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Automorphism groups of finite posets

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ABSTRACT

For any finite group G, we construct a finite poset (or equivalently, a finite T_0-space) X, whose group of automorphisms is isomorphic to G. If the order of the group is n and it has r generators, X has $n(r+2)$ points. This construction improves previous results by G. Birkhoff and M.C. Thornton. The relationship between automorphisms and homotopy types is also analyzed.

1. Introduction

It is well known that any finite group G can be realized as the automorphism group of a finite poset. In 1946 Birkhoff [1] proved that if the order of G is n, G can be realized as the automorphisms of a poset with $n(n+1)$ points. In 1972 Thornton [2] improved slightly Birkhoff's result: He obtained a poset of $n(2r+1)$ points, when the group is generated by r elements. Following Birkhoff's and Thornton's ideas, we exhibit here a simple proof of the following fact which improves their results.

Theorem. *Given a group G of finite order n with r generators, there exists a poset X with $n(r+2)$ points such that* $\mathrm{Aut}(X) \simeq G$.

The proof of the theorem uses basic topology. Recall that there exists a one-to-one correspondence between finite posets and finite T_0-topological spaces. Given a finite poset X, the subsets $U_x = \{y \in X \mid y \leq x\}$ constitute a basis for a topology on the set X. Conversely, given a T_0-topology on the set X, one can define a partial order given by $x \leq y$ if x is contained in every open set which contains y. It is easy to see that these applications are mutually inverse. Therefore we regard finite posets and finite T_0-spaces as the same objects. Order preserving functions correspond to continuous maps and lower sets to open sets. A finite poset is connected if and only if it is connected as a topological space. For further details see [3].

2. The proof

Let $\{h_1, h_2, \ldots, h_r\}$ be a set of r generators of G. We define the poset $X = G \times \{-1, 0, \ldots, r\}$ with the following order:

- $(g, i) \leq (g, j)$ if $-1 \leq i \leq j \leq r$,
- $(gh_i, -1) \leq (g, j)$ if $1 \leq i \leq j \leq r$.

Define $\phi : G \to \mathrm{Aut}(X)$ by $\phi(g)(h, i) = (gh, i)$. It is easy to see that $\phi(g) : X \to X$ is order preserving and that it is an automorphism with inverse $\phi(g^{-1})$. Therefore ϕ is a well defined homomorphism. Clearly ϕ is a monomorphism since $\phi(g) = 1$ implies $(g, -1) = \phi(g)(e, -1) = (e, -1)$.

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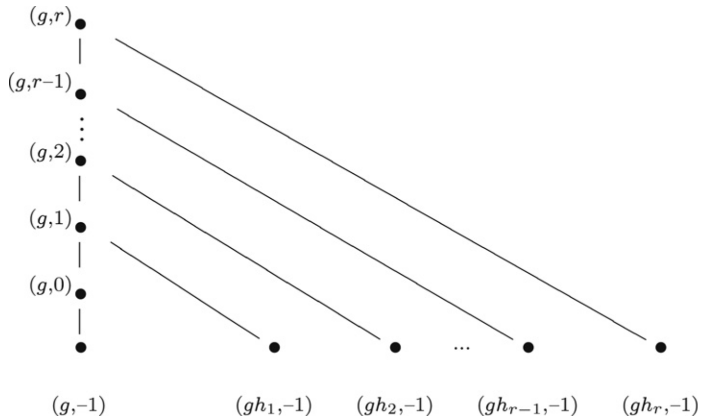

Fig. 1. $U_{(g,r)}$.

It remains to show that ϕ is an epimorphism. Let $f : X \to X$ be an automorphism. Since $(e, -1)$ is minimal in X, so is $f(e, -1)$ and therefore $f(e, -1) = (g, -1)$ for some $g \in G$. We will prove that $f = \phi(g)$.

Let $Y = \{x \in X \mid f(x) = \phi(g)(x)\}$. Y is non-empty since $(e, -1) \in Y$. We prove first that Y is an open subspace of X. Suppose $x = (h, i) \in Y$. Then the restrictions

$$f|_{U_x}, \phi(g)|_{U_x} : U_x \to U_{f(x)}$$

are isomorphisms. On the other hand, there exists a unique automorphism $U_x \to U_x$ since the unique chain of $i+2$ elements must be fixed by any such automorphism. Thus, $f|_{U_x}^{-1}\phi(g)|_{U_x} = 1_{U_x}$, and then $f|_{U_x} = \phi(g)|_{U_x}$, which proves that $U_x \subseteq Y$. Similarly we see that $Y \subseteq X$ is closed. Assume $x = (h, i) \notin Y$. Since $f \in \mathrm{Aut}(X)$, it preserves the height $ht(y)$ of any point y. In particular $ht(f(x)) = ht(x) = i + 1$ and therefore $f(x) = (k, i) = \phi(kh^{-1})(x)$ for some $k \in G$. Moreover $k \neq gh$ since $x \notin Y$. As above, $f|_{U_x} = \phi(kh^{-1})|_{U_x}$, and since $kh^{-1} \neq g$ we conclude that $U_x \cap Y = \emptyset$.

We prove now that X is connected. It suffices to show that any two minimal elements of X are in the same connected component. Given $h, k \in G$, we have $h = kh_{i_1}h_{i_2}\dots h_{i_m}$ for some $1 \leq i_1, i_2, \dots, i_m \leq r$. On the other hand, $(kh_{i_1}h_{i_2}\dots h_{i_s}, -1)$ and $(kh_{i_1}h_{i_2}\dots h_{i_{s+1}}, -1)$ are connected via $(kh_{i_1}h_{i_2}\dots h_{i_s}, -1) < (kh_{i_1}h_{i_2}\dots h_{i_s}, r) > (kh_{i_1}h_{i_2}\dots h_{i_{s+1}}, -1)$. This implies that $(k, -1)$ and $(h, -1)$ are in the same connected component.

Finally, since X is connected and Y is closed, open and non-empty, $Y = X$, i.e. $f = \phi(g)$. Therefore ϕ is an epimorphism, and then $G \simeq \mathrm{Aut}(X)$. □

3. Homotopy types

If the generators $h_1, h_2, \dots, h_r$ are non-trivial, the open sets $U_{(g,r)}$ look as in Fig. 1. In that case it is not hard to prove that the finite space X constructed above is weak homotopy equivalent to a wedge of $n(r-1)+1$ circles, or in other words, that the order complex of X is homotopy equivalent to a wedge of $n(r-1)+1$ circles. The space X deformation retracts to the subspace $Y = G \times \{-1, r\}$ of its minimal and maximal points. A retraction is given by the map $f : X \to Y$, defined as $f(g, i) = (g, r)$ if $i \geq 0$ and $f(g, -1) = (g, -1)$. Now, the order complex $\mathcal{K}(Y)$ of Y is a connected simplicial complex of dimension 1, so its homotopy type is completely determined by its Euler characteristic. This complex has $2n$ vertices and $n(r+1)$ edges, which means that it has the homotopy type of a wedge of $1 - \chi(\mathcal{K}(Y)) = n(r-1)+1$ circles.

On the other hand, note that in general, the automorphism group of a finite space does not say much about its homotopy type as we state in the following

Remark. Given a finite group G and a finite space X, there exists a finite space Y which is homotopy equivalent to X and such that $\mathrm{Aut}(Y) \simeq G$.

We make this construction in two steps. First, we find a finite T_0-space $\tilde{X}$ homotopy equivalent to X and such that $\mathrm{Aut}(\tilde{X}) = 0$. To do this, assume that X is T_0 and consider a linear extension $x_1, x_2, \dots, x_n$ of the poset X. Now, for each $1 \leq k \leq n$ attach a chain of length kn to X with minimum x_{n-k+1}. The resulting space $\tilde{X}$ deformation retracts to X and every automorphism $f : \tilde{X} \to \tilde{X}$ must fix the unique chain C_1 of length n^2 (with minimum x_1). Therefore f restricts to a homeomorphism $\tilde{X} \smallsetminus C_1 \to \tilde{X} \smallsetminus C_1$ which must fix the unique chain C_2 of length $n(n-1)$ of $\tilde{X} \smallsetminus C_1$ (with minimum x_2). Applying this reasoning repeatedly, we conclude that f fixes every point of $\tilde{X}$. On the other hand, we know that there exists a finite T_0-space Z such that $\mathrm{Aut}(Z) \simeq G$.

Now the space Y is constructed as follows. Take one copy of $\tilde{X}$ and of Z, and put every element of Z under $x_1 \in \tilde{X}$. Clearly Y deformation retracts to $\tilde{X}$. Moreover, if $f : Y \to Y$ is an automorphism, $f(x_1) \notin Z$ since $f(x_1)$ cannot be comparable with x_1 and distinct from it. Since there is only one chain of n^2 elements in $\tilde{X}$, it must be fixed by f. In particular $f(x_1) = x_1$, and then $f|_Z : Z \to Z$. Thus f restricts to automorphisms of $\tilde{X}$ and of Z and therefore $\mathrm{Aut}(Y) \simeq \mathrm{Aut}(Z) \simeq G$.

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