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Note

A note on the chromatic number of a dense random graph

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ABSTRACT

Let $G_{n,p}$ denote the random graph on n labeled vertices, where each edge is included with probability p independent of the others. We show that for all constant p

$$\chi(G_{n,p}) \geq \frac{n}{2\log_{\frac{1}{1-p}} n - 2\log_{\frac{1}{1-p}} \log_{\frac{1}{1-p}} n - 2\log_{\frac{1}{1-p}} 2 + o(1)}$$

holds with high probability. This improves the best known lower bound for the chromatic number of dense random graphs and shows in particular that the estimate $\chi(G_{n,p}) \geq \frac{n}{\alpha(G_{n,p})}$, where α denotes the independence number, is not tight.

1. Introduction

Let $G_{n,p}$ denote the random graph on n labeled vertices, where each edge is included independently of the others with probability p. The *chromatic number* $\chi(G_{n,p})$, which is the minimum number of colors needed to color the vertices of the graph $G_{n,p}$, such that no two adjacent vertices are colored the same, is one of the most studied parameters in the theory of random graphs, see e.g. [8,11,3,12].

While it is easy to show that the chromatic number of a random graph $G_{n,p}$ satisfies with high probability[1] $n/(2\log_b n) \leq \chi(G_{n,p}) \leq (n + o(n))/\log_b n$, where we shall abbreviate in the remainder of the paper $b = 1/(1-p)$, the asymptotic gap between lower and upper bound was only closed in 1988 in a celebrated paper by Bollobás [3], who proved that the correct asymptotic answer is given by the lower bound. Here we state a theorem by McDiarmid, who slightly refined Bollobás's proof of the upper bound.

Theorem 1.1 (*[6,12]*). *Let $0 < p < 1$ be fixed and denote by $\alpha(G_{n,p})$ the independence number of the random graph $G_{n,p}$. Define the auxiliary function*

$$r_p(n) = 2\log_b n - 2\log_b \log_b n + 2\log_b(e/2),$$

where $b = \frac{1}{1-p}$. With high probability it holds that

$$\left\lfloor r_p(n) + 1 - \frac{2\log_b \log_b n}{\log_b n} \right\rfloor \leq \alpha(G_{n,p}) \leq \left\lfloor r_p(n) + 1 + \frac{2\log_b \log_b n}{\log_b n} \right\rfloor$$

and

$$\frac{n}{r_p(n) + 1 + o(1)} \leq \chi(G_{n,p}) \leq \frac{n}{r_p(n) - \frac{1}{2} - \frac{1}{1-\sqrt{1-p}} + o(1)}.$$

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[1] We shall say that a sequence of events $\mathcal{E}_n$ occurs *with high probability (whp)*, if $\lim_{n\to\infty} \Pr[\mathcal{E}_n] = 1$.

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The lower bound in the above theorem is obtained by the estimate $\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil$, which is valid for all graphs $G = (V, E)$. The upper bound is much harder to obtain – in order to prove it, Bollobás showed that in *every* sufficiently large subset of the vertices of the random graph, an independent set of appropriate size can be found with high probability.

The bounds given in Theorem 1.1 characterize precisely the order of magnitude of the chromatic number of the random graph. On the other hand, if we subtract the lower bound from the upper bound, an easy calculation yields that the remaining gap is of order $\Theta\left(n(\log n)^{-2}\right)$. This shows that the bounds from Theorem 1.1 – that are still the best known bounds – are far from best possible, as it was shown by Shamir and Spencer [13] that the chromatic number of the random graph is concentrated in an interval of size roughly $\mathcal{O}(\sqrt{n})$. More precisely, they showed that

$$\Pr\left[\left|\chi(G_{n,p}) - \mathbb{E}\left[\chi(G_{n,p})\right]\right| \geq \omega_n \cdot \sqrt{n}\right] \leq \mathrm{e}^{-\Theta(\omega_n^2)},$$

where ω_n is a function such that $\omega_n \to \infty$ when $n \to \infty$. Therefore, we know that the chromatic number takes with high probability less than $2\omega_n\sqrt{n}$ different values. In fact, in a recent paper Bollobás [5] asks about the *concentration* of the chromatic number and speculates that it might well be possible that the chromatic number of a random graphs $G_{n,p}$ is concentrated within an interval of size $\log n$ or less.

Note that the above results concern dense random graphs, i.e. the case that the edge probability p is a fixed constant. For sparse random graphs much more is known. For example, Achlioptas and Naor [1] proved that for every $d > 0$, the chromatic number of the random graph $G_{n,d/n}$ is with high probability either k or $k+1$, where k is the smallest integer such that $2k \log k > d$. For slightly larger $p = p(n)$ it is known from work of Łuczak [10] and Alon and Krivelevich [2] that for all $p \ll n^{-1/2}$ the chromatic number of $G_{n,p}$ is concentrated on an interval of length two, without specifying, however, where this interval might be located. For $p \leq n^{-3/4-\delta}$, where $\delta > 0$ is an arbitrary constant, Coja-Oghlan, Panagiotou, and Steger [7] determined a set of three consecutive integers that contain this interval.

When p is very small, for instance $\frac{d}{n}$, where $d > 1$, it can easily be seen that the general lower bound $\lceil n/\alpha(G_{n,p}) \rceil$ *cannot* be tight. Indeed, note that $G_{n,d/n}$ has whp roughly $\mathrm{e}^{-d}n$ isolated vertices, and it thus follows that the largest independent set in $G_{n,d/n}$ contains substantially more vertices than the largest independent set in the remaining graph. The lower bound $\lceil |V(G)|/\alpha(G) \rceil$ for the chromatic number thus fails to give a satisfactory estimate in this case. On the other hand, such a situation does not occur in the case of dense random graphs: due to the result of Bollobás [3], *any* sufficiently large subset of the vertices of $G_{n,p}$ has a large independent set. It therefore seems quite plausible that the bound $\lceil |V(G)|/\alpha(G) \rceil$ actually does provide a good lower bound in the case of dense random graphs. In this paper we show that this, however, is not true.

More precisely, in this work we improve the lower bound of Theorem 1.1, which in turn implies that the estimate $\chi(G) \geq \lceil |V(G)|/\alpha(G) \rceil$ is *not tight* for dense random graphs.

Theorem 1.2. *Let $0 < p < 1$ be fixed and let $b = \frac{1}{1-p}$. With high probability it holds that*

$$\chi(G_{n,p}) \geq \frac{n}{2\log_b n - 2\log_b \log_b n - 2\log_b 2 + o(1)} = \frac{n}{r_p(n) - \log_b e + o(1)}, \tag{1.1}$$

where $r_p(n)$ is defined in Theorem 1.1.

Note that although Theorem 1.2 narrows the gap between the known bounds for the values that the chromatic number of the random graph can take with high probability, the size of this gap is still of order $\Theta\left(n(\log n)^{-2}\right)$. It therefore remains an interesting question to investigate if the bounds in Theorem 1.1 can be further sharpened, cf. also Bollobás [5].

2. The proof

In order to prove Theorem 1.2 we apply the first moment method, which states that if a random variable X attains only non-negative values, then $\Pr[X > 0] \leq \mathbb{E}[X]$. We refer the reader to the excellent books [4,9] for a general exposition and numerous applications of the first moment method. In our application, X will be the number of ℓ-partitions of $G_{n,p}$ that induce valid colorings, for some $\ell = \ell(n)$. The main part of the proof consists of carefully estimating the asymptotic value of $\mathbb{E}[X]$ so that we can then show that this value tends to zero for some specifically chosen value of $\ell = \ell(n)$.

In the following we shall assume without loss of generality that $\ell = o(n)$. We can write any partition that induces a valid coloring as a partition $(V_1, \ldots, V_\ell)$ of the vertex set of $G_{n,p}$ with the property that $V_i \cap V_j = \emptyset$ for all $i \neq j$ and $V_i \neq \emptyset$ for all i. Note that with this notation, every coloring corresponds to precisely $\ell!$ distinct partitions (as there are precisely $\ell!$ ways to permute the colors). Now, by setting $n_i := |V_i|$ for $1 \leq i \leq \ell$, the expectation of X can be written as

$$\mathbb{E}[X] = \frac{1}{\ell!} \cdot \sum_{\substack{n_1+\cdots+n_\ell=n \\ \forall i: n_i > 0}} \binom{n}{n_1, \ldots, n_\ell} \cdot (1-p)^{\sum_{i=1}^{\ell}\binom{n_i}{2}}, \tag{2.1}$$

where the multinomial coefficient counts the number of partitions with the given sizes of the (color) classes, the exponent of $(1-p)$ counts the number of edges which cannot be included in the graph, as they would connect two vertices belonging to the same independent set, and finally the $\ell!$ accounts for the fact that in the sum we have counted each coloring (resp. *ordered* partition) for all permutations of the colors.

Let $\beta = \beta(n, \ell) = \lfloor \frac{n}{\ell} \rfloor$, and let $r = r(n, \ell) = n - \beta\ell$ denote the remainder of n and ℓ. Moreover, let

$$\mathcal{D}(n, \ell) = \left\{ (d_1, \ldots, d_\ell) \,\middle|\, \sum_{i=1}^{\ell} d_i = r \text{ and } \forall 1 \le i \le \ell : 0 < d_i + \beta \le n \right\}.$$

Note that the d_i are allowed to obtain negative values in the above definition. For a given partition $(V_1, \ldots, V_\ell)$, write $n_i = \beta + d_i$ and let $\mathbf{d} = (d_1, \ldots, d_\ell)$. Observe that this mapping is a bijection between the sets $\mathcal{D}(n, \ell)$ and $\{(n_1, \ldots, n_\ell) \mid \sum_{i=1}^{\ell} n_i = n \text{ and } n_i > 0\}$. With this notation, as

$$\sum_{1\le i\le \ell} \binom{n_i}{2} = \sum_{1\le i\le \ell} \binom{\beta + d_i}{2} = \frac{1}{2}\left(-n + \sum_{1\le i\le \ell} (\beta^2 + 2\beta d_i + d_i^2)\right) = -\frac{n}{2} + \frac{n^2 - r^2}{2\ell} + \frac{1}{2}\sum_{1\le i\le \ell} d_i^2,$$

we can rewrite (2.1) as

$$\mathbb{E}[X] = \frac{1}{\ell!} \sum_{\mathbf{d}\in\mathcal{D}(n,\ell)} \binom{n}{\beta + d_1, \ldots, \beta + d_\ell} \cdot (1-p)^{\frac{n^2-r^2}{2\ell} - \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{\ell} d_i^2}. \tag{2.2}$$

In order to obtain an upper bound for the above expression, we first observe that the multinomial coefficient is maximized for $\mathbf{d}^* = (1, \ldots, 1, 0, \ldots, 0)$, which has ones at the first r positions. Furthermore, if $\ell = o(n)$, by using *Stirling*'s formula for the approximation of the factorial function

$$x^x e^{-x}\sqrt{2\pi x} \le x! \le x^x e^{-x}\sqrt{2\pi x} \cdot e^{\frac{1}{12x}},$$

valid for all natural numbers x, we obtain with $\lfloor \frac{n}{\ell} \rfloor = \frac{n-r}{\ell}$

$$\binom{n}{\beta + d_1^*, \ldots, \beta + d_\ell^*} \le 2\ell^{n+\ell/2}(2\pi n)^{-\frac{\ell-1}{2}} \cdot \prod_{1\le i\le \ell} \frac{1}{(1 - \frac{r+\ell d_i^*}{n})^{\beta + d_i^* + 1/2}} \le \ell^{n+\frac{\ell}{2}}(2\pi n)^{-\frac{\ell-1}{2}} \cdot 2^{\Theta(\ell)}. \tag{2.3}$$

Let $q = 1 - p$ and note that $b = q^{-1}$; in order to handle the remaining terms in (2.2), we estimate

$$\sum_{\mathbf{d}\in\mathcal{D}(n,\ell)} q^{\frac{1}{2}\sum_{i=1}^{\ell} d_i^2} \le \sum_{\mathbf{d}\in\mathbb{Z}^\ell} b^{-\frac{1}{2}\sum_{i=1}^{\ell} d_i^2} = S(p)^\ell,$$

where S denotes the rapidly converging sum $\sum_{d\in\mathbb{Z}} b^{-d^2/2}$. By putting everything together, and using again Stirling's formula to estimate the $\ell!$-term, we obtain the following upper bound on $\mathbb{E}[X]$ from (2.2).

$$\mathbb{E}[X] \le \left(\frac{e}{\ell}\right)^\ell \sqrt{2\pi\ell} \cdot \ell^{n+\frac{\ell}{2}}(2\pi n)^{-\frac{\ell-1}{2}} \cdot q^{\frac{n^2-r^2}{2\ell} - \frac{n}{2}} \cdot S^\ell \cdot 2^{\Theta(\ell)} = \ell^{n-\frac{\ell}{2}} n^{-\frac{\ell}{2}} \cdot b^{-\frac{n^2}{2\ell} + \frac{n}{2}} \cdot 2^{\Theta(\ell)}. \tag{2.4}$$

We now show that for the claimed values of ℓ the above bound for the expected number of colorings of the random graph $G_{n,p}$ with ℓ colors tends to zero, i.e. $\chi(G_{n,p}) > \ell$ holds with high probability. This will complete the proof of the theorem.

Define the function $\alpha_p(n) = 2\log_b n - 2\log_b\log_b n - C$, where we assume $C = \Theta(1)$, and let $\ell = \frac{n}{\alpha_p(n)}$. With this notation, we obtain from (2.4) the upper bound

$$\log_b \mathbb{E}[X] \le \left(n - \frac{\ell}{2}\right)\log_b \ell - \frac{\ell \log_b n}{2} - \frac{n\alpha_p(n) - n}{2} + \Theta(\ell),$$

which can be bounded from above by

$$\left(n - \frac{n}{2\alpha_p(n)}\right)(\log_b n - \log_b \alpha_p(n)) - \frac{n\log_b n}{2\alpha_p(n)} - n\left(\log_b n - \log_b\log_b n - \frac{C+1}{2}\right) + \Theta(\ell).$$

But this simplifies with $\log_b(\alpha_p(n)) = \log_b\log_b n + \log_b 2 - \Theta\left(\frac{\log\log n}{\log n}\right)$ to

$$\log_b \mathbb{E}[X] \le -n\left(\log_b 2 + \frac{\log_b n}{\alpha_p(n)} - \frac{C+1}{2}\right) + o(n) \le -n\left(\log_b 2 - \frac{C}{2}\right) + o(n).$$

Therefore, whenever $C \le 2\log_b 2 - o(1)$ the expected number of colorings of $G_{n,p}$ goes to zero.

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