Discrete Mathematics 309 (2009) 3092–3101

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Chromatic classes of 2-connected $(n, n+4)$-graphs with three triangles and one induced 4-cycle

Y.H. Peng [a,b,*], G.C. Lau [b,c]

[a] Department of Mathematics, Universiti Putra Malaysia, 43400UPM Serdang, Malaysia
[b] Institute for Mathematical Research, Universiti Putra Malaysia, 43400UPM Serdang, Malaysia
[c] Faculty of I.T. and Quantitative Sc., Universiti Teknologi MARA, Segamat Campus, Johor, Malaysia

A R T I C L E I N F O

Article history:
Received 29 May 2003
Accepted 13 August 2008
Available online 16 September 2008

Keywords:
Chromatic polynomial
Chromatically unique graph
Chromatically equivalent graph
Relative-closed

A B S T R A C T

For a graph G, let $P(G, \lambda)$ be its chromatic polynomial. Two graphs G and H are chromatically equivalent, denoted $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph G is chromatically unique if $P(H, \lambda) = P(G, \lambda)$ implies that $H \cong G$. In this paper, we shall determine all chromatic equivalence classes of 2-connected $(n, n+4)$-graphs with three triangles and one induced 4-cycle, under the equivalence relation '$\sim$'. As a by product of these, we obtain various new families of chromatically-equivalent graphs and chromatically-unique graphs.

1. Introduction

Let $P(G, \lambda)$ (or simply $P(G)$) denote the chromatic polynomial of a simple graph G. Two graphs G and H are *chromatically equivalent* (simply χ-equivalent), denoted $G \sim H$, if $P(G) = P(H)$. A graph G is *chromatically unique* (simply χ-unique) if $P(H) = P(G)$ implies that $H \cong G$. Let $\langle G \rangle$ denote the equivalence class determined by the graph G under $\sim$. Clearly, G is χ-unique if and only if $\langle G \rangle = \{G\}$. A graph H is called a *relative* of G if there is a sequence of non-isomorphic graphs $G = H_1, H_2, \ldots, H_k = H$, such that each H_i is a K_{r_i}-gluing of some graphs (say X_i and Y_i) and that H_{i+1} is obtained from H_i by forming a K_{r_i}-gluing of X_i and Y_i for $1 \le i \le k-1$. We say H is a graph of *type* G if H is a relative of G or $H \cong G$. A family $\mathcal{S}$ of graphs is said to be *relative-closed* (simply χ_r-closed) if

(i) no two graphs in $\mathcal{S}$ are relative of each other; and
(ii) for any graph $G \in \mathcal{S}$, $P(H, \lambda) = P(G, \lambda)$ implies that $H \in \mathcal{S}$ or H is a relative of a graph in $\mathcal{S}$.

If $\mathcal{S}$ is a relative-closed family, then the chromatic equivalence class of each graph G in $\mathcal{S}$ can be determined by studying the chromaticity of each graph G in $\mathcal{S}$.

If G is a graph of order n and size m, we say G is an (n, m)-graph. The chromatic equivalence classes of 2-connected $(n, n+i)$-graph have been fully determined for $i = 0, 1$ in [2,6], and partially determined for $i = 2, 3$ in [3,4,8]. Peng and Lau have also characterized and classified all chromatic equivalence classes of 2-connected $(n, n+4)$-graph with at least four triangles in [7]. In this paper, we determine all equivalence classes of 2-connected $(n, n+4)$-graphs with three triangles and one induced 4-cycle. As a by-product of these, we obtain various new families of χ-equivalent graphs and χ-unique graphs. The readers may refer to [1] for terms and notation used but not defined here.

* Corresponding author at: Department of Mathematics, Universiti Putra Malaysia, 43400UPM, Serdang, Selangor, Malaysia.
E-mail address: yhpeng@fsas.upm.edu.my (Y.H. Peng).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.08.016

2. Notation and basic results

Let C_n (or n-cycle) be the cycle of order n. An *induced 4-cycle* is the cycle C_4 without chords. The following are some useful known results and techniques for determining the chromatic polynomial of a graph. Throughout this paper, all graphs are assumed to be connected unless stated otherwise.

Lemma 2.1 (*Fundamental Reduction Theorem (Whitney [10])*). *Let G be a graph and e an edge of G. Then*

$$P(G) = P(G - e) - P(G \cdot e)$$

where $G - e$ is the graph obtained from G by deleting e, and $G \cdot e$ is the graph obtained from G by identifying the end vertices of e.

Let G_1 and G_2 be graphs, each containing a complete subgraph K_p with p vertices. If G is the graph obtained from G_1 and G_2 by identifying the two subgraphs K_p, then G is called a K_p-gluing of G_1 and G_2. Note that a K_1-gluing and a K_2-gluing are also called a vertex-gluing and an edge-gluing, respectively.

Lemma 2.2 (*Zykov[11]*). *Let G be a K_r-gluing of G_1 and G_2. Then*

$$P(G) = \frac{P(G_1)P(G_2)}{P(K_r)}.$$

Lemma 2.2 implies that all K_r-gluing of G_1 and G_2 are χ-equivalent. It follows from Lemma 2.2 that if H is a relative of G, then $H \sim G$.

The following necessary conditions for two graphs G and H to be χ-equivalence are well known (see for example [4]).

Lemma 2.3. *Let G and H be two χ-equivalent graphs. Then G and H have, respectively, the same number of vertices, edges and triangles. If both G and H do not contain K_4, then they have the same number of induced 4-cycles.*

A generalized θ-graph is a 2-connected graph, consisting of three edge-disjoint paths between two vertices of degree 3. All other vertices have degree two. These paths have lengths x, y and z respectively, where $x \geq y \geq z$. The graph denoted by $\theta_{x,y,z}$ is of order $x + y + z - 1$ and size $x + y + z$ (see [6]). We shall denote K_2 as C_2 for convenience.

Lemma 2.4.

(i) $P(C_n) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$, $n \geq 2$.

(ii) $$P(\theta_{x,y,z}) = \begin{cases} \dfrac{P(C_{x+1})P(C_{y+1})P(C_{z+1})}{\lambda^2(\lambda-1)^2} + \dfrac{P(C_x)P(C_y)P(C_z)}{\lambda^2} & \text{if } z \neq 1 \\ \dfrac{P(C_{x+1})P(C_{y+1})}{\lambda(\lambda-1)} & \text{if } z = 1. \end{cases}$$

For integers x, y, z, n and λ, let us write

$$Q_n(\lambda) = \sum_{i=0}^{n-2} (-1)^i (\lambda - 1)^{n-2-i}$$

and

$$M_{x,y,z}(\lambda) = Q_{x+1}(\lambda)Q_{y+1}(\lambda)Q_{z+1}(\lambda) + (\lambda - 1)^2 Q_x(\lambda)Q_y(\lambda)Q_z(\lambda).$$

Note that when $\lambda = 1$, we have $Q_n(1) = (-1)^n$ and $M_{x,y,z}(1) = (-1)^{x+y+z+1}$. Lemma 2.4 can then be written as the following lemma.

Lemma 2.5 (*[4]*).

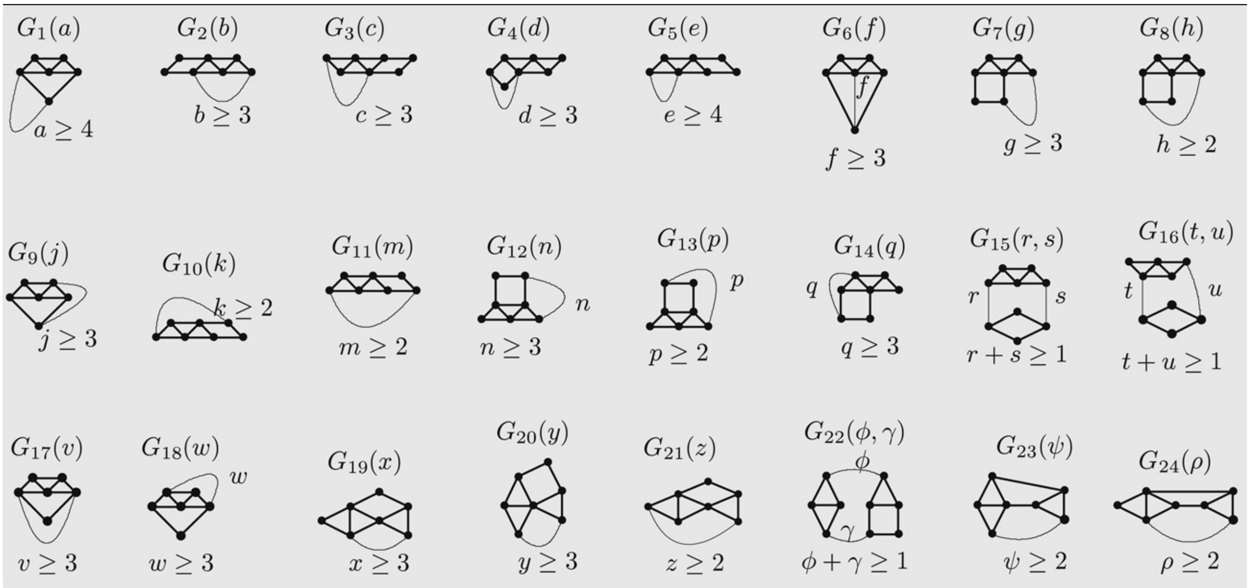
(i) $P(C_n) = \lambda(\lambda - 1)Q_n(\lambda)$.
(ii) $P(\theta_{x,y,z}) = \lambda(\lambda - 1)M_{x,y,z}(\lambda)$.

We also need the following lemma.

Lemma 2.6 (*Whitehead and Zhao [9]*). *A graph G contains a cut-vertex if and only if $(\lambda - 1)^2 \mid P(G)$.*

Lemma 2.6 also implies that if $H \sim G$, then H is 2-connected if and only if G is so.

Table 1
24 types of 2-connected $(n, n+4)$-graphs with exactly three triangles and one induced 4-cycle



The light lines of the graphs refer to the paths of indicated length.

3. Classification of graphs

Let $\mathcal{G}$ denote the χ_r-closed family of 2-connected $(n, n+4)$-graphs with exactly 3 triangles and one induced C_4. In [5], we classified all the 24 types of graph $G \in \mathcal{G}$ as shown in Table 1. Since the approach used to classify all the graphs G is rather long and repetitive, we shall not discuss it here. The reader may refer to Theorems 1 and 2 in [5] for detail derivation of the graphs.

We are now ready to study the chromaticity of all types of 2-connected $(n, n+4)$-graphs having exactly three triangles and one induced C_4. We first note that if $H \sim G_i (1 \leq i \leq 24)$ in Table 1, then H must be of type G_j for some $1 \leq j \leq 24$ in Table 1 as well. For convenience, we shall say that the graph G_i, or any of its relatives, is of type (i).

We now present our main result in the following theorem.

Theorem 3.1. *For a graph G, let $\langle G \rangle = \{H \mid H \sim G\}$. We have*

1. *$H \in \langle G_1(a) \rangle$ if and only if H is of type $G_1(a)$.*
2. *$H \in \langle G_2(b) \rangle$ if and only if H is of type $G_2(b)$.*
3. *$H \in \langle G_3(c) \rangle$ if and only if H is of type $G_3(c)$.*
4. *$H \in \langle G_4(d) \rangle$ if and only if H is of type $G_4(d)$.*
5. *$H \in \langle G_5(e) \rangle$ if and only if H is of type $G_5(e)$.*
6. *$\langle G_6(f) \rangle = \{G_6(f), G_{18}(f)\}$.*
7. *$\langle G_7(g) \rangle = \{G_7(g), G_{20}(g)\}$.*
8. *$H \in \langle G_8(h) \rangle$ if and only if $H \cong G_8(h)$ or $G_{22}(\phi, \gamma)$ with $\phi + \gamma = h - 1$, or H is of type $G_{24}(h)$.*
9. *$G_9(j)$ is χ-unique.*
10. *$\langle G_{10}(k) \rangle = \{G_{10}(k), G_{21}(k)\}$.*
11. *$G_{11}(m)$ is χ-unique.*
12. *$H \in \langle G_{12}(n) \rangle$ if and only if H is of type $G_{12}(n)$ or $G_{19}(n)$.*
13. *$H \in \langle G_{13}(p) \rangle$ if and only if H is of type $G_{13}(p)$.*
14. *$H \in \langle G_{14}(q) \rangle$ if and only if H is of type $G_{14}(q)$.*
15. *$\langle G_{15}(r, s) \rangle = \{G_{15}(r', s')$ with $r' + s' = r + s\}$.*
16. *$H \in \langle G_{16}(t, u) \rangle$ if and only if H is of type $G_{16}(t', u')$ with $t + u = t' + u'$.*
17. *$G_{17}(v)$ is χ-unique.*
18. *$\langle G_{18}(w) \rangle = \{G_6(w), G_{18}(w)\}$.*
19. *$H \in \langle G_{19}(x) \rangle$ if and only if H is of type $G_{12}(x)$ or $G_{19}(x)$.*
20. *$\langle G_{20}(y) \rangle = \{G_7(y), G_{20}(y)\}$.*
21. *$\langle G_{21}(z) \rangle = \{G_{10}(z), G_{21}(z)\}$.*
22. *$H \in \langle G_{22}(\phi, \gamma) \rangle$ if and only if $H \cong G_8(\phi + \gamma + 1)$ or $G_{22}(\phi', \gamma')$ with $\phi' + \gamma' = \phi + \gamma$, or H is of type $G_{24}(\phi + \gamma + 1)$.*
23. *$G_{23}(\psi)$ is χ-unique.*
24. *$H \in \langle G_{24}(\rho) \rangle$ if and only if $H \cong G_8(\rho)$ or $G_{22}(\phi, \gamma)$ with $\phi + \gamma = \rho - 1$, or H is of type $G_{24}(\rho)$.*

4. Chromatic polynomials of the graphs

Before proving our main result, we present here some useful information about the chromatic polynomial of G_i ($1 \leq i \leq 24$). Let $W(n,k)$ be the graph of order n obtained from the wheel W_n by deleting all but k consecutive spokes. Using Lemma 2.1, we have $P(W(n,4)) = (\lambda-2)(\lambda-3)P(C_{n-2}) + (\lambda-2)P(C_{n-3})$ and $P(W(n,3)) = (\lambda-2)[P(C_{n-1}) - P(C_{n-2})]$ which will be used in computing the chromatic polynomials of the graphs in Table 1.

Lemma 4.1.

(1) $$\begin{aligned} P(G_1) &= P(C_{a+1})P(W(6,4))/\lambda(\lambda-1) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^3-6\lambda^2+13\lambda-11)Q_{a+1}(\lambda) \\ &= \lambda(\lambda-1)N_1(\lambda), \end{aligned}$$

where $N_1(\lambda) = (\lambda-2)(\lambda^3-6\lambda^2+13\lambda-11)Q_{a+1}(\lambda)$
and $N_1(1) = (-1)(1-6+13-11)(-1)^{a+1} = 3(-1)^{a+1}$.

(2) $$\begin{aligned} P(G_2) &= P(C_4)P(W(b+4,4))/\lambda(\lambda-1) \\ &= P(C_4)[(\lambda-2)(\lambda-3)P(C_{b+2}) + (\lambda-2)P(C_{b+1})]/\lambda(\lambda-1) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)[(\lambda-3)Q_{b+2}(\lambda)+Q_{b+1}(\lambda)] \\ &= \lambda(\lambda-1)N_2(\lambda), \end{aligned}$$

where $N_2(\lambda) = (\lambda-2)(\lambda^2-3\lambda+3)[(\lambda-3)Q_{b+2}(\lambda)+Q_{b+1}(\lambda)]$
and $N_3(1) = (-1)(1-3+3)[(-2)(-1)^{b+2}+(-1)^{b+1}] = 3(-1)^b$.

(3) $$\begin{aligned} P(G_3) &= [(\lambda-2)^2P(C_{c+2})P(C_4)/\lambda(\lambda-1)] - [(\lambda-2)^2P(C_{c+1})P(C_4)/\lambda(\lambda-1)] \\ &= \lambda(\lambda-1)(\lambda-2)^2(\lambda^2-3\lambda+3)[Q_{c+2}(\lambda)-Q_{c+1}(\lambda)] \\ &= \lambda(\lambda-1)N_3(\lambda), \end{aligned}$$

where $N_3(\lambda) = (\lambda-2)^2(\lambda^2-3\lambda+3)[Q_{c+2}(\lambda)-Q_{c+1}(\lambda)]$
and $N_3(1) = (-1)^2(1-3+3)[(-1)^{c+2}-(-1)^{c+1}] = 2(-1)^c$.

(4) $$\begin{aligned} P(G_4) &= (\lambda-2)^3P(\theta_{d,2,2}) \\ &= \lambda(\lambda-1)(\lambda-2)^3M_{d,2,2}(\lambda) \\ &= \lambda(\lambda-1)N_4(\lambda), \end{aligned}$$

where $N_4(\lambda) = (\lambda-2)^3M_{d,2,2}(\lambda)$
and $N_4(1) = (-1)^3(-1)^{d+5} = (-1)^d$.

(5) $$\begin{aligned} P(G_5) &= (\lambda-2)^3P(C_{e+1})P(C_4)/\lambda(\lambda-1) \\ &= (\lambda-2)^3(\lambda^2-3\lambda+3)P(C_{e+1}) \\ &= \lambda(\lambda-1)(\lambda-2)^3(\lambda^2-3\lambda+3)Q_{e+1}(\lambda) \\ &= \lambda(\lambda-1)N_5(\lambda), \end{aligned}$$

where $N_1(\lambda) = (\lambda-2)^3(\lambda^2-3\lambda+3)Q_{e+1}(\lambda)$
and $N_5(1) = (-1)^3(1-3+3)(-1)^{e+1} = (-1)^e$.

(6) $$\begin{aligned} P(G_6) &= (\lambda-2)^2P(\theta_{f,2,2}) - (\lambda-2)^2P(C_{f+2}) + (\lambda-2)(\lambda-3)P(C_{f+1}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)M_{f,2,2}(\lambda) - (\lambda-2)Q_{f+2}(\lambda) + (\lambda-3)Q_{f+1}(\lambda)] \\ &= \lambda(\lambda-1)N_6(\lambda), \end{aligned}$$

where $N_6(\lambda) = (\lambda-2)[(\lambda-2)M_{f,2,2}(\lambda) - (\lambda-2)Q_{f+2}(\lambda) + (\lambda-3)Q_{f+1}(\lambda)]$
and $N_6(1) = (-1)[(-1)(-1)^{f+5} - (-1)(-1)^{f+2} + (-2)(-1)^{f+1}] = 4(-1)^{f+1}$.

(7) $$\begin{aligned} P(G_7) &= (\lambda-2)^2P(\theta_{g+1,3,1}) - [(\lambda-3)P(\theta_{g+1,3,1}) + P(\theta_{g,2,2})] \\ &= (\lambda^2-5\lambda+7)P(\theta_{g+1,3,1}) - P(\theta_{g,2,2}) \\ &= \lambda(\lambda-1)[(\lambda^2-5\lambda+7)M_{g+1,3,1}(\lambda) - M_{g,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_7(\lambda), \end{aligned}$$

where $N_7(\lambda) = (\lambda^2-5\lambda+7)M_{g+1,3,1}(\lambda) - M_{g,2,2}(\lambda)$
and $N_7(1) = (1-5+7)(-1)^{g+6} - (-1)^{g+5} = 4(-1)^g$.

(8) $$\begin{aligned} P(G_8) &= (\lambda-2)^2P(\theta_{h+1,2,2}) - [(\lambda-3)P(\theta_{h+1,2,2}) + P(\theta_{h,3,1})] \\ &= (\lambda^2-5\lambda+7)P(\theta_{h+1,2,2}) - P(\theta_{h,3,1}) \\ &= \lambda(\lambda-1)[(\lambda^2-5\lambda+7)M_{h+1,2,2}(\lambda) - M_{h,3,1}(\lambda)] \\ &= \lambda(\lambda-1)N_8(\lambda), \end{aligned}$$

where $N_8(\lambda) = (\lambda^2 - 5\lambda + 7)M_{h+1,2,2}(\lambda) - M_{h,3,1}(\lambda)$
and $N_8(1) = (1 - 5 + 7)(-1)^{h+6} - (-1)^{h+5} = 4(-1)^h$.

(9) $$\begin{aligned} P(G_9) &= (\lambda-2)[P(\theta_{j+1,2,2}) - P(\theta_{j,3,1})] - [P(\theta_{j+1,2,2}) - P(\theta_{j,3,1}) - (\lambda-2)^2 P(C_{j+1})] \\ &= (\lambda-3)[P(\theta_{j+1,2,2}) - P(\theta_{j,3,1})] + (\lambda-2)^2 P(C_{j+1}) \\ &= \lambda(\lambda-1)(\lambda-3)[M_{j+1,2,2}(\lambda) - M_{j,3,1}(\lambda)] + \lambda(\lambda-1)(\lambda-2)^2 Q_{j+1}(\lambda) \\ &= \lambda(\lambda-1)N_9(\lambda), \end{aligned}$$

where $N_9(\lambda) = (\lambda-3)[M_{j+1,2,2}(\lambda) - M_{j,3,1}(\lambda)] + (\lambda-2)^2 Q_{j+1}(\lambda)$
and $N_9(1) = (-2)[(-1)^{j+6} - (-1)^{j+5}] + (-1)^2(-1)^{j+1} = 5(-1)^{j+1}$.

(10) $$\begin{aligned} P(G_{10}) &= [P(C_4)P(W(k+4,3))/\lambda(\lambda-1)] - (\lambda-1)P(W(k+4,3)) + P(W(k+4,4)) \\ &= (\lambda-2)(\lambda^2-3\lambda+3)[P(C_{k+3}) - P(C_{k+2})] - (\lambda-1)(\lambda-2)[P(C_{k+3}) \\ &\quad - P(C_{k+2})] + (\lambda-2)(\lambda-3)P(C_{k+2}) + (\lambda-2)P(C_{k+1}) \\ &= (\lambda-2)^3[P(C_{k+3}) - P(C_{k+2})] + (\lambda-2)(\lambda-3)P(C_{k+2}) + (\lambda-2)P(C_{k+1}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)^2 Q_{k+3}(\lambda) - (\lambda^2-5\lambda+7)Q_{k+2}(\lambda) + Q_{k+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{10}(\lambda), \end{aligned}$$

where $N_{10}(\lambda) = (\lambda-2)[(\lambda-2)^2 Q_{k+3}(\lambda) - (\lambda^2-5\lambda+7)Q_{k+2}(\lambda) + Q_{k+1}(\lambda)]$
and $N_{10}(1) = (-1)[(-1)^2(-1)^{k+3} - (1-5+7)(-1)^{k+2} + (-1)^{k+1}] = 5(-1)^k$.

(11) $$\begin{aligned} P(G_{11}) &= (\lambda-2)P(W(m+5,3)) - (\lambda-2)P(W(m+4,4)) \\ &= (\lambda-2)^2 P(C_{m+4}) - (\lambda-2)^2 P(C_{m+3}) - (\lambda-2)^2(\lambda-3)P(C_{m+2}) - (\lambda-2)^2 P(C_{m+1}) \\ &= \lambda(\lambda-1)(\lambda-2)^2[Q_{m+4}(\lambda) - Q_{m+3}(\lambda) - (\lambda-3)Q_{m+2}(\lambda) - Q_{m+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{11}(\lambda), \end{aligned}$$

where $N_{11}(\lambda) = (\lambda-2)^2[Q_{m+4}(\lambda) - Q_{m+3}(\lambda) - (\lambda-3)Q_{m+2}(\lambda) - Q_{m+1}(\lambda)]$
and $N_{11}(1) = (-1)^2[(-1)^{m+4} - (-1)^{m+3} - (-2)(-1)^{m+2} - (-1)^{m+1}] = 5(-1)^m$.

(12) $$\begin{aligned} P(G_{12}) &= (\lambda-2)[P(\theta_{n+2,2,2}) - P(\theta_{n+1,3,1})] - (\lambda-2)^2 P(\theta_{n,3,1}) \\ &= \lambda(\lambda-1)(\lambda-2)[M_{n+2,2,2}(\lambda) - M_{n+1,3,1}(\lambda) - (\lambda-2)M_{n,3,1}(\lambda)] \\ &= \lambda(\lambda-1)N_{12}(\lambda), \end{aligned}$$

where $N_{12}(\lambda) = (\lambda-2)[M_{n+2,2,2}(\lambda) - M_{n+1,3,1}(\lambda) - (\lambda-2)M_{n,3,1}(\lambda)]$
and $N_{12}(1) = (-1)[(-1)^{n+7} - (-1)^{n+6} - (-1)(-1)^{n+5}] = 3(-1)^n$.

(13) $$\begin{aligned} P(G_{13}) &= (\lambda-2)[P(\theta_{p+2,2,2}) - P(\theta_{p+1,3,1})] - (\lambda-2)^2 P(\theta_{p,2,2}) \\ &= \lambda(\lambda-1)(\lambda-2)[M_{p+2,2,2}(\lambda) - M_{p+1,3,1}(\lambda) - (\lambda-2)M_{p,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{13}(\lambda), \end{aligned}$$

where $N_{13}(\lambda) = (\lambda-2)[M_{p+2,2,2}(\lambda) - M_{p+1,3,1}(\lambda) - (\lambda-2)M_{p,2,2}(\lambda)]$
and $N_{13}(1) = (-1)[(-1)^{p+7} - (-1)^{p+6} - (-1)(-1)^{p+5}] = 3(-1)^p$.

(14) $$\begin{aligned} P(G_{14}) &= (\lambda-2)^2[P(\theta_{q+1,2,2}) - P(\theta_{q,3,1})] \\ &= \lambda(\lambda-1)(\lambda-2)^2[M_{q+1,2,2}(\lambda) - M_{q,3,1}(\lambda)] \\ &= \lambda(\lambda-1)N_{14}(\lambda), \end{aligned}$$

where $N_{14}(\lambda) = (\lambda-2)^2[M_{q+1,2,2}(\lambda) - M_{q,3,1}(\lambda)]$
and $N_{14}(1) = (-1)^2[(-1)^{q+6} - (-1)^{q+5}] = 2(-1)^q$.

(15) $$\begin{aligned} P(G_{15}) &= (\lambda-2)^2 P(\theta_{r+s+2,2,2}) - (\lambda-2)P(\theta_{r+s+2,2,2}) + (\lambda-2)P(\theta_{r+s+1,2,2}) \\ &= (\lambda-2)(\lambda-3)P(\theta_{r+s+2,2,2}) + (\lambda-2)P(\theta_{r+s+1,2,2}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)M_{r+s+2,2,2}(\lambda) + M_{r+s+1,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{15}(\lambda), \end{aligned}$$

where $N_{15}(\lambda) = (\lambda-2)[(\lambda-3)M_{r+s+2,2,2}(\lambda) + M_{r+s+1,2,2}(\lambda)]$
and $N_{15}(1) = (-1)[(-2)(-1)^{r+s+7} + (-1)^{r+s+6}] = 3(-1)^{r+s+1}$.

(16) $$\begin{aligned} P(G_{16}) &= (\lambda-2)^2 P(\theta_{t+u+2,2,2}) - (\lambda-2)^2 P(\theta_{t+u+1,2,2}) \\ &= \lambda(\lambda-1)(\lambda-2)^2[M_{t+u+2,2,2}(\lambda) - M_{t+u+1,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{16}(\lambda), \end{aligned}$$

where $N_{16}(\lambda) = (\lambda-2)^2[M_{t+u+2,2,2}(\lambda) - M_{t+u+1,2,2}(\lambda)]$
and $N_{16}(1) = (-1)^2[(-1)^{t+u+7} - (-1)^{t+u+6}] = 2(-1)^{t+u+1}$.

(17) $$\begin{aligned} P(G_{17}) &= (\lambda-2)^2P(\theta_{v,2,2}) - (\lambda-2)P(\theta_{v,2,2}) + (\lambda-2)^2P(C_{v+1}) \\ &= (\lambda-2)(\lambda-3)P(\theta_{v,2,2}) + (\lambda-2)^2P(C_{v+1}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-3)M_{v,2,2}(\lambda) + (\lambda-2)Q_{v+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{17}(\lambda), \end{aligned}$$

where $N_{17}(\lambda) = (\lambda-2)[(\lambda-3)M_{v,2,2}(\lambda) + (\lambda-2)Q_{v+1}(\lambda)]$

and $N_{17}(1) = (-1)[(-2)(-1)^{v+5} + (-1)(-1)^{v+1}] = 3(-1)^{v+1}$.

(18) $$\begin{aligned} P(G_{18}) &= [P(C_4)P(W(w+3,3))/\lambda(\lambda-1)] - (\lambda-1)P(W(w+3,3)) + [P(K_4)P(C_{w+1})/\lambda(\lambda-1)] \\ &= [(\lambda^2-3\lambda+3) - (\lambda-1)]P(W(w+3,3)) + (\lambda-2)(\lambda-3)P(C_{w+1}) \\ &= (\lambda-2)^2[P(\theta_{w,2,2}) - P(C_{w+2})] + (\lambda-2)(\lambda-3)P(C_{w+1}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)M_{w,2,2}(\lambda) - (\lambda-2)Q_{w+2}(\lambda) + (\lambda-3)Q_{w+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{18}(\lambda), \end{aligned}$$

where $N_{18}(\lambda) = (\lambda-2)[(\lambda-2)M_{w,2,2}(\lambda) - (\lambda-2)Q_{w+2}(\lambda) + (\lambda-3)Q_{w+1}(\lambda)]$

and $N_{18}(1) = (-1)[(-1)(-1)^{w+5} - (-1)(-1)^{w+2} + (-2)(-1)^{w+1}] = 4(-1)^{w+1}$.

(19) $$\begin{aligned} P(G_{19}) &= (\lambda-2)[P(\theta_{x+2,2,2}) - P(\theta_{x+1,3,1}) - (\lambda-2)P(\theta_{x,3,1})] \\ &= \lambda(\lambda-1)(\lambda-2)[M_{x+2,2,2}(\lambda) - M_{x+1,3,1}(\lambda) - (\lambda-2)M_{x,3,1}(\lambda)] \\ &= \lambda(\lambda-1)N_{19}(\lambda), \end{aligned}$$

where $N_{19}(\lambda) = (\lambda-2)[M_{x+2,2,2}(\lambda) - M_{x+1,3,1}(\lambda) - (\lambda-2)M_{x,3,1}(\lambda)]$

and $N_{19}(1) = (-1)[(-1)^{x+7} - (-1)^{x+6} - (-1)(-1)^{x+5}] = 3(-1)^x$.

(20) $$\begin{aligned} P(G_{20}) &= (\lambda-2)^2P(\theta_{y+1,3,1}) - [(\lambda-3)P(\theta_{y+1,3,1}) + P(\theta_{y,2,2})] \\ &= (\lambda^2-5\lambda+7)P(\theta_{y+1,3,1}) - P(\theta_{y,2,2}) \\ &= \lambda(\lambda-1)[(\lambda^2-5\lambda+7)M_{y+1,3,1}(\lambda) - M_{y,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{20}(\lambda), \end{aligned}$$

where $N_{20}(\lambda) = (\lambda^2-5\lambda+7)M_{y+1,3,1}(\lambda) - M_{y,2,2}(\lambda)$

and $N_{20}(1) = (1-5+7)(-1)^{y+6} - (-1)^{y+5} = 4(-1)^y$.

(21) $$\begin{aligned} P(G_{21}) &= [P(C_4)P(W(z+4,3))/\lambda(\lambda-1)] - (\lambda-1)P(W(z+4,3)) + P(W(z+4,4)) \\ &= (\lambda-2)(\lambda^2-3\lambda+3)[P(C_{z+3}) - (\lambda-1)(\lambda-2)[P(C_{z+3}) - P(C_{z+2})] \\ &\quad + P(C_{z+2})] - (\lambda-2)(\lambda-3)P(C_{z+2}) + (\lambda-2)P(C_{z+1}) \\ &= (\lambda-2)^3[P(C_{z+3}) - P(C_{z+2})] + (\lambda-2)(\lambda-3)P(C_{z+2}) + (\lambda-2)P(C_{z+1}) \\ &= \lambda(\lambda-1)(\lambda-2)[(\lambda-2)^2Q_{z+3}(\lambda) - (\lambda^2-5\lambda+7)Q_{z+2}(\lambda) + Q_{z+1}(\lambda)] \\ &= \lambda(\lambda-1)N_{21}(\lambda), \end{aligned}$$

where $N_{21}(\lambda) = (\lambda-2)[(\lambda-2)^2Q_{z+3}(\lambda) - (\lambda^2-5\lambda+7)Q_{z+2}(\lambda) + Q_{z+1}(\lambda)]$

and $N_{21}(1) = (-1)[(-1)^2(-1)^{z+3} - (1-5+7)(-1)^{z+2} + (-1)^{z+1}] = 5(-1)^z$.

(22) $$\begin{aligned} P(G_{22}) &= (\lambda-2)[P(\theta_{\phi+\gamma+3,2,2}) - P(\theta_{\phi+\gamma+2,3,1}) - P(\theta_{\phi+\gamma+2,3,1}) + P(\theta_{\phi+\gamma+1,2,2})] \\ &= (\lambda-2)[P(\theta_{\phi+\gamma+3,2,2}) - 2P(\theta_{\phi+\gamma+2,3,1}) + P(\theta_{\phi+\gamma+1,2,2})] \\ &= \lambda(\lambda-1)(\lambda-2)[M_{\phi+\gamma+3,2,2}(\lambda) - 2M_{\phi+\gamma+2,3,1}(\lambda) + M_{\phi+\gamma+1,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{22}(\lambda), \end{aligned}$$

where $N_{22}(\lambda) = (\lambda-2)[M_{\phi+\gamma+3,2,2}(\lambda) - 2M_{\phi+\gamma+2,3,1}(\lambda) + M_{\phi+\gamma+1,2,2}(\lambda)]$

and $N_{22}(1) = (-1)[(-1)^{\phi+\gamma+8} - 2(-1)^{\phi+\gamma+7} + (-1)^{\phi+\gamma+6}] = 4(-1)^{\phi+\gamma+1}$.

(23) $$\begin{aligned} P(G_{23}) &= (\lambda-2)[P(\theta_{\psi+2,3,1}) - P(\theta_{\psi+1,2,2})] - [P(\theta_{\psi,3,3}) - 2P(\theta_{\psi+1,2,2}) + P(C_{\psi+3})] \\ &= (\lambda-2)P(\theta_{\psi+2,3,1}) - (\lambda-4)P(\theta_{\psi+1,2,2}) - P(\theta_{\psi,3,3}) - P(C_{\psi+3}) \\ &= \lambda(\lambda-1)[(\lambda-2)M_{\psi+2,3,1}(\lambda) - (\lambda-4)M_{\psi+1,2,2}(\lambda) - M_{\psi,3,3}(\lambda) - Q_{\psi+3}(\lambda)] \\ &= \lambda(\lambda-1)N_{23}(\lambda), \end{aligned}$$

where $N_{23}(\lambda) = (\lambda-2)M_{\psi+2,3,1}(\lambda) - (\lambda-4)M_{\psi+1,2,2}(\lambda) - M_{\psi,3,3}(\lambda) - Q_{\psi+3}(\lambda)$

and $N_{23}(1) = (-1)(-1)^{\psi+7} - (-3)(-1)^{\psi+6} - (-1)^{\psi+7} - (-1)^{\psi+3} = 6(-1)^{\psi}$.

(24) $$\begin{aligned} P(G_{24}) &= (\lambda-2)[P(\theta_{\rho+2,2,2}) - P(\theta_{\rho+1,3,1}) - P(\theta_{\rho+1,3,1}) + P(\theta_{\rho,2,2})] \\ &= \lambda(\lambda-1)(\lambda-2)[M_{\rho+2,2,2}(\lambda) - 2M_{\rho+1,3,1}(\lambda) + M_{\rho,2,2}(\lambda)] \\ &= \lambda(\lambda-1)N_{24}(\lambda), \end{aligned}$$

where $N_{24}(\lambda) = (\lambda-2)[M_{\rho+2,2,2}(\lambda) - 2M_{\rho+1,3,1}(\lambda) + M_{\rho,2,2}(\lambda)]$

and $N_{24}(1) = (-1)[(-1)^{\rho+7} - 2(-1)^{\rho+6} + (-1)^{\rho+5}] = 4(-1)^{\rho}$.

Proof. The computation of the chromatic polynomials in this lemma is straight-forward using Lemmas 2.1, 2.2, 2.4 and 2.5. □

Lemma 4.2. *Let $\mathcal{G}_1 = \{G_4, G_5\}$, $\mathcal{G}_2 = \{G_3, G_{14}, G_{16}\}$, $\mathcal{G}_3 = \{G_1, G_2, G_{12}, G_{13}, G_{15}, G_{17}, G_{19}\}$, $\mathcal{G}_4 = \{G_6, G_7, G_8, G_{18}, G_{20}, G_{22}, G_{24}\}$, $\mathcal{G}_5 = \{G_9, G_{10}, G_{11}, G_{21}\}$ and $\mathcal{G}_6 = \{G_{23}\}$. Then, for each $G \in \mathcal{G}_i$, $i = 1, 2, 3, 4, 5, 6$, $H \sim G$ implies that H must be of type G or G' for a G' in $\mathcal{G}_i$.*

Proof. It follows directly from Lemma 4.1 that if $i \neq j$, $G_p \in \mathcal{G}_i$ and $G_q \in \mathcal{G}_j$, then $|N_p(1)| = i \neq j = |N_q(1)|$. Note that $N_p(\lambda)$ and $N_q(\lambda)$ are as defined in Lemma 4.1 □

From Lemma 4.1, we can also get the following useful information.

Lemma 4.3. (1) *$G_6(f) \sim G_{18}(w)$ if and only if $f = w$.*
(2) *$G_7(g) \sim G_{20}(y)$ if and only if $g = y$.*
(3) *$G_{10}(k) \sim G_{21}(z)$ if and only if $k = z$.*
(4) *$G_{12}(n) \sim G_{19}(x)$ if and only if $n = x$.*
(5) *$G_{22}(\phi, \gamma) \sim G_{24}(\rho)$ if and only if $\rho - 1 = \phi + \gamma$.*

Proof. The sufficiency of each part above follows directly from Lemma 4.1. To prove the necessity, we note that if $G_i \sim G_j$ for each part above, then both G_i and G_j must have the same order. It can then be checked that each of the above claims hold. □

Lemma 4.4. (1) *(a) $P(G_3) \neq P(G_{14})$, (b) $P(G_3) \neq P(G_{16})$.*
(2) *$P(G_4) \neq P(G_5)$.*
(3) *(a) $P(G_7) \neq P(G_8)$, (b) $P(G_8) \neq P(G_{20})$.*
(4) *(a) $P(G_{10}) \neq P(G_{11})$, (b) $P(G_{11}) \neq P(G_{21})$.*

Proof. (1) $P(G_3) = (\lambda - 1)^2(\lambda - 2)P(W(c + 3, 3)) - (\lambda - 2)^2 P(W(c + 3, 3))$,
$P(G_{14}) = (\lambda - 1)(\lambda - 2)^3 P(C_{q+2}) - (\lambda - 2)^2 P(W(q + 3, 3))$. and
$P(G_{16}) = (\lambda - 1)(\lambda - 2)P(W(t + u + 5, 3)) - (\lambda - 2)^2 P(W(t + u + 4, 3))$
(a) If $P(G_3) = P(G_{14})$, Lemma 2.3 implies that $c = q$. So, $(\lambda - 1)P(W(c + 3, 3)) = (\lambda - 2)^2 P(C_{c+2})$, a contradiction since $(\lambda - 2)^2 P(C_{c+2})$ is divisible by $(\lambda - 2)^2$ but not $(\lambda - 1)P(W(c + 3, 3))$.
(b) If $P(G_3) = P(G_{16})$, Lemma 2.3 implies that $c - 1 = t + u$. So, $(\lambda - 1)P(W(c + 3, 3)) = P(W(c + 4, 3))$, a contradiction since $(\lambda - 1)P(W(c + 3, 3))$ is divisible by $(\lambda - 1)^2$ but not $P(W(c + 4, 3))$
(2) $P(G_4) = (\lambda - 2)^3 P(\theta_{d,2,2})$ and $P(G_5) = (\lambda - 2)^3 P(\theta_{e,3,1})$. If $P(G_4) = P(G_5)$, Lemma 2.3 implies that $d = e$. So, $P(\theta_{e,3,1}) = P(\theta_{e,2,2})$, a contradiction since both $\theta_{e,3,1}$ and $\theta_{e,2,2}$ are χ-unique graphs that are not isomorphic.
(3) $P(G_7) = (\lambda - 1)(\lambda - 2)^3 P(C_{g+2}) - P(W(g + 5, 5))$ and
$P(G_8) = (\lambda - 1)P(W(h + 5, 4)) - P(W(h + 5, 5))$.
(a) If $P(G_7) = P(G_8)$, by Lemma 2.3, $g = h$. So, $(\lambda - 2)^3 P(C_{g+2}) = P(W(g + 5, 4))$, a contradiction since $(\lambda - 2)^3 P(C_{g+2})$ is divisible by $(\lambda - 2)^3$ but not $P(W(g + 5, 4))$.
(b) If $P(G_8) = P(G_{20})$, by Lemma 2.3, $h = y$. By Lemma 4.3 and the above result, we conclude that $P(G_8) \neq P(G_{20})$.
(4) $P(G_{10}) = (\lambda - 1)(\lambda - 2)P(W(k + 4, 3)) - P(W(k + 5, 4)) + (\lambda - 2)P(W(k + 3, 3))$ and
$P(G_{11}) = (\lambda - 1)P(W(m + 5, 4)) - P(W(m + 5, 4)) + (\lambda - 2)P(W(m + 3, 3))$.
(a) If $P(G_{10}) = P(G_{11})$, by Lemma 2.3, $k = m$. So, $(\lambda - 2)P(W(m + 4, 3)) = P(W(m + 5, 4))$, a contradiction since $(\lambda - 2)P(W(m + 4, 3))$ is divisible by $(\lambda - 2)^2$ but not $P(W(m + 5, 4))$.
(b) If $P(G_{11}) = P(G_{21})$, by Lemma 2.3, $m = z$. By Lemma 4.3 and the above result, we conclude that $P(G_{11}) \neq P(G_{21})$. □

Let $\omega = \lambda - 1$ and $[\omega^n]P(G_i)$ be the coefficient of ω^n in $P(G_i)$. Using Lemmas 2.4 and 4.1, and Software Maple, we then have the following straight-forward lemma.

Lemma 4.5.

$$\begin{aligned}(1)\quad P(G_1) &= (\lambda - 2)(\lambda^3 - 6\lambda^2 + 13\lambda - 11)P(C_{a+1}) \\ &= \omega(\omega - 1)(\omega^3 - 3\omega^2 + 4\omega - 3)(\omega^a + (-1)^{a+1})\end{aligned}$$

and $[\omega^2]P(G_1) = 7(-1)^a$.

$$\begin{aligned}(2)\quad P(G_2) &= (\lambda - 2)(\lambda^2 - 3\lambda + 3)[(\lambda - 3)P(C_{b+2}) + P(C_{b+1})] \\ &= \omega(\omega - 1)(\omega^2 - \omega + 1)[(\omega - 2)(\omega^{b+1} + (-1)^b) + (\omega^b + (-1)^{b+1})]\end{aligned}$$

and $[\omega^2]P(G_2) = 7(-1)^{b+1}$.

$$\begin{aligned}(3)\quad P(G_6) &= (\lambda - 2)^2 P(\theta_{f,2,2}) - (\lambda - 2)^2 P(C_{f+2}) + (\lambda - 2)(\lambda - 3)P(C_{f+1}) \\ &= (\lambda - 2)^4 P(C_{f+1}) + (\lambda - 1)^2(\lambda - 2)^2 P(C_f) - (\lambda - 2)^2 P(C_{f+2}) + (\lambda - 2)(\lambda - 3)P(C_{f+1}) \\ &= \omega(\omega - 1)^4(\omega^f + (-1)^{f+1}) + \omega^3(\omega - 1)^2(\omega^{f-1} + (-1)^f) \\ &\quad - \omega(\omega - 1)^2(\omega^{f+1} + (-1)^f) + \omega(\omega - 1)(\omega - 2)(\omega^f + (-1)^{f+1})\end{aligned}$$

and $[\omega^2]P(G_6) = 9(-1)^f$.

(4) $$\begin{aligned} P(G_7) &= (\lambda^2-5\lambda+7)P(\theta_{g+1,3,1}) - P(\theta_{g,2,2}) \\ &= (\lambda^2-5\lambda+7)(\lambda^2-3\lambda+3)P(C_{g+2}) - (\lambda-2)^2P(C_{g+1}) - (\lambda-1)^2P(C_g) \\ &= \omega(\omega^2-3\omega+3)(\omega^2-\omega+1)(\omega^{g+1}+(-1)^g) - \omega(\omega-1)^2(\omega^g+(-1)^{g+1}) - \omega^3(\omega^{g-1}+(-1)^g) \end{aligned}$$

and $[\omega^2]P(G_7) = 8(-1)^{g+1}$.

(5) $$\begin{aligned} P(G_8) &= (\lambda^2-5\lambda+7)P(\theta_{h+1,2,2}) - P(\theta_{h,3,1}) \\ &= (\lambda^2-5\lambda+7)[(\lambda-2)^2P(C_{h+2}) + (\lambda-1)^2P(C_{h+1})] - (\lambda^2-3\lambda+3)P(C_{h+1}) \\ &= \omega(\omega-1)^2(\omega^2-3\omega+3)(\omega^{h+1}+(-1)^h) + \omega^3(\omega^2-3\omega+3) \\ &\quad \times(\omega^h+(-1)^{h+1}) - \omega(\omega^2-\omega+1)(\omega^h+(-1)^{h+1}) \\ &= \omega^{h+6} - 4\omega^{h+5} + 7\omega^{h+4} - 7\omega^{h+3} + 4\omega^{h+2} - \omega^{h+1} + 2\omega^4(-1)^{h+1} \\ &\quad + 8\omega^3(-1)^h + 10\omega^2(-1)^{h+1} + 4\omega(-1)^h. \end{aligned}$$

and $[\omega^2]P(G_8) = 10(-1)^{h+1}$.

(6) $$\begin{aligned} P(G_{12}) &= (\lambda-2)[P(\theta_{n+2,2,2}) - P(\theta_{n+1,3,1})] - (\lambda-2)^2P(\theta_{n,3,1}) \\ &= (\lambda-2)^3P(C_{n+3}) + (\lambda-1)^2(\lambda-2)P(C_{n+2}) \\ &\quad - (\lambda-2)(\lambda^2-3\lambda+3)P(C_{n+2}) - (\lambda-2)^2(\lambda^2-3\lambda+3)P(C_{n+1}) \\ &= (\lambda-2)^3P(C_{n+3}) + (\lambda-2)^2P(C_{n+2}) - (\lambda-2)^2(\lambda^2-3\lambda+3)P(C_{n+1}) \\ &= \omega(\omega-1)^3(\omega^{n+2}+(-1)^{n+1}) + \omega(\omega-1)^2(\omega^{n+1}+(-1)^n) \\ &\quad - \omega(\omega-1)^2(\omega^2-\omega+1)(\omega^n+(-1)^{n+1}) \end{aligned}$$

and $[\omega^2]P(G_{12}) = 8(-1)^{n+1}$.

(7) $$\begin{aligned} P(G_{13}) &= (\lambda-2)[P(\theta_{p+2,2,2}) - P(\theta_{p+1,3,1})] - (\lambda-2)^2P(\theta_{p,2,2}) \\ &= (\lambda-2)^3P(C_{p+3}) + (\lambda-1)^2(\lambda-2)P(C_{p+2}) - (\lambda-2)(\lambda^2-3\lambda+3) \\ &\quad \times P(C_{p+2}) - (\lambda-2)^4P(C_{p+1}) - (\lambda-1)^2(\lambda-2)^2P(C_p) \\ &= (\lambda-2)^3P(C_{p+3}) + (\lambda-2)^2P(C_{p+2}) - (\lambda-2)^4P(C_{p+1}) - (\lambda-1)^2(\lambda-2)^2P(C_p) \\ &= \omega(\omega-1)^3(\omega^{p+2}+(-1)^{p+1}) + \omega(\omega-1)^2(\omega^{p+1}+(-1)^p) \\ &\quad - \omega(\omega-1)^4(\omega^p+(-1)^{p+1}) - \omega^3(\omega-1)^2(\omega^{p-1}+(-1)^p) \end{aligned}$$

and $[\omega^2]P(G_{13}) = 9(-1)^{p+1}$.

(8) $$\begin{aligned} P(G_{15}) &= (\lambda-2)(\lambda-3)P(\theta_{r+s+2,2,2}) + (\lambda-2)P(\theta_{r+s+1,2,2}) \\ &= (\lambda-2)^3(\lambda-3)P(C_{r+s+3}) + (\lambda-1)^2(\lambda-2)(\lambda-3)P(C_{r+s+2}) \\ &\quad + (\lambda-2)^3P(C_{r+s+2}) + (\lambda-1)^2(\lambda-2)P(C_{r+s+1}) \\ &= \omega(\omega-1)^3(\omega-2)(\omega^{r+s+2}+(-1)^{r+s+1}) + \omega^3(\omega-1)(\omega-2) \\ &\quad \times(\omega^{r+s+1}+(-1)^{r+s}) + \omega(\omega-1)^3(\omega^{r+s+1}+(-1)^{r+s}) + \omega^3(\omega-1)(\omega^{r+s}+(-1)^{r+s+1}) \end{aligned}$$

and $[\omega^2]P(G_{15}) = 10(-1)^{r+s}$.

(9) $$\begin{aligned} P(G_{17}) &= (\lambda-2)(\lambda-3)P(\theta_{v,2,2}) + (\lambda-2)^2P(C_{v+1}) \\ &= (\lambda-2)^3(\lambda-3)P(C_{v+1}) + (\lambda-1)^2(\lambda-2)(\lambda-3)P(C_v) + (\lambda-2)^2P(C_{v+1}) \\ &= \omega(\omega-1)^3(\omega-2)(\omega^v+(-1)^{v+1}) + \omega^3(\omega-1)(\omega-2) \\ &\quad \times(\omega^{v-1}+(-1)^v) + \omega(\omega-1)^2(\omega^v+(-1)^{v+1}) \end{aligned}$$

and $[\omega^2]P(G_{17}) = 9(-1)^v$.

(10) $$\begin{aligned} P(G_{22}) &= (\lambda-2)[P(\theta_{\phi+\gamma+3,2,2}) - 2P(\theta_{\phi+\gamma+2,3,1}) + P(\theta_{\phi+\gamma+1,2,2})] \\ &= (\lambda-2)^3P(C_{\phi+\gamma+4}) + (\lambda-1)^2(\lambda-2)P(C_{\phi+\gamma+3}) - 2(\lambda-2)(\lambda^2-3\lambda+3) \\ &\quad \times P(C_{\phi+\gamma+3}) + (\lambda-2)^3P(C_{\phi+\gamma+2}) + (\lambda-1)^2(\lambda-2)P(C_{\phi+\gamma+1}) \\ &= (\lambda-2)^3P(C_{\phi+\gamma+4}) - (\lambda-2)(\lambda^2-4\lambda+5)P(C_{\phi+\gamma+3}) \\ &\quad + (\lambda-2)^3P(C_{\phi+\gamma+2}) + (\lambda-1)^2(\lambda-2)P(C_{\phi+\gamma+1}) \\ &= \omega(\omega-1)^3(\omega^{\phi+\gamma+3}+(-1)^{\phi+\gamma}) - \omega(\omega-1)(\omega^2-2\omega+2)(\omega^{\phi+\gamma+2}+(-1)^{\phi+\gamma+1}) \\ &\quad + \omega(\omega-1)^3(\omega^{\phi+\gamma+1}+(-1)^{\phi+\gamma}) + \omega^3(\omega-1)(\omega^{\phi+\gamma}+(-1)^{\phi+\gamma+1}) \\ &= \omega^{\phi+\gamma+7} - 4\omega^{\phi+\gamma+6} + 7\omega^{\phi+\gamma+5} - 7\omega^{\phi+\gamma+4} + 4\omega^{\phi+\gamma+3} - \omega^{\phi+\gamma+2} + 2\omega^4(-1)^{\phi+\gamma} \\ &\quad + 8\omega^3(-1)^{\phi+\gamma+1} + 10\omega^2(-1)^{\phi+\gamma} + 4\omega(-1)^{\phi+\gamma+1}. \end{aligned}$$

Lemma 4.6. *$G_8 \sim G_{22}(\phi,\gamma) \sim G_{22}(\phi',\gamma') \sim G_{24}(\rho)$ if and only if $h-1 = \phi+\gamma = \phi'+\gamma' = \rho-1$.*

Proof. It follows directly from Lemmas 2.1 and 2.3, 4.3(5), 4.5(5) and 4.5(10). □

5. Proof of the main theorem

We are now ready to prove our main theorem (Theorem 3.1).

(1) Let $H \sim G_1(a)$. By Lemma 4.2, H must be of type (1), (2), (12), (13), (15), (17) or (19). If $H = G_1(a')$, Lemma 2.3 implies that $a' = a$. If $H = G_2(b)$, Lemma 2.3 implies that $a = b+1$. We note that $P(G_2) = (\lambda-2)(\lambda-3)P(\theta_{b+1,3,1}) + (\lambda-2)P(\theta_{b,3,1})$, whereas $P(G_1) = (\lambda-2)(\lambda-3)P(\theta_{a,3,1}) + (\lambda-2)P(\theta_{a,2,1})$. So, $P(G_2) = P(G_1)$ implies that $\theta_{b,3,1} \sim \theta_{b+1,2,1}$, a contradiction since both of $\theta_{b,3,1}$ and $\theta_{b+1,2,1}$ are χ-unique and non-isomorphic. Therefore, $P(G_2) \neq P(G_1)$. Lemma 4.5 implies that $[\omega^2]P(G_1) \neq [\omega^2]P(G_{12})$ or $[\omega^2]P(G_{13})$ or $[\omega^2]P(G_{15})$ or $[\omega^2]P(G_{17})$. Thus, H cannot be of type (12), (13), (15) or (17). If $H = G_{19}(x)$, Lemma 2.3 implies that $a = x + 1$. Since Lemma 4.3 implies that $G_{12}(x) \sim G_{19}(x)$, we conclude that $G_1 \not\sim G_{19}$. Thus, H is of type G_1.

(2) Let $H \sim G_2(b)$. By Lemma 4.2 and the above result, H must be of type (2), (12), (13), (15), (17) or (19). If $H = G_2(b')$, Lemma 2.3 implies that $b' = b$. Lemma 4.5 implies that $[\omega^2]P(G_2) \neq [\omega^2]P(G_{12})$ or $[\omega^2]P(G_{13})$ or $[\omega^2]P(G_{15})$ or $[\omega^2]P(G_{17})$. Thus, H cannot be of type (12), (13), (15) or (17). If $H = G_{19}(x)$, Lemma 2.3 implies that $b = x$. Since Lemma 4.3 implies that $G_{12}(x) \sim G_{19}(x)$, we conclude that $G_2 \not\sim G_{19}$. Thus, H is of type G_2.

(3) Let $H \sim G_3(c)$. By Lemma 4.2, H must be of type (3), (14) or (16). Lemma 4.4 further implies that H must be of type (3). If $H = G_3(c')$, Lemma 2.3 implies that $c' = c$. Thus, H is of type G_3.

(4) Let $H \sim G_4(d)$. By Lemma 4.2, H is of type (4) or (5). If $H = G_4(d')$, Lemma 2.3 implies that $d' = d$. By Lemma 4.4, $P(G_5) \neq P(G_4)$. Thus, H is of type G_4.

(5) Let $H \sim G_5(e)$. By Lemma 4.2 and the above result, H must be of type (5). If $H = G_5(e')$, Lemma 2.3 implies that $e' = e$. Thus, H is of type G_5.

(6) Let $H \sim G_6(g)$. By Lemma 4.2, H must be of type (6), (7), (8), (18), (20), (22) or (24). If $H = G_6(f')$ or $G_{18}(w)$, Lemmas 2.3 and 4.3 implies that $f' = f = w$. Suppose $H = G_7$. Note that Lemma 4.5 implies that $[\omega^2]P(G_6) \neq [\omega^2]P(G_7)$ while Lemma 4.3 further implies that $P(G_7(g)) = P(G_{20}(g))$. Thus, H cannot be of type (7) or (20). If $P(G_6) = P(G_8) = P(G_{22}) = P(G_{24})$, then Lemma 2.3 implies that $f - 2 = h - 1 = \rho - 1 = \phi + \gamma$. However, Lemma 4.5 implies that $[\omega^2]P(G_6) \neq [\omega^2]P(G_8)$ and Lemma 4.6 implies that $P(G_8(h)) = P(G_{22}(h)) = P(G_{24}(\phi, \gamma))$ where $\phi + \gamma = h - 1$. Therefore, H cannot be of type (8), (22) or (24). Hence, $\langle G_6(f)\rangle = \{G_6(f), G_{18}(f)\}$.

(7) Let $H \sim G_7(g)$. By Lemma 4.2 and the above result, H must be of type (7), (8), (20), (22) or (24). If $H = G_7(g')$ or $G_{20}(y)$, Lemmas 2.3 and 4.3 imply that $g' = g = y$. Note that Lemma 4.5 implies that $[\omega^2]P(G_7) \neq [\omega^2]P(G_8)$. By Lemma 4.6, we conclude that H cannot be of type (8), (22) or (24). Hence, $\langle G_7(g)\rangle = \{G_7(g), G_{20}(g)\}$.

(8) Let $H \sim G_8(h)$. By Lemma 4.2 and the above results, H must be of type (8), (22) or (24). The result then follows from Lemma 4.6.

(9) Let $H \sim G_9(j)$. By Lemma 4.2, H must be of type (9), (10), (11) or (21). If $H = G_9(j')$, Lemma 2.3 implies that $j' = j$. If $H = G_{10}(k)$, Lemma 2.3 implies that $k+1 = j$. We note that $P(G_{10}) = (\lambda-3)[P(\theta_{k+2,3,1}) - P(\theta_{k+1,2,2})] + (\lambda-2)P(\theta_{k,3,1})$ and $P(G_9) = (\lambda-3)[P(\theta_{j+1,3,1}) - P(\theta_{j,2,2})] + (\lambda-2)P(\theta_{j,2,1})$. So, $P(G_9) = P(G_{10})$ implies that $\theta_{k,3,1} \sim \theta_{k+1,2,1}$, a contradiction since $\theta_{k,3,1}$ and $\theta_{k+1,2,1}$ are χ-unique and non-isomorphic. Lemma 4.3 further implies that H cannot be of type (21). If $H = G_{11}(m)$, Lemma 2.3 implies that $m+1 = j$. We note that $P(G_{11}) = (\lambda-3)[P(\theta_{m+2,3,1}) - P(\theta_{m+1,2,2})] + (\lambda-2)P(\theta_{m,2,2})$. So, $P(G_9) = P(G_{11})$ implies that $\theta_{m,2,2} \sim \theta_{m+1,2,1}$, a contradiction since $\theta_{m,2,2}$ and $\theta_{m+1,2,1}$ are χ-unique and non-isomorphic. Thus, G_9 is χ-unique.

(10) Let $H \sim G_{10}(k)$. By Lemma 4.2 and the above result, H must be of type (10), (11) or (21). If $H = G_{10}(k')$ or $G_{21}(z)$, Lemma 2.3 and 4.3 imply that $k' = k = z$. Lemma 4.4 further implies that $P(G_{10}) \neq P(G_{11})$. Thus, $\langle G_{10}(k)\rangle = \{G_{10}(k), G_{21}(k)\}$.

(11) Let $H \sim G_{11}(m)$. By Lemma 4.2 and the above results, we conclude that H must be of type (11). If $H = G_{11}(m')$, Lemma 2.3 implies that $m' = m$. Thus, G_{11} is χ-unique.

(12) Let $H \sim G_{12}(n)$. By Lemma 4.2 and the above results, H must be of type (12), (13), (15), (17) or (19). If $H = G_{12}(p')$ or $G_{19}(x)$, Lemma 2.3 and 4.3 imply that $n' = n = x$. Note that Lemma 4.5 implies that $[\omega^2]P(G_{12}) \neq [\omega^2]P(G_{13})$ or $[\omega^2]P(G_{15})$ or $[\omega^2]P(G_{17})$. Thus, H cannot be of type (13), (15) or (17). Hence, $H \in \langle G_{12}(n)\rangle$ if and only if H is of type $G_{12}(n)$ or $G_{19}(n)$.

(13) Let $H \sim G_{13}(p)$. By Lemma 4.2 and the above results, H must be of type (13), (15) or (17). If $H = G_{13}(p')$, Lemma 2.3 implies that $p' = p$. Lemma 4.5 implies that $[\omega^2]P(G_{13}) \neq [\omega^2]P(G_{15})$. Thus, H cannot be of type (15). If $H = G_{17}(v)$, Lemma 2.3 implies that $v = p + 1$. Note that $P(G_{13}) = (\lambda-2)(\lambda-3)P(\theta_{p+1,2,2}) + (\lambda-2)P(\theta_{p,3,1})$ and $P(G_{17}) = (\lambda-2)(\lambda-3)P(\theta_{v,2,2}) + (\lambda-2)P(\theta_{v,2,1})$. So, $P(G_{13}) = P(G_{17})$ implies that $\theta_{p+1,2,1} \sim \theta_{p,3,1}$, a contradiction since $\theta_{p+1,2,1}$ and $\theta_{p,3,1}$ are χ-unique and non-isomorphic. Thus, H is of type G_{13}.

(14) Let $H \sim G_{14}(q)$. By Lemma 4.2 and the above result, H must be of type (14) or (16). If $H = G_{14}(q')$, Lemma 2.3 implies that $q' = q$. If $H = G_{16}(q)$, Lemma 2.3 implies that $q = t + u + 1$ and Lemma 4.1(14) and (16) then imply that $\theta_{q,2,2} \sim \theta_{q,3,1}$, a contradiction since both $\theta_{q,2,2}$ and $\theta_{q,3,1}$ are χ-unique and non-isomorphic. Thus, H is of type G_{14}.

(15) Let $H \sim G_{15}(r, s)$. By Lemma 4.2, H must be of type (15) or (17). If $H = G_{15}(r', s')$, Lemma 2.3 implies that $r' + s' = r + s$. Using Lemma 2.1, it is easy to show that $G_{15}(r, s) \sim G_{15}(r', s')$ if $r' + s' = r + s$. Lemma 4.5 implies that $[\omega^2]P(G_{15}) \neq [\omega^2]P(G_{17})$. Thus, H cannot be of type (17). Hence, $\langle G_{15}(r, s)\rangle = \{G_{15}(r', s')$ with $r + s = r' + s'\}$.

(16) Let $H \sim G_{16}(t, u)$. By Lemma 4.2 and the above results, H must be of type (16). If $H = G_{16}(t', u')$, Lemma 2.3 implies that $t' + u' = t + u$. Using Lemma 2.1, it is easy to show that $G_{16}(t, u) \sim G_{16}(t', u')$ if $t' + u' = t + u$. Hence, $\langle G_{16}(t, u)\rangle = \{G_{16}(t', u')$ with $t + u = t' + u'\}$.

(17) Let $H \sim G_{17}(v)$. By Lemma 4.2 and the above results, H must be of type (17). If $H = G_{17}(v')$, Lemma 2.3 implies that $v' = v$. Thus, G_{17} is χ-unique.

(18) The result follows from (6) above.

(19) The result follows from (12) above.

(20) The result follows from (7) above.

(21) The result follows from (10) above.

(22) The result follows from (8) above.

(23) Let $H \sim G_{23}(\psi)$. By Lemma 4.2, H must be of type (23). If $H = G_{23}(\psi')$, Lemma 2.3 implies that $\psi' = \psi$. Thus, $G_{23}(\psi)$ is χ-unique.

(24) The result follows from (8) above.

This completes the proof of our main theorem. □

Acknowledgment

The authors would like to express their sincere thanks to the referees for their helpful and valuable comments.

References

[1] M. Behzad, G. Chartrand, L. Lesniak-Foster, Graphs and Digraphs, Wadsworth, Belmont, CA, 1979.
[2] C.Y. Chao, E.G. Whitehead Jr., On chromatic equivalence of graphs, in: Y. Alavi, D.R. Lick (Eds.), Theory and Applications of Graph, in: Lecture Notes in Math., vol. 642, Springer, New York, 1978, pp. 121–131.
[3] F.M. Dong, K.L. Teo, K.M. Koh, A note on chromaticity of some 2-connected $(n, n+3)$-graphs, Discrete Math. 243 (2002) 217–221.
[4] K.M. Koh, K.L. Teo, Chromatic classes of 2-connected $(n, n+3)$-graphs with at least two triangles, Discrete Math. 127 (1994) 243–258.
[5] G.C. Lau, Y.H. Peng, Relative-closed family of graphs with exactly three triangles (submitted for publication).
[6] B. Loerinc, Chromatic uniqueness of the generalized θ-graphs, Discrete Math. 23 (1978) 313–316.
[7] Y.H. Peng, G.C. Lau, Chromatic classes of 2-connected $(n, n+4)$-graphs with at least four triangles, Discrete Math. 278 (2004) 209–218.
[8] K.L. Teo, K.M. Koh, Chromatic classes of certain 2-connected $(n, n+2)$-graphs, Ars Combin. 32 (1991) 65–76.
[9] E.G. Whitehead Jr., L.C. Zhao, Cutpoints and the chromatic polynomial, J. Graph Theory 8 (1984) 371–377.
[10] H. Whitney, The coloring of graphs, Ann. Math. 33 (1932) 688–718.
[11] A.A. Zykov, On some properties of linear complexes, Amer. Math. Soc. Transl. 79 (1952), Translated from Math. Sb. 24(66) (1949) 163–188.