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A new Bartholdi zeta function of a digraph II

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ABSTRACT

We introduce a new type of the Bartholdi zeta function of a digraph D. Furthermore, we define a new type of the Bartholdi L-function of D, and give a determinant expression of it. We show that this L-function of D is equal to the L-function of D defined in [H. Mizuno, I. Sato, A new Bartholdi zeta function of a digraph, Linear Algebra Appl. 423 (2007) 498–511]. As a corollary, we obtain a decomposition formula for a new type of the Bartholdi zeta function of a group covering of D by new Bartholdi L-functions of D.

1. Introduction

Graphs treated here are finite. Let $G = (V(G), E(G))$ be a connected graph (possibly multiple edges and loops) with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v. Set $R(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$. For $e = (u, v) \in R(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$.

A *path P of length n* in G is a sequence $P = (e_1, \ldots, e_n)$ of n arcs such that $e_i \in R(G)$, $t(e_i) = o(e_{i+1})(1 \le i \le n-1)$, where indices are treated mod n. Set $|P| = n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$*-path*. We say that a path $P = (e_1, \ldots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some $i(1 \le i \le n-1)$. A (v, w)-path is called a v*-cycle* (or v*-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \ldots, e_n)$ is the cycle $C^{-1} = (e_n^{-1}, \ldots, e_1^{-1})$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are said to be *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r times around a cycle B. Such a cycle is called a *power* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara zeta function* of a graph G is a function of $t \in \mathbf{C}$ with $|t|$ sufficiently small, defined by

$$\mathbf{Z}(G, t) = \mathbf{Z}_G(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G (see [7]).

Ihara zeta functions of graphs originated from Ihara zeta functions of regular graphs by Ihara [7]. Originally, Ihara presented p-adic Selberg zeta functions of discrete groups. Let Γ be a torsion-free discrete cocompact subgroup of $PGL(2, k_p)$, where k_p is a p-adic number field over a finite field. Ihara defined a zeta function associated with Γ as an analogue of the

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Selberg zeta function for a discrete cocompact subgroup of $PGL(2, \mathbf{R})$, and showed that its reciprocal is an explicit polynomial. Serre [17] pointed out that the Ihara zeta function is the zeta function of the quotient T/Γ (a finite regular graph) of the one-dimensional Bruhat–Tits building T (an infinite regular tree) associated with $GL(2, k_p)$. Furthermore, in [8], Ihara discovered an identity between the zeta function of T/Γ and a certain Shimura curve reduced modulo the prime number p.

A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [19,20]. Hashimoto [5] treated multivariable zeta functions of bipartite graphs. Bass [2] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial.

Theorem 1 (*Bass*). *Let G be a connected graph. Then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}(G, t)^{-1} = (1 - t^2)^{r-1} \det(\mathbf{I} - t\mathbf{A}(G) + t^2(\mathbf{D} - \mathbf{I})),$$

where r and $\mathbf{A}(G)$ are the Betti number and the adjacency matrix of G, respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$ where $V(G) = \{v_1, \ldots, v_n\}$.

Various proofs of Bass' theorem were given by Stark and Terras [18], Foata and Zeilberger [3], Kotani and Sunada [9], Hoffman [6] and Northshield [13].

Let G be a connected graph. We say that a path $P = (e_1, \ldots, e_n)$ has a *bump* at $t(e_i)$ if $e_{i+1} = e_i^{-1}$ $(1 \le i \le n)$. The *cyclic bump count* $cbc(\pi)$ of a cycle $\pi = (\pi_1, \ldots, \pi_n)$ is

$$cbc(\pi) = |\{i = 1, \ldots, n \mid \pi_i = \pi_{i+1}^{-1}\}|,$$

where $\pi_{n+1} = \pi_1$. Then the *Bartholdi zeta function* of G is a function of $u, t \in \mathbf{C}$ with $|u|, |t|$ sufficiently small, defined by

$$\zeta_G(u, t) = \zeta(G, u, t) = \prod_{[C]} (1 - u^{cbc(C)} t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime cycles of G (see [1]). If $u = 0$, then the Bartholdi zeta function of G is the Ihara zeta function of G.

Bartholdi [1] gave a determinant expression of the Bartholdi zeta function of a graph.

Theorem 2 (*Bartholdi*). *Let G be a connected graph with n vertices and m unoriented edges. Then the reciprocal of the Bartholdi zeta function of G is given by*

$$\zeta(G, u, t)^{-1} = (1 - (1 - u)^2 t^2)^{m-n} \det(\mathbf{I} - t\mathbf{A}(G) + (1 - u)(\mathbf{D} - (1 - u)\mathbf{I})t^2).$$

In the case of $u = 0$, Theorem 2 implies Theorem 1.

Mizuno and Sato [12] considered a new zeta function of a digraph, and defined a new zeta function of a digraph by using not an infinite product but a determinant.

Let D be a connected graph with n vertices $v_1, \ldots, v_n$ and m arcs. Then we consider an $n \times n$ matrix $\mathbf{W} = \mathbf{W}(D) = (w_{ij})_{1 \le i,j \le n}$ with the ij entry the complex variable w_{ij} if $(v_i, v_j) \in A(D)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(D)$ is called the *weighted matrix* of D. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(D)$ and $w(e) = w_{ij}$, $e = (v_i, v_j) \in A(D)$. Then $w : A(D) \longrightarrow \mathbf{C}$ is called a *weight* of D. For each path $P = (e_1, \ldots, e_r)$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P) = w(e_1) \ldots w(e_r)$.

Let D be a connected digraph with n vertices and m arcs, and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Two $m \times m$ matrices $\mathbf{B} = \mathbf{B}(D) = (\mathbf{B}_{e,f})_{e,f \in A(D)}$ and $\mathbf{J}_0 = \mathbf{J}_0(D) = (\mathbf{J}_{e,f})_{e,f \in A(D)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \qquad \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then a *weighted Bartholdi zeta function* of D is defined by

$$\zeta_1(D, w, u, t) = \det(\mathbf{I}_n - t(\mathbf{B} - (1 - u)\mathbf{J}_0))^{-1}.$$

If $w = \mathbf{1}$, i.e., $w(e) = 1$ for any $e \in A(D)$, then the weighted Bartholdi zeta function of D is the Bartholdi zeta function of D (see [11]). If $u = 0$ and $D = D_G$ is the symmetric digraph corresponding to a graph G, then the weighted Bartholdi zeta function of D is the zeta function $\mathbf{Z}_1(G, w, t)$ of G (see [15]). Furthermore, in the case of $D = D_G$, we have $\zeta_1(D_G, \mathbf{1}, u, t) = \zeta(G, u, t)$ and $\zeta_1(D_G, \mathbf{1}, 0, t) = \mathbf{Z}(G, t)$.

We define two $n \times n$ matrices $\mathbf{W}_1 = \mathbf{W}_1(D) = (a_{uv})$ and $\mathbf{W}_0$ as follows:

$$a_{uv} = \begin{cases} w(u, v) & \text{if both } (u, v) \text{ and } (v, u) \in A(D), \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{W}_0 = \mathbf{W}_0(D) = \mathbf{W}(D) - \mathbf{W}_1.$$

Let $V(D) = \{v_1, \ldots, v_n\}$. Then an $n \times n$ matrix $\mathbf{S} = (s_{ij})$ is the diagonal matrix defined by

$$s_{ii} = \sum_{e, e^{-1} \in A(D); o(e)=v_i} w(e).$$

Set

$$s(v_i) = s_{ii}, \quad 1 \le i \le n.$$

Theorem 3 (*Mizuno and Sato*). *Let D be a connected digraph, and let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Furthermore, let $m_1 = |\{e \in A(D) \mid e^{-1} \in A(D)\}|/2$. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\zeta_1(D, w, u, t)^{-1} = (1 - (1-u)^2 t^2)^{m_1 - n} \times \det(\mathbf{I}_n - t\mathbf{W}_1(D) - (1 - (1-u)^2 t^2) t \mathbf{W}_0(D) + (1-u) t^2 (\mathbf{S} - (1-u)\mathbf{I}_n)).$$

where $n = |V(D)|$.

In Section 2, we define a new type of the Bartholdi zeta function of a digraph D, and give a decomposition formula of a new type of the Bartholdi zeta function of a group covering of D. In Section 3, we define a new type of the Bartholdi L-function of D, and present a determinant expression for a new type of the Bartholdi L-function of D. Furthermore, we show that this L-function of D is equal to the L-function of D defined in [12]. As a corollary, we show that a new type of the Bartholdi zeta function of a group covering of D is a product of new Bartholdi L-functions of D.

For a general theory of the representation of groups and graph coverings, the reader is referred to [16,4], respectively.

2. New Bartholdi zeta functions of digraphs

We consider a new zeta function of a digraph, and define a new zeta function of a digraph by using not an infinite product but a determinant.

Let D be a connected graph with n vertices $v_1, \ldots, v_n$ and m arcs, and let $w : A(D) \longrightarrow \mathbf{C}$ be a weight of D.

Let $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. An $m \times m$ matrix $\mathbf{B}' = \mathbf{B}'(D) = (\mathbf{B}'_{e,f})_{e,f \in A(D)}$ is defined as follows:

$$\mathbf{B}'_{e,f} = \begin{cases} w(e) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then a *weighted Bartholdi zeta function* of D is defined by

$$\zeta_2(D, w, u, t) = \det(\mathbf{I}_n - t(\mathbf{B}' - (1-u)\mathbf{J}_0))^{-1}.$$

We can generalize the notion of a Γ-covering of a graph to a simple digraph. Let D be a connected digraph and Γ a finite group. Then a mapping $\alpha : A(D) \longrightarrow \Gamma$ is called a *pseudo-ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in A(D)$ such that $(v, u) \in A(D)$. The pair (D, α) is called an *ordinary voltage digraph*. The *derived digraph D^α* of the ordinary voltage digraph (D, α) is defined as follows: $V(D^\alpha) = V(D) \times \Gamma$ and $((u, h), (v, k)) \in A(D^\alpha)$ if and only if $(u, v) \in A(D)$ and $k = h\alpha(u, v)$. The digraph D^α is called a *Γ-covering* of D. Note that a Γ-covering of the symmetric digraph corresponding to a graph G is a Γ-covering of G (c.f., [4]).

Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment. In the Γ-covering D^α, set $v_g = (v, g)$ and $e_g = (e, g)$, where $v \in V(D), e \in A(D), g \in \Gamma$. For $e = (u, v) \in A(D)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(D)$ be a weighted matrix of D. Then we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(D^\alpha) = (\tilde{w}(u_g, v_h))$ of D^α *derived from* $\mathbf{W}$ as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u, v) & \text{if } (u, v) \in A(D) \text{ and } h = g\alpha(u, v), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$ be the block diagonal sum of square matrices $\mathbf{M}_1, \ldots, \mathbf{M}_s$. If $\mathbf{M}_1 = \mathbf{M}_2 = \cdots = \mathbf{M}_s = \mathbf{M}$, then we write $s \circ \mathbf{M} = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_s$. The *Kronecker product* $\mathbf{A} \bigotimes \mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

We give a decomposition formula for the weighted Bartholdi zeta function ζ_2 of a group covering of a digraph D.

Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment. Let $w : A(D) \longrightarrow \mathbf{C}$ be a weight of D. Then two matrices $\tilde{\mathbf{B}} = \mathbf{B}(D^\alpha) = (\tilde{b}(e_g, f_h))$ and $\tilde{\mathbf{J}} = \mathbf{J}(D^\alpha) = (\tilde{c}(e_g, f_h))$ of D^α are given by

$$\tilde{b}(e_g, f_h) := \begin{cases} w(e) & \text{if } t(e_g) = o(f_h), \\ 0 & \text{otherwise,} \end{cases} \qquad \tilde{c}(e_g, f_h) := \begin{cases} 1 & \text{if } e_g^{-1} = f_h, \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, let the matrix $\mathbf{B}_g = (b_{ef}^{(g)})$ be defined by

$$b_{ef}^{(g)} := \begin{cases} w(e) & \text{if } \alpha(e) = g \text{ and } t(e) = o(f), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let the matrix $\mathbf{J}_g = (c_{ef}^{(g)})$ be defined by

$$c_{ef}^{(g)} := \begin{cases} 1 & \text{if } \alpha(e) = g \text{ and } e^{-1} = f, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4. *Let D be a connected graph with l arcs, $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Suppose that the Γ-covering D^α of D is connected. Then the reciprocal of the weighted Bartholdi zeta function of D^α is*

$$\zeta_2(D^\alpha, \tilde{w}, u, t)^{-1} = \zeta_2(D, w, u, t)^{-1} \cdot \prod_{i=2}^{k} \det\left(\mathbf{I}_{lf_i} - t \sum_{h \in \Gamma} \rho_i(h) \bigotimes (\mathbf{B}_h - (1-u)\mathbf{J}_h)\right)^{f_i}.$$

Proof. Let $A(D) = \{e_1, \ldots, e_l\}$ and $\Gamma = \{1 = g_1, g_2, \ldots, g_m\}$. Arrange arcs of D^α in m blocks: $(e_1, 1), \ldots, (e_l, 1); (e_1, g_2), \ldots, (e_l, g_2); \cdots; (e_1, g_m), \ldots, (e_l, g_m)$. We consider the matrix $\tilde{\mathbf{B}} - \tilde{\mathbf{J}}$ under this order. For $h \in \Gamma$, let $\mathbf{P}_h = (p_{ij}^{(h)})$ be the permutation matrix of h. Suppose that $p_{ij}^{(h)} = 1$, i.e., $g_j = g_i h$. Then $t(e, g_i) = o(f, g_j)$ if and only if $t(e) = o(f)$ and $(o(f), g_j) = o(f, g_j) = t(e, g_i) = (t(e), g_i\alpha(e))$, i.e., $\alpha(e) = g_i^{-1} g_j = g_i^{-1} g_i h = h$. Thus we have

$$\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}} = \sum_{h \in \Gamma} \mathbf{P}_h \bigotimes (\mathbf{B}_h - (1-u)\mathbf{J}_h).$$

Let ρ be the right regular representation of Γ. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be all inequivalent irreducible representations of Γ, and f_i the degree of ρ_i for each i, where $f_1 = 1$. Then we have $\rho(h) = \mathbf{P}_h$ for $h \in \Gamma$. Furthermore, there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\rho(h)\mathbf{P} = (1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)$ for each $h \in \Gamma$ (see [16]). Putting $\mathbf{F} = (\mathbf{P}^{-1} \bigotimes \mathbf{I}_l)(\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}})(\mathbf{P} \bigotimes \mathbf{I}_l)$, we have

$$\mathbf{F} = \sum_{h \in \Gamma} \{(1) \oplus f_2 \circ \rho_2(h) \oplus \cdots \oplus f_k \circ \rho_k(h)\} \bigotimes (\mathbf{B}_h - (1-u)\mathbf{J}_h).$$

Note that $\mathbf{B}' - (1-u)\mathbf{J}_0 = \sum_{h \in \Gamma} (\mathbf{B}_h - (1-u)\mathbf{J}_h)$ and $1 + f_2^2 + \cdots + f_k^2 = m$. Therefore it follows that

$$\begin{aligned} \zeta_2(D^\alpha, \tilde{w}, u, t)^{-1} &= \det(\mathbf{I}_{lm} - t(\tilde{\mathbf{B}} - (1-u)\tilde{\mathbf{J}})) \\ &= \det(\mathbf{I}_l - t(\mathbf{B}' - (1-u)\mathbf{J}_0)) \prod_{i=2}^{k} \det\left(\mathbf{I}_{lf_i} - t \sum_{h} \rho_i(h) \bigotimes (\mathbf{B}_h - (1-u)\mathbf{J}_h)\right)^{f_i}. \quad \square \end{aligned}$$

3. *L*-functions of digraphs

Let D be a connected graph with n vertices and l arcs, $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment. For each path $P = (e_1, \ldots, e_r)$ of G, set $\alpha(P) = \alpha(e_1) \cdots \alpha(e_r)$. This is called the *net voltage* of P. Furthermore, let ρ be a unitary representation of Γ and d its degree.

The *L-function* of D associated with ρ and α is defined by

$$\zeta_2(D, w, u, t, \rho, \alpha) = \det\left(\mathbf{I}_{ld} - t \sum_{h \in \Gamma} \rho(h) \bigotimes (\mathbf{B}_h - (1-u)\mathbf{J}_h)\right)^{-1}.$$

If $\rho = 1$(the identity representation of Γ), then the L-function of D is the weighted Bartholdi zeta function $\zeta_2(D, w, u, t)$ of D.

Let $1 \le i, j \le n$. Then, the (i, j)-block $\mathbf{F}_{i,j}$ of a $dn \times dn$ matrix $\mathbf{F}$ is the submatrix of $\mathbf{F}$ consisting of $d(i-1)+1, \ldots, di$ rows and $d(j-1)+1, \ldots, dj$ columns. Two $ld \times ld$ matrices $\mathbf{B}_\rho = ((\mathbf{B}_\rho)_{e,f})_{e,f \in A(D)}$ and $\mathbf{J}_\rho = ((\mathbf{J}_\rho)_{e,f})_{e,f \in A(D)}$ are defined as follows:

$$(\mathbf{B}_\rho)_{e,f} = \begin{cases} w(e)\rho(\alpha(e)) & \text{if } t(e) = o(f), \\ \mathbf{0}_d & \text{otherwise,} \end{cases} \qquad (\mathbf{J}_\rho)_{e,f} = \begin{cases} \rho(\alpha(e)) & \text{if } f = e^{-1}, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, the matrix $\mathbf{W}_{0,g} = (a_{uv}^{(g)})$ is defined as follows:

$$a_{uv}^{(g)} := \begin{cases} w(u, v) & \text{if } (u, v) \in A(D), (v, u) \notin A(D) \text{ and } \alpha(u, v) = g, \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in \Gamma$, the matrix $\mathbf{W}_{1,g} = (b_{uv}^{(g)})$ is defined as follows:

$$b_{uv}^{(g)} := \begin{cases} w(u,v) & \text{if } (u,v), (v,u) \in A(D) \text{ and } \alpha(u,v) = g, \\ 0 & \text{otherwise.} \end{cases}$$

A determinant expression for the L-function of D associated with ρ and α is given as follows.

Theorem 5. *Let D be a connected digraph with ν vertices and ϵ arcs, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Set $\epsilon_1 = |\{e \in A(D) \mid \mathrm{e}^{-1} \in A(D)\}|/2$. Furthermore, let ρ be a representation of Γ, and d the degree of ρ. Then the reciprocal of the L-function of D associated with ρ and α is*

$$\zeta_2(D, w, u, t, \rho, \alpha)^{-1} = \det(\mathbf{I}_{\epsilon d} - (\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)t) = (1-(1-u)^2t^2)^{(\epsilon_1-\nu)d} \det\Bigg(\mathbf{I}_{\nu d} - t\sum_{h\in\Gamma}\rho(h)\bigotimes \mathbf{W}_{1,h}$$
$$- (1-(1-u)^2t^2)t\sum_{h\in\Gamma}\rho(h)\bigotimes\mathbf{W}_{0,h} + (1-u)t^2\left(\mathbf{I}_d\bigotimes(\mathbf{S}-(1-u)\mathbf{I}_\nu)\right)\Bigg).$$

Proof. The argument is an analogue of Bass' method [2].

At first, since $\mathbf{B}_\rho = \sum_{g\in\Gamma}\mathbf{B}_g \bigotimes \rho(g)$ and $\mathbf{J}_\rho = \sum_{g\in\Gamma}\mathbf{J}_g \bigotimes \rho(g)$, we have

$$\det(\mathbf{I}_{\epsilon d} - (\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho)t) = \det\left(\mathbf{I}_{\epsilon d} - t\sum_{g\in\Gamma}\rho(g)\bigotimes(\mathbf{B}_g - (1-u)\mathbf{J}_g)\right).$$

Let $V(D) = \{v_1, \ldots, v_\nu\}$ and, let $A(D) = \{e_1, \ldots, e_{\epsilon_0}, e_{\epsilon_0+1}, \ldots, e_{\epsilon_0+\epsilon_1}, e_{\epsilon_0+\epsilon_1+1}, \ldots, e_{\epsilon_0+2\epsilon_1}\}$ such that $\mathrm{e}_i^{-1} \notin A(D)$ for $1 \le i \le \epsilon_0$ and $e_{\epsilon_0+\epsilon_1+j} = \mathrm{e}_{\epsilon_0+j}^{-1}$ for $1 \le j \le \epsilon_1$. Note that $\epsilon = \epsilon_0 + 2\epsilon_1$.

Let $\mathbf{K} = (\mathbf{K}_{i,j})_{1\le i\le\epsilon; 1\le j\le\nu}$ be the $\epsilon d \times \nu d$ matrix defined by

$$\mathbf{K}_{i,j} := \begin{cases} w(e_i)\rho(\alpha(e_i))\,\mathbf{I}_d & \text{if } t(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Define the $\epsilon d \times \nu d$ matrix $\mathbf{L} = (\mathbf{L}_{i,j})_{1\le i\le\epsilon; 1\le j\le\nu}$ by

$$\mathbf{L}_{i,j} := \begin{cases} \mathbf{I}_d & \text{if } o(e_i) = v_j, \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Then we have

$$\mathbf{K}^t\mathbf{L} = \mathbf{B}_\rho = \sum_{g\in\Gamma}\mathbf{B}_g\bigotimes\rho(g) \tag{1}$$

and

$${}^t\mathbf{L}\mathbf{K} = \sum_{g\in\Gamma}(\mathbf{W}_{0,g} + \mathbf{W}_{1,g})\bigotimes\rho(g) = \mathbf{W}_{0,\rho} + \mathbf{W}_{1,\rho}, \tag{2}$$

where

$$\mathbf{W}_{i,\rho} = \sum_{g\in\Gamma}\mathbf{W}_{i,g}\bigotimes\rho(g) \quad \text{for } i = 0, 1.$$

Let $\mathbf{H} = (\mathbf{H}_{i,j})_{1\le i\le\epsilon; 1\le j\le\nu}$ be the $\epsilon d \times \nu d$ matrix:

$$\mathbf{H}_{i,j} := \begin{cases} (1-u)t\mathbf{I}_d & \text{if } o(e_i) = v_j \text{ and } \mathrm{e}_i^{-1} \notin A(D), \\ \rho(\alpha(e_i)) & \text{if } t(e_i) = v_j \text{ and } \mathrm{e}_i^{-1} \in A(D), \\ \mathbf{0}_d & \text{otherwise.} \end{cases}$$

Then we have

$${}^t\overline{\mathbf{H}}\mathbf{K} = \mathbf{S}\bigotimes\mathbf{I}_d + (1-u)t\mathbf{W}_{0,\rho}, \tag{3}$$

where ${}^t\overline{\mathbf{H}}$ is the conjugate transpose of $\mathbf{H}$.

Now, let

$$\mathbf{M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & w(f_1^{-1})\rho(\alpha(f_1)) \oplus \cdots \oplus w(f_{\epsilon_1}^{-1})\rho(\alpha(f_{\epsilon_1})) \\ \mathbf{0} & w(f_1)\rho(\alpha(f_1))^{-1} \oplus \cdots \oplus w(f_{\epsilon_1})\rho(\alpha(f_{\epsilon_1}))^{-1} & \mathbf{0} \end{bmatrix}$$

and

$$\mathbf{N} = \mathbf{B}_\rho - \mathbf{M},$$

where $f_i = e_{\epsilon_0+i}$ for $1 \le i \le \epsilon_1$. Furthermore, let

$$\mathbf{M}_0 = ((1-u)t\mathbf{I}_{\epsilon_0 d} \oplus \mathbf{0}_{2\epsilon_1 d}) + \mathbf{J}_\rho.$$

Then we have

$$\mathbf{K}^t\overline{\mathbf{H}} = \mathbf{N}\mathbf{M}_0 + (\mathbf{0}_{\epsilon_0 d} \oplus w(e_{\epsilon_0+1})\mathbf{I}_d \oplus \cdots \oplus w(e_{2\epsilon_1})\mathbf{I}_d). \tag{4}$$

We introduce two $(\epsilon+\nu)d \times (\epsilon+\nu)d$ matrices as follows:

$$\mathbf{P} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\nu d} & -{}^t\mathbf{L} + (1-u)t^t\overline{\mathbf{H}} \\ \mathbf{0} & \mathbf{I}_{\epsilon d} \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} \mathbf{I}_{\nu d} & {}^t\mathbf{L} - (1-u)t^t\overline{\mathbf{H}} \\ t\mathbf{K} & (1-(1-u)^2t^2)\mathbf{I}_{\epsilon d} \end{bmatrix}.$$

By (2) and (3), we have

$$\mathbf{PQ} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\nu d} - t\,{}^t\mathbf{L}\mathbf{K} + (1-u)t^2\,{}^t\overline{\mathbf{H}}\mathbf{K} & \mathbf{0} \\ t\mathbf{K} & (1-(1-u)^2t^2)\mathbf{I}_{\epsilon d} \end{bmatrix}$$
$$= \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\nu d} - t(\mathbf{W}_{1,\rho} + \mathbf{W}_{0,\rho}) + (1-u)t^2\left(\mathbf{S}\bigotimes\mathbf{I}_d + (1-u)t\,\mathbf{W}_{0,\rho}\right) & \mathbf{0} \\ t\mathbf{K} & (1-(1-u)^2t^2)\mathbf{I}_{\epsilon d} \end{bmatrix}.$$

Furthermore,

$$\mathbf{QP} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\nu d} & \mathbf{0} \\ t(1-(1-u)^2t^2)\mathbf{K} & -t\mathbf{K}^t\mathbf{L} + (1-u)t^2\mathbf{K}^t\overline{\mathbf{H}} + (1-(1-u)^2t^2)\mathbf{I}_{\epsilon d} \end{bmatrix}.$$

Note that

$$\mathbf{M}\mathbf{M}_0 = \mathbf{0}_{\epsilon_0 d} \oplus w(e_{\epsilon_0+1})\mathbf{I}_d \oplus \cdots \oplus w(e_{2\epsilon_1})\mathbf{I}_d$$

and

$$\mathbf{J}_\rho\mathbf{M}_0 = \mathbf{0}_{\epsilon_0 d} \oplus \mathbf{I}_{2\epsilon_1 d},$$

By (1) and (4), we have

$$-t\mathbf{K}^t\mathbf{L} + (1-u)t^2\mathbf{K}^t\overline{\mathbf{H}} + (1-(1-u)^2t^2)\mathbf{I}_{\epsilon d}$$
$$= \mathbf{I}_{\epsilon d} - t(\mathbf{N}+\mathbf{M}) + (1-u)t^2(\mathbf{N}\mathbf{M}_0 + \mathbf{M}\mathbf{M}_0) - (1-u)t(\mathbf{M}_0 - \mathbf{J}_\rho) - (1-u)^2t^2\,\mathbf{J}_\rho\mathbf{M}_0$$
$$= (\mathbf{I}_{\epsilon d} - t(\mathbf{N}+\mathbf{M}-(1-u)\mathbf{J}_\rho))(\mathbf{I}_{\epsilon d} - (1-u)t\mathbf{M}_0).$$

Thus,

$$\mathbf{QP} = \begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\nu d} & \mathbf{0} \\ t(1-(1-u)^2t^2)\mathbf{K} & (\mathbf{I}_{\epsilon d} - t(\mathbf{N}+\mathbf{M}-(1-u)\mathbf{J}_\rho))(\mathbf{I}_{\epsilon d} - (1-u)t\,\mathbf{M}_0) \end{bmatrix}.$$

Since $\det(\mathbf{PQ}) = \det(\mathbf{QP})$, we have

$$(1-(1-u)^2t^2)^{\epsilon d}\det\left(\mathbf{I}_{\nu d} - t\,\mathbf{W}_{1,\rho} - (1-(1-u)^2t^2)t\mathbf{W}_{0,\rho} + (1-u)\left(\mathbf{S}\bigotimes\mathbf{I}_d - (1-u)\mathbf{I}_{\nu d}\right)t^2\right)$$
$$= (1-(1-u)^2t^2)^{\nu d}\det(\mathbf{I}_{\epsilon d} - t(\mathbf{B}_\rho - (1-u)\mathbf{J}_\rho))\det(\mathbf{I}_{\epsilon d} - (1-u)t\,\mathbf{M}_0).$$

Now,

$$\det(\mathbf{I}_{\epsilon d} - (1-u)t\,\mathbf{M}_0)$$
$$= \det\left(\begin{bmatrix} \mathbf{I}_{\epsilon_0 d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\epsilon_1 d} & (1-u)t\{\rho(\alpha(f_1)) \oplus \cdots \oplus \rho(\alpha(f_{\epsilon_1}))\} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\epsilon_1 d} \end{bmatrix}\right)\det\Big((1-(1-u)^2t^2)\mathbf{I}_{\epsilon_0 d}$$
$$\oplus \begin{bmatrix} \mathbf{I}_{\epsilon_1 d} & -(1-u)t\{\rho(\alpha(f_1)) \oplus \cdots \oplus \rho(\alpha(f_{\epsilon_1}))\} \\ -(1-u)t\{\rho(\alpha(f_1))^{-1} \oplus \cdots \oplus \rho(\alpha(f_{\epsilon_1}))^{-1}\} & \mathbf{I}_{\epsilon_1 d} \end{bmatrix}\Big)$$
$$= \det\left(\begin{bmatrix} (1-(1-u)^2t^2)\mathbf{I}_{\epsilon_0 d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (1-(1-u)^2t^2)\mathbf{I}_{\epsilon_1 d} & \mathbf{0} \\ \mathbf{0} & * & \mathbf{I}_{\epsilon_1 d} \end{bmatrix}\right) = (1-(1-u)^2t^2)^{(\epsilon_0+\epsilon_1)d}.$$

Therefore it follows that

$$(1-(1-u)^2t^2)^{\epsilon d}\det\left(\mathbf{I}_{\nu d}-t\,\mathbf{W}_{1,\rho}-(1-(1-u)^2t^2)t\mathbf{W}_{0,\rho}+(1-u)\left(\mathbf{S}\bigotimes\mathbf{I}_d-(1-u)\mathbf{I}_{\nu d}\right)t^2\right)$$
$$=(1-(1-u)^2t^2)^{(\epsilon_0+\epsilon_1+\nu)d}\det(\mathbf{I}_{\epsilon d}-t(\mathbf{B}_\rho-(1-u)\mathbf{J}_\rho)).$$

Hence

$$\det(\mathbf{I}_{\epsilon d}-t(\mathbf{B}_\rho-(1-u)\mathbf{J}_\rho))=(1-(1-u)^2t^2)^{(\epsilon_1-\nu)d}\det\left(\mathbf{I}_{\nu d}-t\sum_{g\in\Gamma}\rho(g)\bigotimes\mathbf{W}_{1,g}\right.$$
$$\left.-(1-(1-u)^2t^2)t\sum_{g\in\Gamma}\rho(g)\bigotimes\mathbf{W}_{0,g}+(1-u)t^2\left(\mathbf{I}_d\bigotimes(\mathbf{S}-(1-u)\mathbf{I}_\nu)\right)\right).\quad\square$$

By Theorem 5 and [12, Theorem 8], the following result holds.

Corollary 1. *Let D be a connected digraph, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Furthermore, let ρ be a representation of Γ. Then the L-function of D is equal to that of D defined in [12]:*

$$\zeta_2(D,w,u,t,\rho,\alpha)=\zeta_1(D,w,u,t,\rho,\alpha).$$

If $\rho = 1$ then by Theorems 3 and 5, we have the following result.

Corollary 2. *Let D be a connected digraph with n vertices, and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Set $m_1 = |\{e \in A(D) \mid e^{-1} \in A(D)\}|/2$. Then the reciprocal of the weighted Bartholdi zeta function of D is given by*

$$\begin{aligned}\zeta_2(D,w,u,t)^{-1}&=(1-(1-u)^2t^2)^{m_1-n}\\&\quad\times\det(\mathbf{I}_n-t\mathbf{W}_1(D)-(1-(1-u)^2t^2)t\mathbf{W}_0(D)+(1-u)t^2(\mathbf{S}-(1-u)\mathbf{I}_n))\\&=\zeta_1(D,w,u,t)^{-1}.\end{aligned}$$

By Theorems 4 and 5, the following result holds.

Corollary 3. *Let D be a connected digraph, Γ a finite group, $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment and $\mathbf{W} = \mathbf{W}(D)$ a weighted matrix of D. Then we have*

$$\zeta_1(D^\alpha,\tilde{w},u,t)=\zeta_2(D^\alpha,\tilde{w},u,t)=\prod_\rho\zeta_2(D,w,u,t,\rho,\alpha)^{\deg\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

In the case that $w(e) = 1$ for each $e \in A(D)$, we obtain a decomposition formula for the Bartholdi zeta function of a group covering of a digraph by Sato [14].

Corollary 4 (Sato). *Let D be a connected digraph, Γ a finite group and $\alpha : A(D) \longrightarrow \Gamma$ a pseudo-ordinary voltage assignment. Suppose that the Γ-covering D^α of D is connected. Then we have*

$$\zeta(D^\alpha,u,t)=\prod_\rho\zeta_D(u,t,\rho,\alpha)^{\deg\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

If $u = 0$ and $D = D_G$ is the symmetric digraph corresponding to a graph G, then, we obtain a decomposition formula for the zeta function of a regular covering of a graph by Sato [15].

Corollary 5 (Sato). *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\mathbf{Z}_1(G^\alpha,\tilde{w},t)=\prod_\rho\mathbf{Z}_1(G,w,t,\rho,\alpha)^{\deg\rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

If $w = \mathbf{1}$ and $D = D_G$, then we obtain a decomposition formula for the Bartholdi zeta function of a regular covering of a graph G (see [10]).

Corollary 6 (*Mizuno and Sato*). *Let G be a connected graph, Γ a finite group and $\alpha : R(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\zeta(G^{\alpha}, u, t) = \prod_{\rho} \zeta(G, u, t, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

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