j+\nu_k+2(s+1)N}\big)\frac{(t^{-1}\zeta_{(\mu)j}\zeta_{(\mu)k}q^{1+\nu_j+\nu_k+2(s+1)N})_\infty}{(t\,\zeta_{(\mu)j}\zeta_{(\mu)k}\,q^{\nu_j+\nu_k+2(s+1)N})_\infty}\right].\end{aligned} \tag{7.15}$$

Since $(x)_\infty\to1$ if $x\to0$, from (7.14) and (7.15), $|T^NI_2(q^\nu\zeta_{(\mu)})T^NI_3(q^\nu\zeta_{(\mu)})|$ is bounded for $N>0$ and $\nu\in D_\mu$. Then Eq. (7.12) indicates that the principal term of the asymptotic behavior of $T^N\langle\chi_\lambda,\zeta_{(\mu)}\rangle$ as $N\to+\infty$ depends on the maximum value of $|T^N\chi_\lambda(q^\nu\zeta_{(\mu)})T^NI_1(q^\nu\zeta_{(\mu)})|$ over $\nu\in D_\mu$. From (7.13) it follows that

$$\frac{\sum_{\nu\in D_\mu-\{0\}}|T^N\chi_\lambda(q^\nu\zeta_{(\mu)})T^NI_1(q^\nu\zeta_{(\mu)})|}{|T^N\chi_\lambda(\zeta_{(\mu)})T^NI_1(\zeta_{(\mu)})|}\to0\quad(N\to+\infty).$$

Since $|T^NI_2(\zeta_{(\mu)})T^NI_3(\zeta_{(\mu)})|\neq0$, the summand in the right-hand side of (7.12) corresponding to $\nu=(0,0,\dots,0)\in D_\mu$ gives the principal term of the asymptotic behavior of $T^N\langle\chi_\lambda,\zeta_{(\mu)}\rangle$ as $N\to+\infty$, so that we have

$$T^N\langle\chi_\lambda,\zeta_{(\mu)}\rangle\sim(1-q)^nT^N\chi_\lambda(\zeta_{(\mu)})T^NI_1(\zeta_{(\mu)})T^NI_2(\zeta_{(\mu)})T^NI_3(\zeta_{(\mu)}). \tag{7.16}$$

Moreover, the asymptotic behavior of each $T^NI_i(\zeta_{(\mu)})$ as $N\to+\infty$ is given by the following:

$$\begin{aligned}T^NI_1(\zeta_{(\mu)})&=\prod_{i=1}^n\big(\zeta_{(\mu)i}q^{(s+1)N}\big)^{s-\alpha_1-\cdots-\alpha_{2s+2}-2(n-i)\tau+sN}\\&=I_1(\zeta_{(\mu)})\times\frac{(\zeta_{(\mu)1}\zeta_{(\mu)2}\dots\zeta_{(\mu)n})^{sN}q^{ns(s+1)N(N+1)}}{(a_1a_2\dots a_{2s+2}t^{n-1})^{n(s+1)N}},\end{aligned} \tag{7.17}$$

$$T^NI_2(\zeta_{(\mu)})=I_2(\zeta_{(\mu)}), \tag{7.18}$$

and

$$T^N I_3(\zeta_{(\mu)}) = \prod_{i=1}^{n}\left[\frac{1-(\zeta_{(\mu)i})^2 q^{2(s+1)N}}{\prod_{m=1}^{s}(a_m\zeta_{(\mu)i}\, q^{2(s+1)N})_\infty}\prod_{m=s+1}^{2s+2}\frac{(\zeta_{(\mu)i}\, a_m^{-1}q^{1+(2s+1)N})_\infty}{(\zeta_{(\mu)i}\, a_m q^N)_\infty}\right]$$
$$\times \prod_{1\leqslant j<k\leqslant n}\big(1-\zeta_{(\mu)j}\zeta_{(\mu)k}\, q^{2(s+1)N}\big)\frac{(t^{-1}\zeta_{(\mu)j}\zeta_{(\mu)k}\, q^{1+2(s+1)N})_\infty}{(t\,\zeta_{(\mu)j}\zeta_{(\mu)k}\, q^{2(s+1)N})_\infty}$$
$$\sim 1 \quad (N\to+\infty). \tag{7.19}$$

Combining (7.16)–(7.19) and (7.10) in Lemma 7.5, we obtain (7.6). The explicit expression of $F(N)$ is given by (7.10), (7.17) and Lemma 7.3. □

8. Proof of the main theorem

In this section, we give the proof of Theorem 3.9.

The following is straightforward from the property $\theta(qx) = -\theta(x)/x$:

$$\theta\big(q^m x\big) = \frac{\theta(x)}{(-x)^m q^{m(m-1)/2}}, \qquad \theta\big(q^{-m} x\big) = \frac{(-x)^m\theta(x)}{q^{m(m+1)/2}} \tag{8.1}$$

for positive integers m.

Lemma 8.1. *Let f be a function of $a_1, a_2, \ldots, a_{2s+2}$. If f satisfies the functional equations*

$$T_{a_i} f = (-a_i)^{s\binom{s+n-1}{s}} f \quad \textit{for } i=1,2,\ldots,2s+2, \tag{8.2}$$

then the following holds for the shift T^N:

$$f = G(N)\times T^N f,$$

where

$$G(N) = (-1)^{s^2\binom{s+n-1}{s}N} q^{-s\binom{s+n-1}{s}[s\binom{sN+N}{2}+(s+2)\binom{sN+1}{2}]}$$
$$\times\left[\frac{(a_{s+1}a_{s+2}\ldots a_{2s+2})^s}{(a_1a_2\ldots a_s)^{s+1}}\right]^{s\binom{s+n-1}{s}N}. \tag{8.3}$$

In particular, if we set

$$C := \prod_{k=1}^{n}\left[\frac{(qt^{-(n-k+1)})_\infty^s}{(qt^{-1})_\infty^s}\frac{\prod_{1\leqslant i<j\leqslant 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\ldots a_{2s+2}^{-1})_\infty}\right]^{\binom{s+k-2}{k-1}} \tag{8.4}$$

then the function J/C satisfies (8.2).

Proof. If we take $f=[\theta(a_1)\theta(a_2)\dots\theta(a_{2s+2})]^{-s\binom{s+n-1}{s}}$, then f satisfies (8.2). It is sufficient to compute $G(N)$ for this f. We apply (8.1) and obtain $f/T^N f=G(N)$.

Next we prove that J/C satisfies (8.2). From Theorem 3.6, the determinant J satisfies that

$$\frac{T_{a_i}J}{J}=(-a_i)^{-s\binom{s+n-1}{s}}\prod_{k=1}^{n}\left[\frac{\prod_{j=1}^{2s+2}(1-t^{n-k}a_ia_j)}{(1-t^{n-k}a_i^2)(1-t^{n+k-2}\prod_{m=1}^{2s+2}a_m)}\right]^{\binom{s+k-2}{k-1}}.$$

On the other hand, from a direct computation of (8.4), it follows that

$$\frac{T_{a_i}C}{C}=a_i^{-2s\binom{s+n-1}{s}}\prod_{k=1}^{n}\left[\frac{\prod_{j=1}^{2s+2}(1-t^{n-k}a_ia_j)}{(1-t^{n-k}a_i^2)(1-t^{n+k-2}\prod_{m=1}^{2s+2}a_m)}\right]^{\binom{s+k-2}{k-1}}.$$

Thus J/C satisfies (8.2). □

Lemma 8.2. *Let*

$$C'=\prod_{k=1}^{n}\left[\frac{(qt^{-(n-k+1)})_\infty^s}{(qt^{-1})_\infty^s}\prod_{1\leqslant i<j\leqslant s}\theta\left(t^{n-k}a_ia_j\right)\prod_{i=1}^{s}\prod_{j=s+1}^{2s+2}\theta\left(t^{n-k}a_ia_j\right)\right]^{\binom{s+k-2}{k-1}}.$$

Then the asymptotic behavior of T^NC as $N\to+\infty$ is

$$T^NC\sim T^NC'=H(N)C'$$

where

$$\begin{aligned}H(N)={}&(-1)^{s^2\binom{s+n-1}{s}N}q^{-\binom{s+n-1}{s}[s(s+2)\binom{N}{2}+\binom{2sN+2N}{2}\binom{s}{2}]}\\&\times t^{-s(s^2+s+1)\binom{s+n-1}{n-2}N}\left[\frac{1}{(a_1a_2\dots a_s)^{2s+1}(a_{s+1}a_{s+2}\dots a_{2s+2})}\right]^{s\binom{s+n-1}{s}N}.\end{aligned}\tag{8.5}$$

Proof. By definition, C/C' is equal to

$$\begin{aligned}&\prod_{k=1}^{n}\left[\frac{\prod_{s+1\leqslant i<j\leqslant 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1})_\infty\prod_{1\leqslant i<j\leqslant s}(t^{n-k}a_ia_j)_\infty}\right.\\&\quad\left.\times\frac{1}{\prod_{i=1}^{s}\prod_{j=s+1}^{2s+2}(t^{n-k}a_ia_j)_\infty}\right]^{\binom{s+k-2}{k-1}},\end{aligned}\tag{8.6}$$

so that $T^N(C/C')$ is equal to

$$\begin{aligned}&\prod_{k=1}^{n}\left[\frac{\prod_{s+1\leqslant i<j\leqslant 2s+2}(qt^{-(n-k)}a_i^{-1}a_j^{-1}q^{2sN})_\infty}{(qt^{-(n+k-2)}a_1^{-1}a_2^{-1}\dots a_{2s+2}^{-1}q^{sN})_\infty\prod_{1\leqslant i<j\leqslant s}(t^{n-k}a_ia_jq^{2(s+1)N})_\infty}\right.\\&\quad\left.\times\frac{1}{\prod_{i=1}^{s}\prod_{j=s+1}^{2s+2}(t^{n-k}a_ia_jq^N)_\infty}\right]^{\binom{s+k-2}{k-1}}.\end{aligned}$$

This implies $T^N(C/C') \to 1$ $(N \to +\infty)$. Thus we have $T^N C \sim T^N C'$. From definition of C', we have

$$\frac{T^N C'}{C'} = \prod_{k=1}^{n} \left[\prod_{1 \leqslant i<j \leqslant s} \frac{\theta(t^{n-k} a_i a_j q^{2(s+1)N})}{\theta(t^{n-k} a_i a_j)} \prod_{i=1}^{s} \prod_{j=s+1}^{2s+2} \frac{\theta(t^{n-k} a_i a_j q^N)}{\theta(t^{n-k} a_i a_j)} \right]^{\binom{s+k-2}{k-1}}.$$

Applying (8.1) to this, we obtain $T^N C'/C' = H(N)$. □

Lemma 8.3. *The relation between $F(N)$, $G(N)$ and $H(N)$ is the following:*

$$F(N)G(N) = H(N).$$

Proof. This is straightforward from (7.7), (8.3) and (8.5). □

Proposition 8.4. *The determinant J of the truncated Jackson integrals is expressed as*

$$J = (1-q)^{n\binom{s+n-1}{n}} \frac{C}{C'} \det\bigl(S_\lambda\bigl(\zeta_{(\mu)}^{-1}\bigr)\bigr)_{\lambda,\mu} \prod_{\mu \in Z} I_1(\zeta_{(\mu)}) I_2(\zeta_{(\mu)}), \tag{8.7}$$

where $I_1(z)$ and $I_2(z)$ are defined in (7.4) *and* (7.5), *respectively. More precisely J is equal to*

$$\begin{aligned}
&\bigl\{(1-q)(q)_\infty\bigr\}^{n\binom{s+n-1}{n}} \bigl(a_1^{s-1} a_2^{s-2} \ldots a_{s-1}\bigr)^{-\binom{s+n-1}{n-1}} t^{-\binom{s}{2}\binom{s+n-1}{n-2}} \prod_{\mu \in Z} I_1(\zeta_{(\mu)}) \\
&\quad \times \prod_{k=1}^{n} \left[\frac{(t)_\infty^s}{(t^{n-k+1})_\infty^s} \frac{\prod_{s+1 \leqslant i<j \leqslant 2s+2} (q t^{-(n-k)} a_i^{-1} a_j^{-1})_\infty}{(q t^{-(n+k-2)} a_1^{-1} a_2^{-1} \ldots a_{2s+2}^{-1})_\infty} \right. \\
&\quad \times \left. \frac{\prod_{1 \leqslant i<j \leqslant s} \theta(t^{-(n-k)} a_i a_j^{-1})}{\prod_{1 \leqslant i<j \leqslant s} (t^{n-k} a_i a_j)_\infty \prod_{i=1}^{s} \prod_{j=s+1}^{2s+2} (t^{n-k} a_i a_j)_\infty} \right]^{\binom{s+k-2}{k-1}}.
\end{aligned}$$

Proof. The former part of Proposition 8.4 is proved as follows:

$$\begin{aligned}
\frac{J}{C} &= G(N) \frac{T^N J}{T^N C} \quad \text{(from Lemma 8.1)} \\
&= G(N) \frac{F(N)}{H(N) C'} \\
&\quad \times (1-q)^{n\binom{s+n-1}{n}} \det\bigl(S_\lambda\bigl(\zeta_{(\mu)}^{-1}\bigr)\bigr)_{\lambda,\mu} \prod_{\mu \in Z} I_1(\zeta_{(\mu)}) I_2(\zeta_{(\mu)}) \\
&\quad \text{(from Proposition 7.2 and Lemma 8.2)} \\
&= \frac{1}{C'} (1-q)^{n\binom{s+n-1}{n}} \det\bigl(S_\lambda\bigl(\zeta_{(\mu)}^{-1}\bigr)\bigr)_{\lambda,\mu} \prod_{\mu \in Z} I_1(\zeta_{(\mu)}) I_2(\zeta_{(\mu)}) \quad \text{(from Lemma 8.3).}
\end{aligned}$$

For the explicit expression of J, we rewrite $I_2(z)$ as follows:

$$I_2(z) = I_2'(z) \prod_{1\leqslant j<k\leqslant n} \frac{\theta(z_j/z_k)}{\theta(tz_j/z_k)} \tag{8.8}$$

where

$$I_2'(z) = \prod_{i=1}^{n}\prod_{\ell=1}^{s}\big(qa_\ell^{-1}z_i\big)_\infty \prod_{1\leqslant j<k\leqslant n} \frac{(qt^{-1}z_j/z_k)_\infty}{(qz_j/z_k)_\infty}\frac{(qt^{-1}z_k/z_j)_\infty}{(qz_k/z_j)_\infty},$$

so that $\prod_{\mu\in Z} I_2(\zeta_{(\mu)})$ is the product of

$$\begin{aligned}\prod_{\mu\in Z} I_2'(\zeta_{(\mu)}) &= (q)_\infty^{n\binom{s+n-1}{n}} \prod_{k=1}^{n}\left[\frac{(qt^{-(n-k+1)})_\infty^s}{(qt^{-1})_\infty^s}\right]^{\binom{s+k-2}{k-1}} \\ &\quad \times \prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s} \frac{\theta(t^{2r-(n-k)}a_ia_j^{-1})}{1-t^{2r-(n-k)}a_ia_j^{-1}}\right]^{\binom{s+k-3}{k-1}}\end{aligned} \tag{8.9}$$

and

$$\begin{aligned}\prod_{\mu\in Z}\prod_{1\leqslant j<k\leqslant n} \frac{\theta(\zeta_{(\mu)j}/\zeta_{(\mu)k})}{\theta(t\zeta_{(\mu)j}/\zeta_{(\mu)k})} &= \prod_{k=1}^{n}\left[\frac{\theta(t)^s}{\theta(t^{n-k+1})^s}\prod_{1\leqslant i<j\leqslant s}\theta\big(t^{-(n-k)}a_ia_j^{-1}\big)\right]^{\binom{s+k-2}{k-1}} \\ &\quad \times \prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s} \frac{1}{\theta(t^{2r-(n-k)}a_ia_j^{-1})}\right]^{\binom{s+k-3}{k-1}}.\end{aligned} \tag{8.10}$$

From Lemma D.3 in Appendix D, it follows that

$$\prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s} t^r a_i\right]^{\binom{s+k-3}{k-1}} = \big(a_1^{s-1}a_2^{s-2}\dots a_{s-1}\big)^{\binom{s+n-1}{n-1}} t^{\binom{s}{2}\binom{s+n-1}{n-2}}. \tag{8.11}$$

Combining (7.11), (8.6)–(8.11), we obtain the explicit expression of J. □

Lemma 8.5. *The product $\prod_{\mu\in Z}\Theta(\zeta_{(\mu)})$ is evaluated as*

$$\begin{aligned}\prod_{\mu\in Z}\Theta(\zeta_{(\mu)}) &= \prod_{\mu\in Z} I_1(\zeta_{(\mu)}) \\ &\quad \times \prod_{k=1}^{n}\left[\frac{\theta(t)^s}{\theta(t^{n-k+1})^s}\frac{\prod_{1\leqslant i<j\leqslant s}\theta(t^{-(n-k)}a_ia_j^{-1})}{\prod_{1\leqslant i<j\leqslant s}\theta(t^{n-k}a_ia_j)\ \prod_{i=1}^{s}\prod_{j=s+1}^{2s+2}\theta(t^{n-k}a_ia_j)}\right]^{\binom{s+k-2}{k-1}} \\ &\quad \times \prod_{k=1}^{n}\left[\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\frac{1}{\theta(t^{2r-(n-k)}a_ia_j^{-1})\theta(t^{n-k}a_ia_j)}\right]^{\binom{s+k-3}{k-1}}.\end{aligned}$$

Proof. From definition (3.11) of $\Theta(z)$, we have

$$\Theta(\zeta_{(\mu)}) = I_1(\zeta_{(\mu)}) \prod_{i=1}^{n} \frac{\theta(\zeta_{(\mu)i}^2)}{\prod_{\ell=1}^{2s+2}\theta(a_\ell \zeta_{(\mu)i})} \prod_{1\leqslant j<k\leqslant n} \frac{\theta(\zeta_{(\mu)j}/\zeta_{(\mu)k})}{\theta(t\zeta_{(\mu)j}/\zeta_{(\mu)k})} \frac{\theta(\zeta_{(\mu)j}\zeta_{(\mu)k})}{\theta(t\zeta_{(\mu)j}\zeta_{(\mu)k})}.$$

The expression of $\prod_{\mu\in Z}\Theta(\zeta_{(\mu)})$ in Lemma 8.5 is obtained from (8.10) and the following explicit calculation:

$$\prod_{\mu\in Z}\prod_{i=1}^{n} \frac{\theta(\zeta_{(\mu)i}^2)}{\prod_{\ell=1}^{2s+2}\theta(a_\ell \zeta_{(\mu)i})} = \prod_{k=1}^{n}\Bigg[\prod_{\ell=1}^{s} \frac{\theta(a_\ell^2 t^{2(n-k)})}{\theta(a_\ell^2 t^{n-k})} \prod_{1\leqslant i<j\leqslant s} \frac{1}{\theta(a_i a_j t^{n-k})^2} \times \frac{1}{\prod_{i=1}^{s}\prod_{j=s+1}^{2s+2}\theta(a_i a_j t^{n-k})}\Bigg]^{\binom{s+k-2}{k-1}}$$

and

$$\prod_{\mu\in Z}\prod_{1\leqslant j<k\leqslant n} \frac{\theta(\zeta_{(\mu)j}\zeta_{(\mu)k})}{\theta(t\zeta_{(\mu)j}\zeta_{(\mu)k})} = \prod_{k=1}^{n}\Bigg[\prod_{\ell=1}^{s} \frac{\theta(a_\ell^2 t^{n-k})}{\theta(a_\ell^2 t^{2(n-k)})} \prod_{1\leqslant i<j\leqslant s} \theta\big(a_i a_j t^{n-k}\big)\Bigg]^{\binom{s+k-2}{k-1}} \times \prod_{k=1}^{n}\Bigg[\prod_{1\leqslant i<j\leqslant s} \frac{1}{\theta(a_i a_j t^{n-k})}\Bigg]^{(n-k+1)\binom{s+k-3}{k-1}}. \qquad \square$$

We can conclude this paper with the following:

Proof of Theorem 3.9. Since the determinant of the regularized truncated Jackson integrals is expressed as

$$\det\big(\langle\langle \chi_\lambda, \zeta_{(\mu)}\rangle\rangle\big)_{\lambda,\mu} = \frac{J}{\prod_{\mu\in Z}\Theta(\zeta_{(\mu)})},$$

we obtain the product expression of the determinant because the explicit expressions of J and $\prod_{\mu\in Z}\Theta(\zeta_{(\mu)})$ are given in Proposition 8.4 and Lemma 8.5, respectively. $\square$

Appendix A. Proof of Proposition 3.3

Proof of Proposition 3.3. Since the Jackson integral is invariant under q-shift, it follows that

$$\int_{\Lambda_\xi} \varphi(z)\Phi(z)\varpi_q = \int_{\Lambda_\xi} T_{z_1}\varphi(z) T_{z_1}\Phi(z)\varpi_q,$$

so that

$$\int_{\Lambda_\xi} \varphi(z)\Phi(z)\varpi_q - \int_{\Lambda_\xi} T_{z_1}\varphi(z)\frac{T_{z_1}\Phi(z)}{\Phi(z)}\Phi(z)\varpi_q = 0.$$

This implies (3.4) by definition (3.2) of ∇. Using Lemma 3.2, for $w \in W$ we have

$$
\begin{aligned}
w \int_{\Lambda_\xi} \Phi(z)\nabla\varphi(z)\varpi_q &= \int_{\Lambda_{(w^{-1}\xi)}} \Phi(z)\nabla\varphi(z)\varpi_q \\
&= \int_{\Lambda_\xi} w\Phi(z)w\nabla\varphi(z)\varpi_q = \int_{\Lambda_\xi} U_w(z)\Phi(z)w\nabla\varphi(z)\varpi_q \\
&= U_w(\xi)\int_{\Lambda_\xi} \Phi(z), w\nabla\varphi(z)\varpi_q.
\end{aligned}
$$

If (3.4) holds, then

$$
\int_{\Lambda_\xi} \Phi(z)\, w\nabla\varphi(z)\varpi_q = \frac{1}{U_w(\xi)} w \int_{\Lambda_\xi} \Phi(z)\nabla\varphi(z)\varpi_q = 0 \quad \text{for } w \in W.
$$

Thus, for the function $\mathcal{A}\nabla\varphi(z) = \sum_{w\in W}(\operatorname{sgn} w)w\nabla\varphi(z)$, we obtain

$$
\int_{\Lambda_\xi} \Phi(z)\mathcal{A}\nabla\varphi(z)\varpi_q = \sum_{w\in W}(\operatorname{sgn} w)\int_{\Lambda_\xi} \Phi(z)w\nabla\varphi(z)\varpi_q = 0,
$$

which completes the proof. □

Appendix B. Proof of Proposition 3.7

We assume that $\varphi(z)$ is W-symmetric and holomorphic on $(\mathbb{C}^*)^n$. Since $\langle\varphi, z\rangle$ has poles lying only in the set

$$
\left\{ z \in (\mathbb{C}^*)^n;\ \prod_{i=1}^{n}\prod_{m=1}^{2s+2} \theta(a_m z_i) \prod_{1\leqslant j<k\leqslant n} \theta(tz_j/z_k)\theta(tz_jz_k) = 0 \right\},
$$

$\langle\varphi, z\rangle$ is written as

$$
\langle\varphi, z\rangle = g(z)\prod_{i=1}^{n} \frac{z_i^s}{\prod_{m=1}^{2s+2} z_i^{\alpha_m}\theta(a_m z_i)} \prod_{1\leqslant j<k\leqslant n} \frac{1}{z_j^{2\tau}\theta(tz_j/z_k)\theta(tz_jz_k)}
$$

where $g(z)$ is a holomorphic function on $(\mathbb{C}^*)^n$. We show $g(z)$ is divisible by

$$
\prod_{i=1}^{n}\theta\big(z_i^2\big) \prod_{1\leqslant j<k\leqslant n} \theta(z_j/z_k)\theta(z_jz_k),
$$

which proves Proposition 3.7.

In order to prove that $g(z)$ is divisible by $\theta(z_i^2)=\theta(z_i)\theta(-z_i)\theta(\sqrt{q}z_i)\theta(-\sqrt{q}z_i)$, it is sufficient to show $\langle\varphi,z\rangle=0$ at $z_i=\pm1,\pm\sqrt{q}$, because $\langle\varphi,z\rangle$ is invariant under the q-shift $z_i\to q^\nu z_i$, $\nu\in\mathbb{Z}$. In the same way, to prove that $\theta(z_j/z_k)\theta(z_jz_k)$ divides $g(z)$, we show $\langle\varphi,z\rangle=0$ at $z_j/z_k=1$ and $z_jz_k=1$.

Let τ_i, σ_{ij} and τ_{ij} be reflections of the coordinates $z=(z_1,z_2,\dots,z_n)$ defined as follows:

$$\tau_i: z_i\leftrightarrow z_i^{-1},\qquad \sigma_{ij}: z_i\leftrightarrow z_j,\qquad \tau_{ij}: z_i\leftrightarrow z_j^{-1}.$$

Lemma B.1. *For $1\leqslant i\leqslant n$, if $z_i=1,\sqrt{q}$, then $U_{\tau_i}(z)=1$, and if $z_i=-1,-\sqrt{q}$, then $U_{\tau_i}(z)=(-1)^{2[\alpha_1+\cdots+\alpha_{2s+2}+(n-i)(n+i-1)\tau]}$. For $1\leqslant i<j\leqslant n$, if $z_i/z_j=1$ or $z_iz_j=1$, then $U_{\sigma_{ij}}(z)=1$ or $U_{\tau_{ij}}(z)=1$, respectively.*

Proof. By definition, $U_{\tau_i}(z)$, $U_{\sigma_{ij}}(z)$ and $U_{\tau_{ij}}(z)$ are written explicitly as follows:

$$U_{\tau_i}(z)=\prod_{m=1}^{2s+2}\left(z_i^2\right)^{\alpha_m-1/2}\frac{\theta(a_mz_i)}{\theta(qa_m^{-1}z_i)}\prod_{k=i+1}^{n}\left(z_i^2\right)^{2\tau-1}\frac{\theta(tz_i/z_k)\theta(tz_iz_k)}{\theta(qt^{-1}z_i/z_k)\theta(qt^{-1}z_iz_k)},$$

$$U_{\sigma_{ij}}(z)=(z_i/z_j)^{2\tau-1}\frac{\theta(tz_i/z_j)}{\theta(qt^{-1}z_i/z_j)}\prod_{k=i+1}^{j-1}(z_i/z_j)^{2\tau-1}\frac{\theta(tz_i/z_k)\theta(tz_k/z_j)}{\theta(qt^{-1}z_i/z_k)\theta(qt^{-1}z_k/z_j)},$$

$$\begin{aligned}U_{\tau_{ij}}(z)={}&\prod_{m=1}^{2s+2}\left(z_i^2z_j^2\right)^{\alpha_m-1/2}\frac{\theta(a_mz_i)\theta(a_mz_j)}{\theta(qa_m^{-1}z_i)\theta(qa_m^{-1}z_j)}\\&\times(z_iz_j)^{2\tau-1}\frac{\theta(tz_iz_j)}{\theta(qt^{-1}z_iz_j)}\prod_{k=i+1}^{j-1}(z_iz_j)^{2\tau-1}\frac{\theta(tz_i/z_k)\theta(tz_kz_j)}{\theta(qt^{-1}z_i/z_k)\theta(qt^{-1}z_kz_j)}.\end{aligned}$$

Since $\theta(x)=\theta(q/x)$ and $\theta(qx)=-\theta(x)/x$, we obtain Lemma B.1. □

Lemma B.2. *For $1\leqslant i\leqslant n$, if $z_i=\pm1,\pm\sqrt{q}$, then $\tau_i\langle\varphi,z\rangle=\langle\varphi,z\rangle$. For $1\leqslant i<j\leqslant n$, if $z_i/z_j=1$ or $z_iz_j=1$, then $\sigma_{ij}\langle\varphi,z\rangle=\langle\varphi,z\rangle$ or $\tau_{ij}\langle\varphi,z\rangle=\langle\varphi,z\rangle$, respectively.*

Proof. By definition, $\tau_i\langle\varphi,z\rangle=\langle\varphi,\tau_iz\rangle$ where $\tau_iz=(z_1,\dots,z_i^{-1},\dots,z_n)$. If $z_i=\pm1$, then $\tau_iz=z$, so that $\langle\varphi,\tau_iz\rangle=\langle\varphi,z\rangle$. Since $\langle\varphi,z\rangle$ is invariant under the q-shift $z_i\to qz_i$, if $z_i=\pm\sqrt{q}$, then $\langle\varphi,\tau_iz\rangle=\langle\varphi,z\rangle$. Since $\sigma_{ij}z=(z_1,\dots,z_j,\dots,z_i,\dots,z_n)$ and $\tau_{ij}z=(z_1,\dots,z_j^{-1},\dots,z_i^{-1},\dots,z_n)$, if $z_i/z_j=1$ or $z_iz_j=1$, then $\sigma_{ij}z=z$ or $\tau_{ij}z=z$, so that $\sigma_{ij}\langle\varphi,z\rangle=\langle\varphi,z\rangle$ or $\tau_{ij}\langle\varphi,z\rangle=\langle\varphi,z\rangle$, respectively. □

Lemma B.3. *If $z_i=\pm1,\pm\sqrt{q}$ $(1\leqslant i\leqslant n)$ or if $z_i/z_j=1$ or $z_iz_j=1$ $(1\leqslant i<j\leqslant n)$, then $\langle\varphi,z\rangle=0$.*

Proof. Suppose $z_i=\pm1,\pm\sqrt{q}$ first. By Lemma B.2 we have $\tau_i\langle\varphi,z\rangle=\langle\varphi,z\rangle$. On the other hand, since $\varphi(z)$ is W-symmetric, we have $\tau_i\langle\varphi,z\rangle=-U_{\tau_i}(z)\langle\varphi,z\rangle$ from (3.8). Thus $(1+U_{\tau_i}(z))\langle\varphi,z\rangle=0$. Since $(1+U_{\tau_i}(z))\neq0$ from Lemma B.1 under the condition $(\mathcal{C})$, we obtain $\langle\varphi,z\rangle=0$. The same holds for both cases $z_i/z_j=1$ and $z_iz_j=1$. □

Appendix C. Proof of Proposition 6.2

Since P is the matrix defined by (6.5), the determinant of P coincides with the signature of the permutation on B corresponding to P. Thus $\det P$ is written as

$$\det P = (-1)^{\mathrm{Inv}}$$

where Inv is the number of inversions of the permutation and is defined by

$$\mathrm{Inv} := \#\{(\lambda,\mu) \in B \times B;\ \lambda \prec \mu,\ \lambda > \mu\}.$$

In order to prove Proposition 6.2, it is sufficient to show

$$\mathrm{Inv} \equiv (s-1)\binom{n+s-1}{s} \pmod 2.$$

To prove this, we state four lemmas.

For positive integers i and j, we set

$$B_i^j := \{(\lambda_1,\lambda_2,\dots,\lambda_i,0,\dots,0) \in \mathcal{P};\ j = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_i > 0\}.$$

We regard B_0^j as the set $\{(0)\}$. The set B_i and B, which are defined in Section 2, are written as

$$B_i = \bigcup_{j=1}^{s-1} B_i^j \quad \text{and} \quad B = \bigcup_{i=0}^{n} B_i = \bigcup_{i=0}^{n}\bigcup_{j=1}^{s-1} B_i^j. \tag{C.1}$$

Lemma C.1. *Put*

$$D_\ell^k := \left\{(\lambda,\mu) \in \left(\bigcup_{j=1}^{k} B_\ell^j\right) \times \left(\bigcup_{j=1}^{k}\bigcup_{i=0}^{\ell-1} B_i^j\right);\ \lambda > \mu\right\} \quad \textit{and} \quad d_\ell^k := \#D_\ell^k.$$

Then

$$\mathrm{Inv} = \sum_{\ell=1}^{n} d_\ell^{s-1}.$$

Proof. By definition of the ordering $\prec$ on B, we have

$$\{(\lambda,\mu) \in B \times B;\ \lambda \prec \mu,\ \lambda > \mu\} = \bigcup_{\ell=1}^{n}\left\{(\lambda,\mu) \in B_\ell \times \left(\bigcup_{i=0}^{\ell-1} B_i\right);\ \lambda > \mu\right\}.$$

From (C.1) and the definition of D_ℓ^k it follows that

$$\bigcup_{\ell=1}^{n}\left\{(\lambda,\mu) \in B_\ell \times \left(\bigcup_{i=0}^{\ell-1} B_i\right);\ \lambda > \mu\right\} = \bigcup_{\ell=1}^{n} D_\ell^{s-1}.$$

This implies Lemma C.1. □

Lemma C.2. *The following relation holds for $k > 1$ and $\ell \geqslant 1$:*

$$d_\ell^k = d_\ell^{k-1} + d_{\ell-1}^k + \binom{k+\ell-2}{\ell-1}^2. \tag{C.2}$$

Proof. We divide the set $(\bigcup_{j=1}^k B_\ell^j) \times (\bigcup_{j=1}^k \bigcup_{i=0}^{\ell-1} B_i^j)$ into the following four sets:

$$\left(\bigcup_{j=1}^{k} B_\ell^j\right) \times \left(\bigcup_{j=1}^{k}\bigcup_{i=0}^{\ell-1} B_i^j\right) = \left[\left(\bigcup_{j=1}^{k-1} B_\ell^j\right) \times \left(\bigcup_{j=1}^{k-1}\bigcup_{i=0}^{\ell-1} B_i^j\right)\right] \cup \left[\left(\bigcup_{j=1}^{k-1} B_\ell^j\right) \times \left(\bigcup_{i=0}^{\ell-1} B_i^k\right)\right]$$
$$\cup \left[B_\ell^k \times \left(\bigcup_{j=1}^{k-1}\bigcup_{i=0}^{\ell-1} B_i^j\right)\right] \cup \left[B_\ell^k \times \left(\bigcup_{i=0}^{\ell-1} B_i^k\right)\right].$$

Then, from the definition of D_ℓ^k, it follows that

$$D_\ell^k = D_\ell^{k-1} \cup Q \cup R \cup S \tag{C.3}$$

where

$$Q := \left\{(\lambda,\mu) \in \left(\bigcup_{j=1}^{k-1} B_\ell^j\right) \times \left(\bigcup_{i=0}^{\ell-1} B_i^k\right);\ \lambda > \mu\right\},$$
$$R := \left\{(\lambda,\mu) \in B_\ell^k \times \left(\bigcup_{j=1}^{k-1}\bigcup_{i=0}^{\ell-1} B_i^j\right);\ \lambda > \mu\right\},$$
$$S := \left\{(\lambda,\mu) \in B_\ell^k \times \left(\bigcup_{i=0}^{\ell-1} B_i^k\right);\ \lambda > \mu\right\}.$$

From (C.3), in order to prove (C.2), it is sufficient to show the following:

$$\#Q = 0, \qquad \#R = \binom{k+\ell-2}{\ell-1}^2 \quad \text{and} \quad \#S = d_{\ell-1}^k.$$

We show $\#Q = 0$ first. Since $\lambda < \mu$ for arbitrary $\lambda \in (\bigcup_{j=1}^{k-1} B_\ell^j)$ and $\mu \in (\bigcup_{i=0}^{\ell-1} B_i^k)$, we have $Q = \phi$, so that $\#Q = 0$. Next, we show $\#R = \binom{k+\ell-2}{\ell-1}^2$. Since $\lambda > \mu$ holds for arbitrary $\lambda \in B_\ell^k$ and $\mu \in (\bigcup_{j=1}^{k-1} \bigcup_{i=0}^{\ell-1} B_i^j)$, we have

$$R = B_\ell^k \times \left(\bigcup_{j=1}^{k-1}\bigcup_{i=0}^{\ell-1} B_i^j\right). \tag{C.4}$$

By definition of B_i^j, we have

$$\#B_\ell^k = \binom{k+\ell-2}{\ell-1}. \tag{C.5}$$

Using (2.3), we have

$$\begin{aligned}\#\left(\bigcup_{j=1}^{k-1}\bigcup_{i=0}^{\ell-1} B_i^j\right) &= \sum_{j=1}^{k-1}\sum_{i=0}^{\ell-1}\binom{i+j-2}{i-1} = \sum_{i=0}^{\ell-1}\sum_{j=1}^{k-1}\binom{i+j-2}{j-1} = \sum_{i=0}^{\ell-1}\binom{i+k-2}{k-2}\\ &= \sum_{i=0}^{\ell-1}\binom{i+k-2}{i} = \binom{k+\ell-2}{\ell-1}.\end{aligned} \tag{C.6}$$

From (C.4)–(C.6), it follows $\#R = \binom{k+\ell-2}{\ell-1}^2$. Finally we show $\#S = d_{\ell-1}^k$. The projection

$$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n) \to (\lambda_2, \lambda_3, \ldots, \lambda_n, 0)$$

gives rise to the bijection from S to $D_{\ell-1}^k$ because the following two conditions are equivalent:

(1) $\lambda = (k, \lambda_2, \lambda_3, \ldots, \lambda_\ell, 0, \ldots, 0) \in B_\ell^k$ and $\mu = (k, \mu_2, \mu_3, \ldots, \mu_\ell, 0, \ldots, 0) \in \bigcup_{i=0}^{\ell-1} B_i^k$ satisfy $\lambda > \mu$,

(2) $(\lambda_2, \lambda_3, \ldots, \lambda_\ell, 0, \ldots, 0) \in \bigcup_{j=1}^{k} B_{\ell-1}^j$ and $(\mu_2, \mu_3, \ldots, \mu_\ell, 0, \ldots, 0) \in \bigcup_{j=1}^{k}\bigcup_{i=0}^{\ell-2} B_i^j$ satisfy $(\lambda_2, \lambda_3, \ldots, \lambda_\ell, 0, \ldots, 0) > (\mu_2, \mu_3, \ldots, \mu_\ell, 0, \ldots, 0)$.

This implies that $\#S = \#D_{\ell-1}^k = d_{\ell-1}^k$. □

Lemma C.3. *The following holds for d_ℓ^k:*

$$d_\ell^k \equiv \ell\binom{k+\ell-1}{\ell} \pmod 2.$$

Proof. Induction on k and ℓ. Lemma C.3 is correct for $d_\ell^1 = \ell$ and $d_1^k = k$. Next we assume the following for the integers i and j satisfying $1 \leqslant i < \ell$ or $1 \leqslant j < k$:

$$d_i^j \equiv i\binom{j+i-1}{i} \pmod 2. \tag{C.7}$$

From Lemma C.2, it follows that

$$\begin{aligned}&d_\ell^k - \ell\binom{k+\ell-1}{\ell}\\ &= \left(d_\ell^{k-1} + d_{\ell-1}^k + \binom{k+\ell-2}{\ell-1}^2\right) - \ell\left(\binom{k+\ell-2}{\ell} + \binom{k+\ell-2}{\ell-1}\right)\\ &\equiv \ell\binom{k+\ell-2}{\ell} + (\ell-1)\binom{k+\ell-2}{\ell-1} + \binom{k+\ell-2}{\ell-1}^2\end{aligned}$$

$$
\begin{aligned}
&-\ell\left(\binom{k+\ell-2}{\ell}+\binom{k+\ell-2}{\ell-1}\right) \quad \bigl(\text{by the assumption (C.7) of induction}\bigr)\\
&=\binom{k+\ell-2}{\ell-1}\left(\binom{k+\ell-2}{\ell-1}-1\right)\\
&\equiv 0 \pmod 2,
\end{aligned}
$$

which completes the proof. □

Lemma C.4. *The sum $\sum_{\ell=1}^{n} d_\ell^k$ is written as follows:*

$$\sum_{\ell=1}^{n} d_\ell^k \equiv k\binom{k+n}{k+1} \pmod 2.$$

In particular,

$$\mathrm{Inv} \equiv (s-1)\binom{n+s-1}{s} \pmod 2.$$

Proof. From Lemma C.3, using (2.3), it follows that

$$
\begin{aligned}
\sum_{\ell=1}^{n} d_\ell^k &\equiv \sum_{\ell=1}^{n} \ell\binom{k+\ell-1}{\ell} = \sum_{\ell=1}^{n} k\binom{k+\ell-1}{k} = k\sum_{\ell=1}^{n}\binom{k+\ell-1}{k}\\
&= k\sum_{\ell=1}^{n}\binom{k+\ell-1}{\ell-1} = k\binom{k+n}{n-1} = k\binom{k+n}{k+1} \pmod 2.
\end{aligned}
$$

In particular, if $k=s-1$, from Lemma C.1, we obtain

$$\mathrm{Inv} = \sum_{\ell=1}^{n} d_\ell^{s-1} \equiv (s-1)\binom{n+s-1}{s} \pmod 2,$$

which completes the proof. □

Appendix D. Proof of Proposition 7.1

This section is devoted to proving Proposition 7.1. Before proving Proposition 7.1 we show three lemmas.

Let $\zeta_{(\mu)}$, $\mu \in Z$ be the points defined in (3.10).

Lemma D.1. *For $\mu=(\mu_1,\mu_2,\dots,\mu_s)$, $\nu=(\nu_1,\nu_2,\dots,\nu_s)\in Z$, suppose that*

$$\mu_1>\nu_1, \qquad \mu_1+\mu_2=\nu_1+\nu_2 \quad \textit{and} \quad \mu_3-\nu_3=\mu_4-\nu_4=\cdots=\mu_s-\nu_s=0.$$

Then $S_\lambda(\zeta_{(\mu)})-S_\lambda(\zeta_{(\nu)})$ is divisible by $(t^{\nu_1}a_1-t^{\mu_2}a_2)$.

Proof. For $\mu, \nu \in Z$ satisfying the above condition, the explicit expressions of the points $\zeta_{(\mu)}$, $\zeta_{(\nu)}$ are the following:

$$\zeta_{(\mu)} = \big(\underbrace{t^{\mu_1-1}a_1, t^{\mu_1-2}a_1, \ldots, ta_1, a_1}_{\mu_1}, \underbrace{t^{\mu_2-1}a_2, t^{\mu_2-2}a_2, \ldots, a_2}_{\mu_2}, \ldots\big),$$

$$\zeta_{(\nu)} = \big(\underbrace{t^{\nu_1-1}a_1, t^{\nu_1-2}a_1, \ldots, a_1}_{\nu_1}, \underbrace{t^{\nu_2-1}a_2, t^{\nu_2-2}a_2, \ldots, ta_2, a_2}_{\nu_2}, \ldots\big).$$

Suppose $\nu_1 \leqslant \mu_2$ first. Regarding $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$ as a function of a_1, we substitute the value $t^{\mu_2-\nu_1}a_2$ for the variable a_1. Since $\mu_1 + \mu_2 = \nu_1 + \nu_2$, the sequences appearing in $\zeta_{(\mu)}$ and $\zeta_{(\nu)}$ are the same up to ordering. Then $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)}) = 0$ since $S_\lambda(z)$ is symmetric, and $(a_1 - t^{\mu_2-\nu_1}a_2)$ divides the polynomial $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$. Moreover we regard $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$ as a function of t^{ν_1} and substitute 0 for the variable t^{ν_1}. Since $\nu_1 \leqslant \mu_2 < \nu_2$, the sequences in $\zeta_{(\mu)}$ and $\zeta_{(\nu)}$ are also the same up to ordering. Then $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)}) = 0$, so that the polynomial $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$ is divisible by t^{ν_1}. Thus, $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$ is divisible by $t^{\nu_1}(a_1 - t^{\mu_2-\nu_1}a_2) = (t^{\nu_1}a_1 - t^{\mu_2}a_2)$. The same holds for $\nu_1 \geqslant \mu_2$. □

Lemma D.2. $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ *is divisible by*

$$\prod_{k=1}^{n}\prod_{r=0}^{n-k}\prod_{1\leqslant i<j\leqslant s}\big(t^r a_i - t^{n-k-r}a_j\big)^{\binom{s+k-3}{k-1}}.$$

Proof. Without loss of generality, it is sufficient to show that $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ is divisible by

$$\prod_{k=1}^{n}\prod_{r=0}^{n-k}\big(t^r a_1 - t^{n-k-r}a_2\big)^{\binom{s+k-3}{k-1}}.$$

For each integer i such that $0 \leqslant i \leqslant n-1$, the number of the elements in the set $\{(\ell_3, \ell_4, \ldots, \ell_s) \in \mathbb{Z}_{\geqslant 0}^{s-2};\ \ell_3 + \ell_4 + \cdots + \ell_s = i\}$ is equal to $\binom{s+i-3}{i}$. For $\mu = (\mu_1, \mu_2, \ell_3, \ell_4, \ldots, \ell_s)$, $\nu = (\nu_1, \nu_2, \ell_3, \ell_4, \ldots, \ell_s) \in Z$, we assume that $\mu_1 > \nu_1$, $\mu_1 + \mu_2 = \nu_1 + \nu_2 = n - i$. The difference of (λ, μ) entry and (λ, ν) entry of the determinant is given by $S_\lambda(\zeta_{(\mu)}) - S_\lambda(\zeta_{(\nu)})$, which is divisible by $(t^{\nu_1}a_1 - t^{\mu_2}a_2)$ by Lemma D.1. Thus, using elementary column operations, the following factor occurs in $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$:

$$\prod_{k=0}^{n-i-1}\prod_{r=0}^{k}\big(t^r a_1 - t^{k-r}a_2\big)^{\binom{s+i-3}{i}}.$$

Taking the product of above expression over $0 \leqslant i \leqslant n-1$, $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ is divisible by

$$\begin{aligned}\prod_{i=0}^{n-1}\prod_{k=0}^{n-i-1}\prod_{r=0}^{k}\big(t^r a_1 - t^{k-r}a_2\big)^{\binom{s+i-3}{i}} &= \prod_{k=0}^{n-1}\prod_{i=0}^{n-k-1}\prod_{r=0}^{k}\big(t^r a_1 - t^{k-r}a_2\big)^{\binom{s+i-3}{i}} \\ &= \prod_{k=0}^{n-1}\prod_{r=0}^{k}\big(t^r a_1 - t^{k-r}a_2\big)^{\sum_{i=0}^{n-k-1}\binom{s+i-3}{i}}\end{aligned}$$

$$= \prod_{k=0}^{n-1} \prod_{r=0}^{k} \bigl(t^r a_1 - t^{k-r} a_2\bigr)^{\binom{s+n-k-3}{n-k-1}}.$$

We therefore conclude Lemma D.2 exchanging k with $n-k$ above. □

We introduce a lexicographic ordering on the set of monomials in the variables $a_1, a_2, \ldots, a_s$, by the rule $a_1^{i_1} a_2^{i_2} \ldots a_s^{i_s} > a_1^{j_1} a_2^{j_2} \ldots a_s^{j_s}$ if

$$i_1 = j_1, \quad i_2 = j_2, \quad \ldots, \quad i_{k-1} = j_{k-1} \quad \text{and} \quad i_k > j_k$$

for some $k \in \{1, 2, \ldots, s\}$. For any polynomial $f(a_1, a_2, \ldots, a_s)$, we define the *dominant monomial of f* by the maximum monomial in the above order among the monomials with the non-zero coefficients and denote it by $d[f]$. Define the *dominant coefficient of f* by the coefficient of the dominant monomial of f and denote it by $cd[f]$. It is easy to see that for any product of two polynomials f and g, $d[fg] = d[f]d[g]$ and $cd[fg] = cd[f]cd[g]$.

Since the dominant monomial of the right-hand side of (7.2) is $(a_1^{s-1} a_2^{s-2} \ldots a_{s-1})^{\binom{s+n-1}{n-1}}$ and its coefficient is $t^{\binom{s}{2}\binom{s+n-1}{n-2}}$, we can conclude Proposition 7.1 from Lemma D.2 by proving the following:

Lemma D.3. *As a polynomial of $a_1, a_2, \ldots, a_s$, the dominant monomial of* $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ *is* $(a_1^{s-1} a_2^{s-2} \ldots a_{s-1})^{\binom{s+n-1}{n-1}}$ *and its coefficient is* $t^{\binom{s}{2}\binom{s+n-1}{n-2}}$.

Proof. We prove this lemma by induction on s (the number of the variables). The dominant monomial of $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ can be evaluated as the product of minors of order $\binom{s+r-2}{r}$ for $r = 0, 1, \ldots, n$ as follows. The $n+1$ minors which consist of the entries in the following diagonal blocks of the matrix $(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ contribute to the dominant monomial of the polynomial $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$. For the minor of order 1 with the (λ, μ) entry where

$$\lambda = (s-1, s-1, \ldots, s-1) \quad \text{and} \quad \mu = (n, 0, 0, \ldots, 0),$$

the dominant monomial and its coefficient of the minor are equal to $a_1^{(s-1)n}$ and $t^{(s-1)\binom{n}{2}}$, respectively, because

$$\bigl(t^{n-1} a_1\bigr)^{s-1} \bigl(t^{n-2} a_1\bigr)^{s-1} \ldots (t a_1)^{s-1} a_1^{s-1} = a_1^{(s-1)n} t^{(s-1)\binom{n}{2}}.$$

Next, in order to obtain the dominant monomial of the minor of order $\binom{s-1}{1}$ with the (λ, μ) entries where

$$\lambda \in \bigl\{(s-1, s-1, \ldots, s-1, \lambda_n) \in B;\ s-2 \geqslant \lambda_n \geqslant 0\bigr\},$$
$$\mu \in \bigl\{(n-1, \mu_2, \mu_3, \ldots, \mu_s) \in Z;\ \mu_2 + \mu_3 + \cdots + \mu_s = 1\bigr\},$$

we can simultaneously replace each of its entries by the partial sum over the monomials containing $(t^{n-2} a_1)^{s-1} \ldots (t a_1)^{s-1} a_1^{s-1} = a_1^{(s-1)(n-1)} t^{(s-1)\binom{n-1}{2}}$. After factoring the powers of a_1 from each column of the minor, the inductive hypothesis can be applied to that minor with

$s-1$ variables $a_2, a_3, \dots, a_s$. Then the dominant monomial of the minor is $a_1^{(s-1)(n-1)\binom{s-1}{1}} \times (a_2^{s-2} a_3^{s-3} \dots a_{s-1})^{\binom{s-1}{0}}$ and its coefficient is $t^{(s-1)\binom{n-1}{2}\binom{s-1}{1}} \times t^{\binom{s-1}{2}\binom{s-1}{-1}}$.

In general, for $r = 0, 1, \dots, n$, we consider the minor of order $\binom{s+r-2}{r}$ with the (λ, μ) entries where

$$\lambda \in \{(s-1, s-1, \dots, s-1, \lambda_{n-r+1}, \dots, \lambda_n) \in B;\ s-2 \geqslant \lambda_{n-r+1} \geqslant \cdots \geqslant \lambda_n \geqslant 0\},$$
$$\mu \in \{(n-r, \mu_2, \mu_3, \dots, \mu_s) \in Z;\ \mu_2 + \mu_3 + \cdots + \mu_s = r\}.$$

Factoring the powers of a_1 from each column of the minor, we can apply the inductive hypothesis to that minor with $s-1$ variables $a_2, a_3, \dots, a_s$. Thus the dominant monomial of the minor is $a_1^{(s-1)(n-r)\binom{s+r-2}{r}} \times (a_2^{s-2} a_3^{s-3} \dots a_{s-1})^{\binom{s+r-2}{r-1}}$ and its coefficient is $t^{(s-1)\binom{n-r}{2}\binom{s+r-2}{r}} \times t^{\binom{s-1}{2}\binom{s+r-2}{r-2}}$, and so on, up to $r = n$ where the factor no longer depends on a_1.

Taking the product over $0 \leqslant r \leqslant n$, the dominant monomial of $\det(S_\lambda(\zeta_{(\mu)}))_{\lambda,\mu}$ is

$$\prod_{r=0}^{n} a_1^{(s-1)(n-r)\binom{s+r-2}{r}} \left(a_2^{s-2} a_3^{s-3} \dots a_{s-1}\right)^{\binom{s+r-2}{r-1}}$$

and its coefficient is

$$\prod_{r=0}^{n} t^{(s-1)\binom{n-r}{2}\binom{s+r-2}{r} + \binom{s-1}{2}\binom{s+r-2}{r-2}}.$$

Using (7.8) in Lemma 7.4 where $j = 0, 1, 2$, the above monomial and coefficient reduce to those in Lemma D.3, respectively. □

References

[1] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia Math. Appl., vol. 71, Cambridge Univ. Press, Cambridge, 1999.

[2] K. Aomoto, q-Analogue of de Rham cohomology associated with Jackson integrals. I, II, Proc. Japan Acad. Ser. A Math. Sci. 66 (7) (1990) 161–164, Proc. Japan Acad. Ser. A Math. Sci. 66 (8) (1990) 240–244.

[3] K. Aomoto, Connection formulas in the q-analog de Rham cohomology, in: Functional Analysis on the Eve of the 21st Century, vol. 1, Brunswick, NJ, 1993, in: Progr. Math., vol. 131, Birkhäuser Boston, Boston, MA, 1995, pp. 1–12.

[4] K. Aomoto, M. Ito, On the structure of Jackson integrals of BC_n type and holonomic q-difference equations, Proc. Japan Acad. Ser. A Math. Sci. 81 (2005) 145–150.

[5] K. Aomoto, M. Ito, Structure of Jackson integrals of type BC_n, Tokyo J. Math. 31 (2008) 449–477.

[6] K. Aomoto, M. Ito, BC_n-type Jackson integral generalized from Gustafson's C_n-type sum, J. Difference Equ. Appl. 14 (2008) 1059–1097.

[7] J.F. van Diejen, Self-dual Koornwinder–Macdonald polynomials, Invent. Math. 126 (1996) 319–339.

[8] J.F. van Diejen, On certain multiple Bailey, Rogers and Dougall type summation formulas, Publ. Res. Inst. Math. Sci. 33 (1997) 483–508.

[9] J.F. van Diejen, Properties of some families of hypergeometric orthogonal polynomials in several variables, Trans. Amer. Math. Soc. 351 (1999) 233–270.

[10] G. Gasper, M. Rahman, Basic Hypergeometric Series, second ed., Encyclopedia Math. Appl., vol. 96, Cambridge Univ. Press, Cambridge, 2004.

[11] R.A. Gustafson, A generalization of Selberg's beta integral, Bull. Amer. Math. Soc. (N.S.) 22 (1990) 97–105.

[12] M. Ishikawa, M. Ito, S. Okada, A compound determinant identity for rectangular matrices, preprint, 2008.

[13] M. Ito, Convergence and asymptotic behavior of Jackson integrals associated with irreducible reduced root systems, J. Approx. Theory 124 (2003) 154–180.

[14] M. Ito, Askey–Wilson type integrals associated with root systems, Ramanujan J. 12 (2006) 131–151.

[15] M. Ito, q-difference shift for a BC_n type Jackson integral arising from elementary symmetric polynomials, Adv. Math. 204 (2006) 619–646.

[16] M. Ito, Another proof of Gustafson's C_n-type summation formula via elementary symmetric polynomials, Publ. Res. Inst. Math. Sci. 42 (2006) 523–549.

[17] M. Ito, A multiple generalization of Slater's transformation formula for a very-well-poised-balanced ${}_{2r}\psi_{2r}$ series, Q. J. Math. 59 (2008) 221–235.

[18] M. Ito, K. Koike, A generalization of Weyl's denominator formula for the classical groups, J. Algebra 302 (2006) 817–825.

[19] M. Ito, S. Okada, An application of Cauchy–Sylvester's theorem on compound determinants to a BC_n-type Jackson integral, submitted to Proceedings of the Conference on Partitions, q-Series and Modular Forms, University of Florida, March 12–16, 2008, to be published in the Springer book series Dev. Math.

[20] M. Ito, Y. Sanada, On the Sears–Slater basic hypergeometric transformations, Ramanujan J. 17 (2008) 245–257.

[21] K.W.J. Kadell, A proof of the q-Macdonald–Morris conjecture for BC_n, Mem. Amer. Math. Soc. 108 (1994) 516.

[22] Y. Komori, M. Noumi, J. Shiraishi, Kernel functions for difference operators of Ruijsenaars type and their applications, arXiv: math.QA/0812.0279.

[23] I.G. Macdonald, Some conjectures for root systems, SIAM J. Math. Anal. 13 (1982) 988–1007.

[24] I.G. Macdonald, Symmetric functions and orthogonal polynomials, in: Dean Jacqueline B. Lewis Memorial Lectures presented at Rutgers University, New Brunswick, NJ, in: Univ. Lecture Ser., vol. 12, Amer. Math. Soc., Providence, RI, 1998.

[25] W.G. Morris, Constant term identities for finite and affine root systems: Conjectures and theorem, PhD thesis, Univ. Wisconsin, Madison, WI, 1982.

[26] A. Okounkov, BC-type interpolation Macdonald polynomials and binomial formula for Koornwinder polynomials, Transform. Groups 3 (1998) 181–207.

[27] A. Okounkov, Combinatorial formula for Macdonald polynomials and generic Macdonald polynomials, Transform. Groups 8 (2003) 293–305.

[28] L.J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, 1966.

[29] V. Tarasov, A. Varchenko, Geometry of q-hypergeometric functions, quantum affine algebras and elliptic quantum groups, Astérisque 246 (1997) 1–135.