

A witness version of the Cops and Robber game

Nancy E. Clarke

Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia, Canada

ARTICLE INFO

Article history:

Received 8 November 2007

Received in revised form 15 September 2008

Accepted 17 September 2008

Available online 7 November 2008

Keywords:

Cop
Partial information
Witness
Pursuit
Structure

ABSTRACT

The games considered are mixtures of Searching and Cops and Robber. The cops have partial information provided via witnesses who report “sightings” of the robber. The witnesses are able to provide information about the robber’s position but not the direction in which he is moving. The robber has perfect information. In the case when sightings occur at regular intervals, we present a recognition theorem for graphs on which a single cop suffices to guarantee a win. In a special case, this recognition theorem provides a characterization.

© 2008 Elsevier B.V. All rights reserved.

1. Introduction

The game of Cops and Robber is a pursuit game played on a reflexive graph, i.e. a graph with a loop at every vertex. There are two opposing sides, a set of $k > 0$ cops and a single robber. The cops begin the game by each choosing a vertex to occupy, and then the robber chooses a vertex. The two sides move alternately, where a move is to slide along an edge or along a loop. The latter is equivalent to passing were the game played on a loopless graph. There is perfect information, and the cops win if any of the cops and the robber occupy the same vertex at the same time, after a finite number of moves. Graphs on which one cop suffices to win are called *copwin* graphs. The game has been considered on infinite graphs but, here, we only consider finite graphs.

In this paper, a variation of the Cops and Robber game is introduced in which the cops no longer have perfect information, but rather receive information about the robber’s position intermittently from witnesses. The robber continues to play with perfect information. In Sections 3 and 4, the cops’ information is provided at regular intervals, say every k units of time. A graph on which a single cop can guarantee a win with such information will be said to be *k-winnable*. More generally, this problem can be considered when “sightings” are reported at irregular intervals. Some preliminary results are given in Section 2. Note that witnesses provide information about the robber’s position only, and are not able to give an indication of the direction in which he is moving. Note also that witness information will be thought to be relayed to the cop immediately, so that the cop has the information in time for his next move. Finally, we will assume that the cop has information about the robber’s first move since, otherwise, the cop can pass on all moves available to him before he receives news of the first sighting.

A copwin graph G can be recognized in polynomial time via a decomposition algorithm that relies only on knowledge of the neighbours of the vertices of G . In fact, copwin graphs have been completely characterized in this way. We present this characterization in the next few paragraphs. A similar decomposition algorithm exists for recognizing tandem-win graphs (a class of graphs on which two cops suffice to win) although, unlike with copwin graphs, it is not a characterization. See [5].

E-mail address: nancy.clarke@acadiau.ca.

The motivation for this paper lies in attempting to discover a similar, or any, recognition algorithm for k -winnable graphs. This is [Theorem 6](#). [Theorem 9](#) shows that, in a special case, our recognition theorem is a characterization.

We now introduce our notation and basic definitions. Recall that, for us, all graphs will be finite, connected and reflexive. For a graph G , we let $V(G)$ denote the set of vertices of G and $E(G)$, the set of edges. For $a, b \in V(G)$, we use $a \sim b$ to indicate that a and b are adjacent ($a \neq b$), and $a \simeq b$ if a is adjacent or equal to b . For $x \in V(G)$, $N(x) = \{y \mid y \sim x\}$ is the *open neighbourhood* of x and $N[x] = N(x) \cup \{x\}$ is the *closed neighbourhood*. For $w, z \in V(G)$, the distance between w and z is the minimum number of edges on a $w - z$ path and is denoted by $d(w, z)$. If $Y \subseteq V(G)$, then $G - Y$ is the graph induced in G by the vertex set $V(G) \setminus Y$.

A mapping $f : V(G) \rightarrow V(H)$ is a *homomorphism* if, for $x, y \in V(G)$, $f(x) \simeq f(y)$ whenever $x \sim y$. A subgraph H of a graph G is a *retract* of G if there is a homomorphism $f : V(G) \rightarrow V(H)$ such that $f(x) = x$, for all $x \in V(H)$. Note that, since G is reflexive, a homomorphism can send two adjacent vertices to the same vertex. Sometimes we need to consider the situation where the cops are playing on a retract H , while the robber is playing on the full graph G . If r is the vertex occupied by the robber and f is a fixed retraction map of G onto H then, by a small abuse of notation and terminology, we refer to $f(r)$ as the *robber's image*.

A vertex u of a graph G is a *corner*, or *irreducible* (so named because of the similarities between copwin graphs and dismantlable, partially-ordered sets), if there exists a vertex v in G such that $N[u] \subseteq N[v]$; also we say that v *dominates* u . A vertex ordering (x_1, x_2, \dots, x_n) on G is a *domination elimination ordering* [2,3] if, for each $i \in \{1, 2, \dots, n-1\}$, there is a $j_i > i$ such that $N_i(x_i) \subseteq N_i[x_{j_i}]$ in $G_i = G - \{x_1, x_2, \dots, x_{i-1}\}$. If, in addition, for each i , $x_i \sim x_{j_i}$, then this domination elimination ordering is a *copwin ordering* [11]. A main result of [11,13–15] is that: *a finite graph G is copwin if and only if G has a copwin ordering*. Structurally, this means that G is copwin if and only if it can be reduced to a singleton by a series of retractions and, in each retraction, there is exactly one vertex that is not fixed and it is irreducible.

Let (x_1, x_2, \dots, x_n) be a copwin ordering of G . For $j = 1, 2, \dots, n-1$, define $f_j : V(G_j) \rightarrow V(G_{j+1})$ to be the retraction map from G_j to G_{j+1} . The corresponding *copwin spanning tree* S_{x_n} is a spanning tree, rooted at x_n , with the property that, for vertices $x, y \in V(G)$, $xy \in E(S_{x_n})$ if and only if $f_j(x) = y$ or $f_j(y) = x$, for some j . We will use this definition in the first example in Section 3.

We conclude our introduction by noting that some preliminary work has been done on a version of the Cops and Robber game in which the cops play with no information. See [10,16]. This model is closely related to the game of searching, first introduced by Parsons in [12] and later surveyed in [1,9]. Consider a strategy that can be used successfully by the cops to win when playing Cops and Robber with no information. We will refer to such a strategy as a *searching strategy*. In general, any cop strategy that has a cop move during a turn in which he has no information will be said to have a *searching component*. Otherwise, a strategy will be said to be *search-free*.

Finally, other variations of the Cops and Robber game in which the cops play with imperfect information have been studied in [4,6–8].

2. Irregular sightings

We begin by considering the most general version of the witness Cops and Robber game, the version in which the cops receive information about the robber's position from witnesses intermittently and do not know when the next piece of information will be available to them. (We note that the time between sightings will be assumed to be finite.) One primary objective is the characterization of the graphs on which a single cop can guarantee a win. We will call such graphs *winnable with irregular information*.

Theorem 1. *Let T be a tree. Then T is winnable with irregular information.*

Proof. Clearly, a single cop can guarantee a win on any tree, regardless of the pattern of sightings. With every sighting, the cop can make at least one move toward the robber along the unique path connecting their positions. When the cop's move is unclear due to a lack of information, the cop passes and waits for the next sighting. The cop wins (eventually) since the robber has no means of moving past the cop and thus will be caught on a leaf, if not sooner, since all graphs considered are finite. \square

More generally, the proof of [Theorem 1](#) may be used to show that: *If G is a graph in which every block is a clique, then G is winnable with irregular information.*

Theorem 2. *A graph G that is winnable with irregular information is copwin.*

Proof. The cop simply ignores any information unnecessary in his winning strategy for playing with irregular information. \square

3. k -regular information

In this section, we suppose that the cops receive information about the robber’s position at regular intervals, say (at least!) every k units of time. Recall that a graph on which a single cop can guarantee a win with k -regular information is said to be k -winnable.

Consider a similar version of the Cops and Robber game in which the robber is permitted to make up to k moves during a turn while the cop continues to move at most once during a turn. (There is perfect information on both sides.) We call the set of all graphs on which a single cop can guarantee a win in this way the set of $(k, 1)$ -win graphs, and note that the set of $(k, 1)$ -win graphs is contained in the set of k -winnable graphs. (In general, if the robber is permitted to make up to m moves during a turn, and the cop n moves, the graphs on which the cop can guarantee a win are (m, n) -win.) In the witness version of the Cops and Robber game, playing with these rules is equivalent to assuming that the cop moves only on turns when he has information. Essentially, this means that the cop’s winning strategy on a k -winnable graph is search-free, and thus captures the robber on a cop move immediately following a sighting.

We begin by noting that the results given in Section 2 for irregular information hold for k -regular information. In particular, as shown in Theorem 2, a k -winnable graph is copwin. Clearly, as the following example shows, the converse does not hold. There are copwin graphs G that are not winnable by one cop with only witness information.

Example. Consider the copwin graph shown in Fig. 1. One copwin spanning tree is shown in bold. This graph is not even 2-winnable since, to win, the cop must move onto the same branch (and vertex) as the robber. Even if the cop begins on the root of the spanning tree, the cop has a choice of two branches after the robber’s first move, and he cannot ensure choosing correctly. In this way, the game can continue indefinitely, as the cop will have a choice of moves following every unknown robber move.

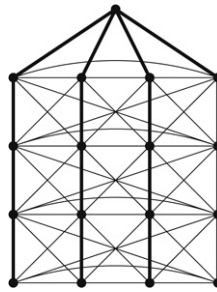


Fig. 1. A copwin graph that is not k -winnable, for any k .

We proceed by noting that not all subgraphs of a k -winnable graph are k -winnable. Consider the graph G shown in Fig. 2. It is easy to see that this graph is 2-winnable. The cop begins on v . Since v is adjacent to all of the other vertices of G , the cop will win on a move immediately following the first sighting. Now G contains C_4 as a subgraph, but C_4 is not copwin and hence not 2-winnable. This is similar to, say, a wheel being copwin while a cycle (of length at least 4) is not. We will now investigate a class of subgraphs that are k -winnable whenever the original graph is.

Theorem 3. Let G be a graph and let G' be a retract of G . Then G' is k -winnable if G is k -winnable.

Proof. Let $f : V(G) \rightarrow V(G')$ be the retraction map from G to G' . Since G is k -winnable, the cop has a winning strategy on G . To win on G' , the cop simply plays his winning strategy as if he was playing on G . The actual moves he makes are the images of these moves under f . Note that since f is a retraction map, if $w, z \in V(G)$ and $d(w, z) \leq k$ in G , then $d(f(w), f(z)) \leq k$ in G' . Using this strategy, the cop captures the image of the robber on G' . Since the robber is actually playing on G' and f is the identity map on G' , the robber’s image coincides with his actual position. Hence the robber is apprehended on G' and, therefore, G' is a k -winnable graph. \square

In particular, we have Corollary 4 which follows the necessary definitions.

Let z be a vertex of a graph G . Define $N^k(z) = \{w \in V(G) | \exists \text{ a } z-w \text{ path of length } k\}$. Further define $N_k(z) = \{w \neq z \in V(G) | d(z, w) \leq k\}$ so that $N_k(z) = \cup_{i=1}^k N^i(z)$, and $N_k[z] = N_k(z) \cup \{z\}$. A vertex u of G is said to be a k -dominated vertex if there exists a vertex v in G such that $N_k[u] \subseteq N[v]$ and, in addition, if $w \in N^k(u)$, then $N(w) \subseteq N_k[u]$. The vertex v is said to k -dominate u . We define a k -corner to be the subgraph induced by $N_k[u] \setminus \{v\}$ for some u and v such that v k -dominates u .

Example. Consider the graph G shown in Fig. 2. The vertex u shown is a 2-dominated vertex with corresponding 2-dominating vertex v . The graph $C_2 = G - v$ is a 2-corner.

Corollary 4. Let G be a graph and let C_k be a k -corner of G . Let $G' = G - C_k$. Then G' is k -winnable if G is k -winnable.

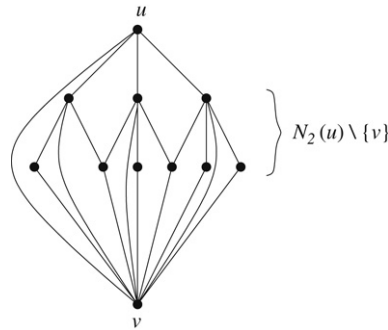


Fig. 2. A graph G with 2-dominated vertex u and 2-dominating vertex v .

Proof. Suppose the k -corner \mathcal{C}_k has k -dominated vertex u with corresponding k -dominating vertex v . Notice that G' is a retract of G with retraction map f defined as follows:

$$f(w) = \begin{cases} v, & \text{for } w \in N_k[u] \\ w & \text{otherwise.} \end{cases} \tag{1}$$

The result follows by Theorem 3. \square

Theorem 5. Let G be a graph and let \mathcal{C}_k be a k -corner of G . Let $G' = G - \mathcal{C}_k$. If G' is k -winnable via a search-free strategy, then G is k -winnable.

Proof. As in the proof of Corollary 4, suppose the k -corner \mathcal{C}_k has k -dominated vertex u with corresponding k -dominating vertex v , and note again that G' is a retract of G with retraction map f defined as in Eq. (1).

Since the game is being played on G , the cop's winning strategy on G' can be thought of as catching the image of the robber. (Since this strategy is search-free, the cop will know when he has captured the robber's image.) Now suppose this image is caught on vertex z . If $z \neq v$, then the robber's image on G' corresponds to his actual position on G , since f is the identity map on G' . The robber is apprehended! Otherwise, the robber's image is apprehended on vertex v . Since it is known that $f(u) = f(w) = f(v) = v$ for $w \in N_k(u)$, the robber is either on u or v in G , or on some $w \in N_k(u)$. If he is on v , his actual position corresponds to his image and he is caught. Otherwise he is on $w \in N_k[u]$, and he will be caught after the next sighting. \square

We note that the proofs of Theorem 3, Corollary 4 and Theorem 5 may be used to show that: A graph G with k -corner \mathcal{C}_k is $(k, 1)$ -win if and only if $G - \mathcal{C}_k$ is $(k, 1)$ -win.

We note also that Theorem 5 does not hold in general, when all winning strategies on G' involve a searching component. This is because it may be possible for the cop to capture the robber's image during a move in which he has no information. In such a case, the cop is unaware that he has captured the robber's image (unless the image coincides with the actual position!).

Theorem 5 leads to a main result.

Theorem 6. Let G be a graph which can be reduced to a singleton by a sequence of k -corner retractions. Then G is k -winnable.

Proof. Let G be a graph. We note that the cop is aware that he has captured the robber's image after the robber's first move, and also that a winning strategy exists for a k -corner that does not involve a searching component. As a result, Theorem 5 applies and guarantees that if \mathcal{C}_k is a k -corner of G , then $G - \mathcal{C}_k$ is k -winnable only if G is k -winnable. \square

We note that the sequence of k -corner retractions given in the statement of Theorem 6 determines a copwin ordering. Consider the k -corners in the order given by the sequence. For each k -corner \mathcal{C}_i , let u_i be the k -dominated vertex with corresponding k -dominating vertex v_i . In the copwin ordering, u_i is retracted onto v_i and then, for all $w \in N_k(u_i)$ in turn, w is retracted onto v_i . (See Fig. 4.) Note that, for ease of notation, we have dropped the subscript k from our k -corner notation. We will continue to do so throughout the remainder of the paper as we instead use subscripts to index the k -corners in sequences of k -corner retractions.

Consider a k -winnable graph G and a fixed sequence $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_p)$ of k -corners of G . Define induced subgraphs $G_j = G - \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_{j-1}\}$, where u_i is the k -dominated vertex in G_i with k -dominating vertex v_i . Further let f_j be the retraction from G_j to G_{j+1} .

The corresponding decomposition tree is the tree S on $V(G)$, rooted at v_p , with $E(S)$ constructed as follows: for $i = 1, 2, \dots, p$, (1) $u_i v_i \in E(S)$ and, (2) if there does not exist $l \in \{i + 1, i + 2, \dots, p\}$ such that $v_i = u_l$ (in which case condition (1) results in the inclusion of edge $v_i v_l$), then there exists $m \in \{i + 1, i + 2, \dots, p\}$ such that $v_i \in N_k(u_m)$ in G_m . Add edge $v_i v_m$ to $E(S)$.

Example. Consider the 2-winnable graph G shown in Fig. 3. The decomposition tree corresponding to one sequence of 2-corner retractions is shown in bold.

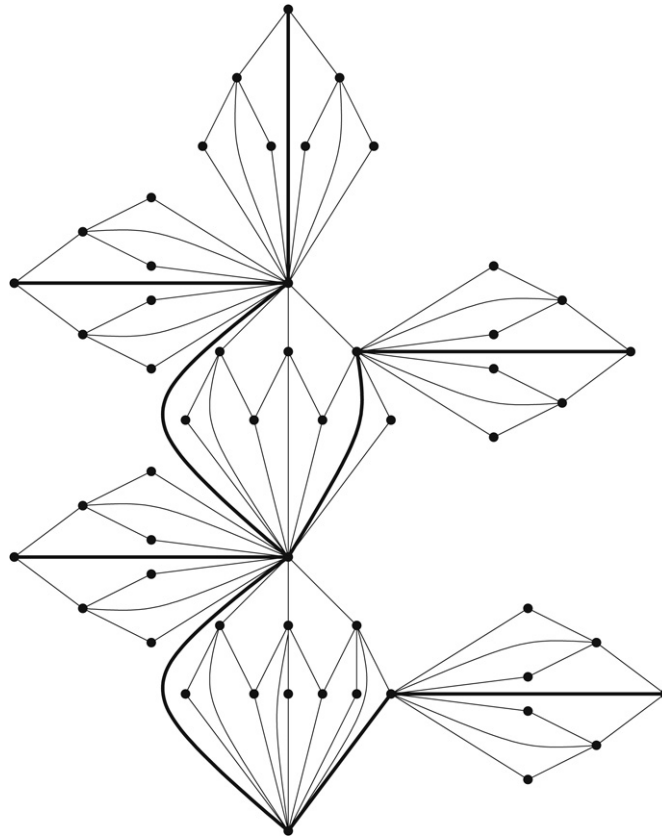


Fig. 3. A 2-winnable graph with decomposition tree shown in bold.

This concept of decomposition tree is analogous to that of copwin spanning tree introduced in [8] and tandem-win decomposition tree introduced in [5]. Note that here, unlike with copwin and tandem-win trees (which involve one-point retractions), we are considering retractions in which multiple vertices are not fixed. In addition, the decomposition tree will normally not span G .

Example. Consider the 2-winnable graph G shown in Fig. 4. A decomposition tree is shown in bold in (a). In (b), the corresponding copwin spanning tree of G is shown.



Fig. 4. A 2-winnable graph with (a) decomposition tree shown in bold and (b) corresponding copwin spanning tree shown in bold.

Suppose we have a sequence of k -corner retractions of a graph G that results in a singleton. We know that a single cop has a winning strategy on G . We now make one such strategy explicit and prove that it is effective in capturing the robber.

Algorithm (k -winnable Strategy). Let $(\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_p)$ represent the k -corners in a sequence of k -corner retractions that reduces G to a singleton v_p . Again, define the induced subgraphs $G_i = G_{i-1} - \mathcal{C}_{i-1}$, where $G_1 = G$ and $G_{p+1} = v_p$, and let $f_i : V(G_i) \rightarrow V(G_{i+1})$ be the retraction from G_i to G_{i+1} . Further, if the robber is on vertex x , define $F_i(x) = f_{i-1} \circ f_{i-2} \circ \dots \circ f_1(x)$ so that F_i is the robber's image on G_i . The robber is always thought to be playing on the graph G , whereas the cop initially moves on the subgraph G_{p+1} . The cop begins on vertex v_p , the vertex on which the cop's position coincides with the robber's image under the mapping F_{p+1} . The cop then wins by capturing F_i on G_i , for each $i = p, p - 1, \dots, 1$, in turn. We note that, in this strategy, the cop does not move during turns for which he has no information; i.e. this strategy is search-free.

Theorem 7. *The k -winnable strategy is effective in capturing the robber.*

Proof. Since the cop does not move during turns for which he has no information, we may assume that the cop and robber move alternately, where the robber may make up to k moves during a turn (i.e. we may consider a $(k, 1)$ -game).

Suppose the robber occupies $x \in V(G)$. By induction, the cop is occupying the robber's image, i.e. $F_j(x)$, on the subgraph G_j and it is the robber's turn to move. Suppose the robber moves to vertex y , where $d(x, y) \leq k$. If $F_j(x) = F_{j-1}(y)$, there is nothing to prove. If $F_j(x) \neq F_{j-1}(y)$ and the robber is not located on a vertex of G_{j-1} , then $F_{j-1}(y) = v_k$, $k < j - 1$, with $v_k \in V(G_{j-1})$ and so $F_j(x) \sim F_{j-1}(y)$. Otherwise, the robber is playing on G_{j-1} so that $y \subseteq N[F_j(x)]$, and the robber is captured immediately or on the cop's next move. Thus, if the cop has captured the image on G_j , he can remain with the image on G_{j-1} by making at most one move. \square

We note that the proof shows that if the cop is playing on the subgraph G_i , then the robber cannot move to (and remain on!) this subgraph without being apprehended immediately or after the next cop move. This is **Corollary 8**.

Corollary 8. *Suppose the cop is playing the k -winnable strategy on the subgraph G_i , and is occupying the robber's image under the mapping F_i . The robber can never occupy a vertex of G_i after his final move before a sighting without the cop being on, or immediately landing on, the same vertex.*

As discussed following the proof of **Theorem 5**, it is not yet known if all k -winnable graphs can be reduced to a singleton by a sequence of k -corner retractions. We now investigate a class of graphs that can be reduced in this way.

4. Triangle-free k -winnable graphs

A graph G is said to be *triangle-free* if it does not contain K_3 as a subgraph. Below we give a characterization of triangle-free k -winnable graphs, when there exists a search-free winning strategy for the cop. Compare Theorems 9 and 10 here with Theorems 8 and 9 in [5].

Theorem 9. *Let G be a triangle-free graph. Then G is k -winnable via a search-free strategy if and only if G can be reduced to a singleton by a sequence of k -corner retractions.*

Proof. Sufficiency of the given reduction scheme is guaranteed by **Theorem 6**, so we need only prove necessity. Suppose a triangle-free graph G is k -winnable via a search-free strategy, and consider the final k robber moves in a game played on G . Since the robber is captured during the cop's next move if he passes on all k of his moves, the robber's position r must be adjacent to the cop's position c . Since the robber is captured if he takes advantage of exactly i of the k opportunities to move, $i = 1, 2, \dots, k$, then $N^i(r) \subseteq N[c]$; i.e. if $w \in N^i(r)$, then $w \simeq c$. If $w \notin N(r)$, then there exists $u \in N^{i-1}(r)$ such that $u \sim w$. But then w, u , and c form a triangle in G . (See **Fig. 5**.) Otherwise, w, r , and c form a triangle in G . And so $N(r) = \{c\}$ and $N^i(r) = \emptyset$, for $i = 2, 3, \dots, k$. Since $G - r$ is k -winnable via a search-free strategy and triangle-free, the result follows. \square

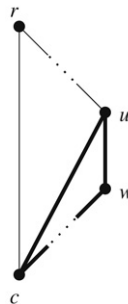


Fig. 5. The final moves made by the robber.

The final result follows easily.

Corollary 10. *If G is triangle-free and k -winnable via a search-free strategy, then G is a tree.*

Finally, we note that the proof of **Theorem 9** establishes a characterization of triangle-free $(k, 1)$ -win graphs, namely: *A triangle-free graph G is $(k, 1)$ -win if and only if G can be reduced to a singleton by a sequence of k -corner retractions.* It follows, of course, that: *A triangle-free $(k, 1)$ -win graph is a tree.*

References

- [1] B. Alspach, Searching and sweeping graphs: A brief survey, *Le Matematiche* 59 (2004) 5–37.
- [2] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph classes: A survey, in: *SIAM Monographs on Discrete Mathematics and Applications*, Society for Industrial and Applied Mathematics, SIAM, Philadelphia, PA, 1999.
- [3] V. Chepoi, On distance-preserving and domination elimination orderings, *SIAM J. Discrete Math.* 11 (3) (1998) 414–436.
- [4] N.E. Clarke, E.L. Connon, Cops, robber, and alarms, *Ars Combin.* 81 (2006) 283–296.
- [5] N.E. Clarke, R.J. Nowakowski, Tandem-win graphs, *Discrete Math.* 299 (2005) 56–64.
- [6] N.E. Clarke, A game of cops and robber played with partial information, *Congr. Numer.* 166 (2004) 145–159.
- [7] N.E. Clarke, Constrained cops and robber, Ph.D. Thesis, Dalhousie University, 2002.
- [8] N.E. Clarke, R.J. Nowakowski, Cops, robber, and photo radar, *Ars Combin.* 56 (2000) 97–103.
- [9] D. Dyer, Sweeping graphs and digraphs, Ph.D. Thesis, Simon Fraser University, 2004.
- [10] D.V. Jeliazkova, Aspects of the cops and robber game played with incomplete information, M.Sc. Thesis, Acadia University, 2006.
- [11] R.J. Nowakowski, P. Winkler, Vertex to vertex pursuit in a graph, *Discrete Math.* 43 (1983) 23–29.
- [12] T.D. Parsons, Pursuit Evasion in a Graph, in: *Lecture Notes in Math.*, vol. 642, Springer, Berlin, 1978.
- [13] A. Quilliot, Jeux et Points Fixes sur les graphes, Thèse de 3ème cycle, Université de Paris VI, 1978, pp. 131–145.
- [14] A. Quilliot, Problèmes de jeux, de point fixe, de connectivité et de représentation sur des graphes, des ensembles ordonnés et des hypergraphes, Thèse d'Etat, Université de Paris VI, 1983.
- [15] A. Quilliot, A short note about pursuit games played on a graph with a given genus, *J. Combin. Theory (B)* 38 (1985) 89–92.
- [16] A. Tang, Zero-visibility cops and robber, M.Sc. Thesis, Dalhousie University, 2004.