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Colouring games on outerplanar graphs and trees

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ABSTRACT

Let f be a graph function which assigns to each graph H a non-negative integer $f(H) \leq |V(H)|$. The f-game chromatic number of a graph G is defined through a two-person game. Let X be a set of colours. Two players, Alice and Bob, take turns colouring the vertices of G with colours from X. A partial colouring c of G is legal (with respect to graph function f) if for any subgraph H of G, the sum of the number of colours used in H and the number of uncoloured vertices of H is at least $f(H)$. Both Alice and Bob must colour legally (i.e., the partial colouring produced needs to be legal). The game ends if either all the vertices are coloured or there are uncoloured vertices with no legal colour. In the former case, Alice wins the game. In the latter case, Bob wins the game. The f-game chromatic number of G, $\chi_g(f, G)$, is the least number of colours that the colour set X needs to contain so that Alice has a winning strategy. Let Acy be the graph function defined as $\mathrm{Acy}(K_2) = 2$, $\mathrm{Acy}(C_n) = 3$ for any $n \geq 3$ and $\mathrm{Acy}(H) = 0$ otherwise. Then $\chi_g(\mathrm{Acy}, G)$ is called the acyclic game chromatic number of G. In this paper, we prove that any outerplanar graph G has acyclic game chromatic number at most 7. For any integer k, let ϕ_k be the graph function defined as $\phi_k(K_2) = 2$ and $\phi_k(P_k) = 3$ (P_k is the path on k vertices) and $\phi_k(H) = 0$ otherwise. This paper proves that if $k \geq 8$ then for any tree T, $\chi_g(\phi_k, T) \leq 9$. On the other hand, if $k \leq 6$, then for any integer n, there is a tree T such that $\chi_g(\phi_k, T) \geq n$.

1. Introduction

Many variations of the chromatic number of graphs have been studied extensively in the literature. As a unification of many variants, Nešetřil and Ossona de Mendez [19] introduced the following generalization of the chromatic number of graphs. Suppose f is a graph function, which assigns to each graph H a non-negative integer $f(H) \leq |V(H)|$ such that $f(H) = f(H')$ if H and H' are isomorphic. An f-colouring of a graph G is a mapping c which assigns to each vertex of G a colour so that any subgraph H of G receives at least $f(H)$ colours. The f-chromatic number, $\chi(f, G)$, is the least number of colours used in an f-colouring of G.

In this paper, when defining a graph function f, we use the convention that whenever $f(H)$ is not given explicitly, then $f(H) = 0$. Let PA, Chi, Acy, Sta, Rel_d be graph functions defined as follows:

$$\mathrm{PA}(C_n) = 2, \quad \forall n \geq 3.$$
$$\mathrm{Chi}(K_2) = 2.$$
$$\mathrm{Acy}(K_2) = 2, \qquad \mathrm{Acy}(C_n) = 3, \quad \forall n \geq 3.$$
$$\mathrm{Sta}(K_2) = 2, \qquad \mathrm{Sta}(P_4) = 3.$$
$$\mathrm{Rel}_d(K_{1,d+1}) = 2.$$

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Then $\chi(\mathrm{PA}, G)$ is the *point-arboricity* of G, that is the smallest size of a vertex partition whose parts induce forests; $\chi(\mathrm{Chi}, G)$ is the same as the chromatic number $\chi(G)$ of G; $\chi(\mathrm{Acy}, G)$ is the *acyclic chromatic number* of G, that is the minimum number of colours needed to colour the vertices so that each colour class is an independent set, and the union of any two colour classes induces a forest; $\chi(\mathrm{Sta}, G)$ is the *star-chromatic number* of G, that is the minimum number of colours needed to colour the vertices so that each colour class is an independent set, and the union of any two colour classes induces a star forest; $\chi(\mathrm{Rel}_d, G)$ is the *d-relaxed chromatic number* of G, that is the minimum number of colours needed to colour the vertices of G so that each colour class induces a subgraph of maximum degree at most d.

We say a graph function f is *dominated* by a graph function f' if for any graph G, an f'-colouring of G is an f-colouring of G. In particular, if for any graph H, there is a subgraph H' of H such that $f(H) \le f'(H')$, then f is dominated by f'. For example, if $d' \ge d$, then $\mathrm{Rel}_{d'}$ is dominated by Rel_d, PA is dominated by Chi, Acy is dominated by Sta (although we have $\mathrm{Acy}(C_3) = 3$ and $\mathrm{Sta}(C_3) = 0$, but $\mathrm{Sta}(K_2) = 2$ implies that in an Sta-colouring of C_3, three colours are used).

A graph H is a *minor* of a graph G if H is isomorphic to a graph H' which can be obtained from a subgraph of G by contracting some edges. A graph G is called *H-minor free* if H is not a minor of G. A class $\mathcal{K}$ of graphs is called a proper minor closed class of graphs, if $G \in \mathcal{K}$ implies that $G' \in \mathcal{K}$ for any minor G' of G, and $\mathcal{K}$ does not contain all finite graphs. Nešetřil and Ossona de Mendez [19] studied the problem that for which graph function f, the parameter $\chi(f, G)$ is bounded by a constant on any proper minor closed class of graphs. For a graph H, the tree-depth $\mathrm{td}(H)$ of H is defined as follows: Suppose T is a rooted tree. The *height* of T is the number of vertices in a longest path from the root to a leaf. The *closure* of T is the graph Q on $V(T)$ in which $x \sim_Q y$ if x is an ancestor of y or y is an ancestor of x. The *tree-depth* of a connected graph G is the smallest height of a rooted tree T such that G is a subgraph of the closure of T. If G is disconnected, then the tree-depth of G is the maximum of the tree-depth of its components. For any integer p, let f_p be the graph function defined as $f_p(H) = \min\{p, \mathrm{td}(H)\}$. Nešetřil and Ossona de Mendez [19] proved that $\chi(f, G)$ is bounded by a constant on any proper minor closed classes of graphs if and only if f is dominated by some f_p. On the other hand, $\chi(f_p, G)$ is not only bounded on all proper minor closed classes of graphs, it is also bounded on some classes of graphs that are not minor closed. Classes of graphs for which $\chi(f_p, G)$ is bounded for all p are characterized in [21] and [27].

In this paper, we are interested in the game version of f-colourings. Suppose G is a graph, f is a graph function and X is a set of colours. The *f-colouring game* on G with colour set X is the following two-person game: Two players, Alice and Bob, take turns colouring the vertices of G, with Alice having the first move. Suppose c is a partial colouring of the graph G. Let C be the set of coloured vertices, and let U be the set of uncoloured vertices. The partial colouring c is *legal* (with respect to f) if for any subgraph H of G, $|c(V(H) \cap C)| + |U \cap V(H)| \ge f(H)$. On each turn, a player colours one uncoloured vertex of G with a colour from X, so that the resulting partial colouring is legal. The game ends if either all the vertices are coloured or there are uncoloured vertices with no legal colour. If all the vertices are coloured, then Alice wins the game. Otherwise Bob is the winner. So Alice and Bob have opposite goals. Alice wants to produce an f-colouring of G, and Bob tries to prevent this from happening. The *f-game chromatic number* of G, $\chi_g(f, G)$, is the minimum number of colours in the colour set X such that Alice has a winning strategy in the f-colouring game. Observe that if $|X| = |V(G)|$, then Alice always wins. So the parameter $\chi_g(f, G)$ is well-defined.

In case f is the graph function defined as $f(K_2) = 2$, then $\chi_g(f, G)$ is just the game chromatic number of G, and is denoted by $\chi_g(G)$. About twenty-five years ago, Steven J. Brams invented the colouring game for plane maps, and asked what is the minimum number of colours needed, so that Alice always has a winning strategy when the game is played on a plane map. Brams' question is equivalent to asking what is the maximum game chromatic number of planar graphs. Brams' game was published by Martin Gardner in his column "Mathematical Games" in *Scientific American* in 1981 [9]. It remained unnoticed by the graph-theoretic community until ten years later, when it was reinvented by Hans L. Bodlaender [1]. He defined the game chromatic number of arbitrary graphs (not just for planar graphs). Since then the problem has been analyzed in combinatorial journals. The benchmark problem in this area is the maximum game chromatic number of planar graphs, which is studied in a sequence of papers [15,4,24,12,26]. The presently best known upper bound for the game chromatic number of planar graphs is 17 [26]. The game chromatic numbers of some other classes of graphs have also been studied in the literature, including those of forests [8], outerplanar graphs [10], partial k-trees [25,17], etc.

If f is the graph function defined as $f(K_{1,d+1}) = 2$, then $\chi_g(f, G)$ is called the *d-relaxed game chromatic number* of G, and is denoted by $\chi_g^{(d)}(G)$. The d-relaxed game chromatic number of graphs was introduced in [2], and has been studied in [5,6,11,22,23]. It is known that if G is a forest, then for $d = 0, 1, 2$, $\chi_g^{(d)}(G) \le 4 - d$ [8,2,11]. If G is an outerplanar graph, then for $d = 0, 1, 2, 3, 4$, $\chi_g^{(d)}(G) \le 7 - d$ and if $d \ge 6$, then $\chi_g^{(d)}(G) \le 2$ [10,2,11,23]. If G is a planar graph, then for $d \ge 93$, $\chi_g^{(d)}(G) \le 6$ [6], and for $d \ge 132$, $\chi_g^{(d)}(G) \le 3$ [7]. If G is a partial k-tree and $d \ge 4k - 1$, then $\chi_g^{(d)}(G) \le k + 1$ [6].

There are some other variations of the game chromatic number that have been studied in the literature. These include the game chromatic number of oriented graphs [20,16], colouring games in which more than one vertex can be coloured in a move [13,14,18], and the marking game which we shall define in Section 2.

Suppose $\mathcal{K}$ is a class of graphs and f is a graph function. A natural question is whether the f-game chromatic number $\chi_g(f, G)$ is bounded by a constant for all $G \in \mathcal{K}$. In comparison to the result of Nešetřil and Ossona de Mendez [19], the following question is of particular interest: Suppose $\mathcal{K}$ is a proper minor closed class of graphs. Is there a function $\xi : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \xi(n) = \infty$ and for any integers p, there is constant κ_p such that for any graph function f with $f(H) \le \min\{p, \xi(\mathrm{td}(H))\}$, we have $\chi_g(f, G) \le \kappa_p$ for any $G \in \mathcal{K}$? If for some graph function f, $\chi_g(f, G)$ is bounded from above for all $G \in \mathcal{K}$, then we would like to find the smallest upper bound.

In this paper, we consider some special graph functions, and consider the case that $\mathcal{K}$ is either the class of outerplanar graphs or the class of forests.

First we consider the case that $f(K_2) = 2$ and $f(C_n) = 3$ for all $n \geq 3$. In other words, Alice's goal is to produce an acyclic colouring of G. For this graph function f, we call $\chi_g(f, G)$ the *acyclic game chromatic number* of G and denote it by $\chi_{ag}(G)$. We shall show that $\chi_{ag}(G)$ is not bounded for K_4-minor free graphs, however, $\chi_{ag}(G) \leq 7$ for any outerplanar graph G. Then we consider graph functions ϕ_k ($k \geq 3$) defined by $\phi_k(K_2) = 2$ and $\phi_k(P_k) = 3$. The question we are interested in is whether $\chi_g(\phi_k, T)$ is bounded by a constant for all trees T. We shall prove that for $k \geq 8$, $\chi_g(\phi_k, T) \leq 9$ for any tree T, and for $k \leq 6$, $\chi_g(\phi_k, T)$ is not bounded by a constant on trees.

2. Acyclic game chromatic number of outerplanar graphs

In this section, we study the acyclic game chromatic number of graphs. First we observe that $\chi_{ag}(G)$ is unbounded for K_4-minor free graphs.

Example 1. For any integer n, there is a K_4-minor free graph G with $\chi_{ag}(G) \geq n$.

Proof. Let G be the graph with vertex set $\{a, b, c, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$ and edge set $\{x_ia, x_ib, y_ib, y_ic : i = 1, 2, \ldots, n\}$. Then G is a K_4-minor free graph. We shall prove that with $n-1$ colours, Bob has a winning strategy. In Bob's first two moves, he makes either a, b or b, c to be coloured by the same colour. This is certainly possible, no matter which vertices are coloured by Alice in her first two moves. Now Bob wins the game, because if all vertices could be coloured, at least two of the x_i's would be coloured by the same colour, and at least two of the y_i's would be coloured by the same colour. In any case, there would be a 2-coloured C_4, and hence this is not an f-colouring of G. ∎

Now we shall prove that $\chi_{ag}(G)$ is bounded for outerplanar graphs. Observe that outerplanar graphs are those graphs that are K_4- and $K_{2,3}$-minor fee. First we prove an easy lemma.

Lemma 1. *Suppose G is an outerplanar graph, C is a cycle of G and uxv are three consecutive vertices of C. Let P_{uv} be the shortest path of $G - x$ connecting u and v. Then all the vertices of P_{uv} are contained in C.*

Proof. Assume $w \in P_{uv}$ is not contained in C. Let z, z' be the two vertices of P_{uv} on the two sides of w in P_{uv} that are closest to w and lie on C. Then the segment of C connecting z, z' contains at least one vertex, say w', because P_{uv} is a shortest path. Now we can contract edges of the union $C \cup P_{uv}$ so that z, z' become adjacent to x, w, w'. So $K_{2,3}$ is a minor of G, contrary to the assumption that G is outerplanar. ∎

The game colouring number of a graph is defined through the following two-person game: Alice and Bob alternately mark vertices of G, with Alice having the first move. Each move of a player marks one unmarked vertex. The game ends when all vertices are marked. When the game ends, let $m : V(G) \to \mathbb{N}$ be defined as $m(v) = k$ if v is marked at the kth move (counting both Alice's moves and Bob's moves). Let $s(v) = |\{u : u \sim v, m(u) < m(v)\}|$ be the number of neighbours of v that are marked before v (we write $u \sim v$ to mean that vertices u and v are adjacent). The *score* of the game is $\max\{s(v) : v \in V(G)\}$. The *game colouring number* of a graph G, denoted by $\mathrm{col}_g(G)$, is the least integer k such that Alice has a strategy in playing the marking game so that the score is at most $k-1$. We shall need the following result proved in [10]:

Theorem 1 (*Guan and Zhu*). *If G is an outerplanar graph, then $\mathrm{col}_g(G) \leq 7$.*

It is also known [18] that there are outerplanar graphs G with $\mathrm{col}_g(G) = 7$. The main result of this section, Theorem 2, shows that the maximum acyclic game chromatic number of outerplanar graphs is bounded from above by the maximum game colouring number of outerplanar graphs.

Theorem 2. *If G is an outerplanar graph, then $\chi_{ag}(G) \leq 7$.*

Proof. Assume G is an outerplanar graph. By Theorem 1, Alice has a strategy for choosing the vertices to be coloured in her moves so that each uncoloured vertex has at most six coloured neighbours. Alice uses this strategy to choose the vertex to be coloured. When a vertex has been chosen to be coloured, Alice uses any legal colour to colour that vertex. It remains to show that any uncoloured vertex x has a legal colour.

In case G is not 2-connected, we add some vertices and edges to obtain a 2-connected outerplanar graph G' such that G is an induced subgraph of G'. Of course, the added vertices (if any) are uncoloured. We assume that G' is embedded in the plane so that all the vertices lie on the facial cycle which is the boundary of the infinite face. Assume x is an uncoloured vertex. Let $u_1, u_2, \ldots, u_s$ be the coloured neighbours of x. We assume that $x, u_1, u_2, \ldots, u_s$ occur in this order in the outer facial cycle. For $i = 1, 2, \ldots, s-1$, let P_i be the shortest path in $G' - x$ connecting u_i and u_{i+1}. By the previous paragraph, $s \leq 6$.

We choose a set S of colours as follows: First of all, S contains all the colours used on $u_1, u_2, \ldots, u_s$. For $i = 1, 2, \ldots, s-1$, we do the following: If u_i and u_{i+1} are coloured with the same colour, and all vertices of P_i are coloured, then choose one colour c_i used on vertices of P_i that is distinct from the colour of u_i, and add colour c_i to S.

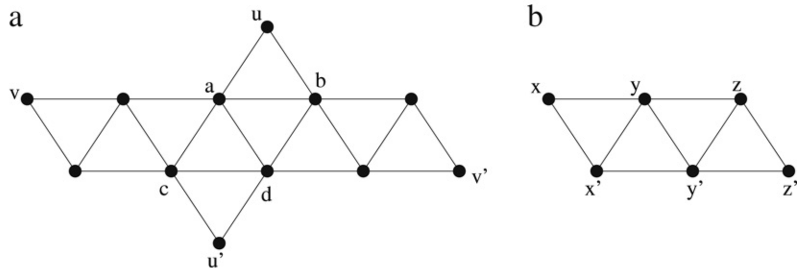

Fig. 1.

By the construction of S, we know that S contains at most $s \leq 6$ colours. As there are seven colours, there is a colour $c \notin S$. We claim that c is a legal colour for x. First of all, c is distinct from all the colours of the coloured neighbours of x. So by colouring x with colour c, there is no monochromatic edge. Assume there is a 2-coloured cycle C. Then C contains x and two coloured neighbours of x. Let u_i, u_j be the two neighbours of x in C, with $i < j$. As $x, u_1, u_2, \ldots, u_s$ are ordered according to the outerplanar embedding of G', $C - x$ contains all the vertices $u_i, u_{i+1}, \ldots, u_j$. By Lemma 1, $C - x$ contains all the vertices on the paths $P_i, P_{i+1}, \ldots, P_{j-1}$. But c is distinct from two colours used on P_i. Hence C cannot be a 2-coloured cycle.

Note that after Alice's move, each uncoloured vertex has at most 5 coloured neighbours. The argument above shows that whichever vertex Bob chooses to colour, there is also a legal colour for that vertex. ■

It is known [25] that series-parallel graphs have game colouring number at most 8. In Example 1, we showed that there are series-parallel graphs whose acyclic game chromatic numbers can be arbitrarily large. So for general graphs G, the difference $\chi_{ag}(G) - \mathrm{col}_g(G)$ can be arbitrarily large.

It follows easily from the definition that for any graph G, $\chi_a(G) \geq \chi(G)$. The question whether $\chi_{ag}(G) \geq \chi_g(G)$ for any graph G remains open. However, once we know a winning strategy for Bob for an ordinary colouring game, it is usually easy to modify it to a winning strategy for Bob in the acyclic colouring game. So a proved lower bound for $\chi_g(G)$ usually can be easily shown to be a lower bound for $\chi_{ag}(G)$. In the following, we show that there are outerplanar graphs G for which $\chi_{ag}(G) \geq 6$. This proof also shows that $\chi_g(G) \geq 6$.

Theorem 3. *There exists an outerplanar graph G with $\chi_{ag}(G) \geq 6$.*

Proof. Let Q be the graph shown in Fig. 1 (a). For each vertex w of Q, add ten degree 1 vertices adjacent to w. The resulting graph is Q'. Take the disjoint union of two copies of Q', and identify the vertex u from both copies into a single vertex. The resulting graph is Q''. Let Q^* be the disjoint union of two copies of Q''. Then Q^* is an outerplanar graph. We shall show that $\chi_{ag}(Q^*) \geq 6$.

Observe that the following is a winning configuration for Bob: *Two adjacent uncoloured vertices x, y have only two legal colours* 4, 5, *and each of x, y has four degree 1 uncoloured neighbours.* In such a state, Bob will colour the degree 1 neighbours of x and y, to force Alice colour x or y. Once Alice colours x with colour 4, then Bob colours a degree 1 neighbour of y with colour 5, and wins the game.

Assume Alice and Bob play the acyclic colouring game on Q^* with five colours 1, 2, 3, 4, and 5. In Bob's first two moves, he guarantees that there is a copy of Q' in which both u, u' are coloured by colour 1, and all the other vertices of this copy of Q' are uncoloured. (As there are four copies of Q', this can be easily done.)

If Alice colours one of the vertices a, b, c, d in this copy of Q' in her next move, by symmetry, we may assume that Alice colours a with colour 2. Then Bob colours the other common neighbour of d and b with colour 3, and we arrive at the winning configuration for Bob described above.

If Alice does not colour any of the vertices a, b, c, d in her next move, then Bob colours either v or v' with colour 1. This will result in a subgraph as in Fig. 1 (b) such that none of the vertices x, y, z, x', y', z' are coloured, and each of them is adjacent to a vertex of colour 1. Bob then colours the degree 1 neighbours of these vertices, to force Alice colour one of these six vertices first. If Alice colours x with colour 2, then Bob will colour y' with colour 3, and we arrive at the winning configuration for Bob again. If Alice colours y with colour 2, then Bob colours z' with colour 3, and we also arrive at a winning configuration for Bob. The other cases are symmetric. So Bob has a winning strategy in this game. ■

3. Colouring trees

In this section, for a positive integer k, we denote by P_k the path on k vertices. Let ϕ_k be the graph function defined as $\phi_k(K_2) = 2$ and $\phi_k(P_k) = 3$.

Theorem 4. *If T is a tree and $k \geq 8$, then $\chi_g(\phi_k, T) \leq 9$.*

In the following, $T = (V, E)$ is a tree, and X is a colour set with $|X| = 9$. We shall only prove Theorem 4 for the case that $k = 8$. The case that $k > 8$ can be proved in the similar way, and we shall point out the difference at appropriate places.

Choose a vertex u of T as a root, and consider T as a rooted tree. Then each vertex v of T other than u has a unique father, which we denote by $f(v)$.

For convenience, we let $f(u) = u$. For a vertex v of T, let $S(v)$ be the set of sons of v, and let $S^2(v) = \cup_{x \in S(v)} S(x)$ be the set of *grandsons* of v. We say y is a *descendant* of x and x is an *ancestor* of y, written as $x < y$, if $x \neq y$ and x lies on the u-y-path.

Suppose the tree T is partially coloured. We denote by C the set of coloured vertices and denote by U the set of uncoloured vertices. For a coloured vertex $v \in C$, we denote the colour of v by $c(v)$. When we write $\beta \neq c(v)$, it means that v is coloured and $\beta \neq c(v)$.

Alice's strategy is a variation of the strategy for playing the ordinary colouring game on T, that is the activation strategy. However, the details of the strategy are complicated. To make it easier for the reader to follow the strategy, let us review the activation strategy Alice uses for playing the ordinary colouring game on T [8,12,15]. Initially, Alice activates and colours the root u. Assume Bob colours a vertex y. Let x be the largest active ancestor of y. Alice activates all the vertices on the x-y-path. She colours x if x is not coloured yet. In case x is coloured already, then Alice activates and colours the least uncoloured vertex. By following this strategy, each uncoloured vertex v has at most two active sons: when the first son of v is activated, Alice activates v. When the second son is activated, Alice colours v. Since each coloured vertex is active, an uncoloured vertex has at most three coloured neighbours: its father and two active sons. Thus with four colours available, every vertex can be coloured.

Roughly speaking, the strategy Alice uses in this paper has two ingredients. Activation is one of them. In this strategy, Alice will still keep record of a set A of *active* vertices, which is a dynamic set created during the game (the detailed strategy will describe which vertex becomes active at the moment). We say a vertex is *activated* when it becomes active. Once a vertex is activated, it remains active forever. The set of active vertices is used in a similar way as in the activation strategy described above: By following the strategy, any uncoloured vertex x has only a few active (and hence only a few coloured) sons and grandsons.

But simply keeping the number of active sons and grandsons of uncoloured vertices small is not enough for Alice to win the game. To describe the dangerous configurations for Alice, we need some definitions.

We call an x-y-path $P = (x, \ldots, y)$ in T a *vertical* path if $x < y$, the vertex x is called the *top* of P and the vertex y is called the *bottom* of P. A path P is called a *bi-coloured path* if all the vertices are coloured by two colours.

Definition 1. A colour β is called *handicapped at vertex* x if there is a vertical bi-coloured path $P = (x_1, x_2, \ldots, x_6)$ with $x = x_1$ and with $c(x_2) = \beta$.

Observe that colour β is illegal for an uncoloured vertex y if the following hold:

1. β is handicapped at $f(y)$.
2. y has a son z coloured with the same colour as $f(y)$.

To win the game, Alice needs to prevent the configuration in which:

- y is uncoloured, $x = f(y)$ is coloured and y has a coloured son z with $c(z) = c(x)$.
- all the colours are either handicapped at x or used by a neighbour of y.

To prevent the configuration described above, Alice's strategy uses the second ingredient: protecting colours.

Definition 2. A colour β is said to be *protected* at vertex x if there is a path $P = (x_1, x_2, x_3, x_4, x_5)$ such that the following hold.

1. $x_3 = x$ is an ancestor of all other vertices of P.
2. $c(x_1) = c(x_3) = c(x_5)$ and $c(x_2) = c(x_4) = \beta$.

Once a colour β is protected at x, it cannot be handicapped at x anymore: if P' is a vertical bi-coloured path on 6 vertices with x as its top and β used by the 2nd, the 4th and the 6th vertices of P', then either $P' \cup \{x_1, x_2\}$ or $P' \cup \{x_4, x_5\}$ is a bi-coloured path on 8 vertices, and hence the partial colouring would not be legal.

Definition 3. Suppose we have a partial colouring c of T. A vertical path $P = (x_1, x_2, \ldots, x_m)$ is called a potential bi-coloured path if two colours are used by coloured vertices of P, and c can be extended to a colouring c' in which P is a bi-coloured path. If P is a potential bi-coloured path and c' is an extension of c in which P is a bi-coloured path, then $c'(x_i)$ is called the *potential colour of* x_i with respect to the potential bi-coloured path P.

Note that when we say β is the potential colour of x_i with respect to the potential bi-coloured path P, the vertex x_i can be either coloured or uncoloured.

Definition 4. 1. A colour β is *in semi-danger* at x if the following hold:

(a) β is not protected at x.
(b) There is a potential bi-coloured vertical path $P = (x_1, x_2, \ldots, x_6)$ which contains exactly four coloured vertices such that $x = x_1$, $x_j = f(x_{j+1})$ and β is the potential colour of x_2.

2. A colour β is *in danger* at x if the following hold:
   (a) β is not protected at x.
   (b) There is a potential bi-coloured vertical path $P = (x_1, x_2, \ldots, x_6)$ which contains exactly five coloured vertices such that $x = x_1, x_j = f(x_{j+1})$ and β is the potential colour of x_2.

If colour β is in semi-danger or in danger at x, then the potential bi-coloured path $P = (x_1, x_2, \ldots, x_6)$ in the definition above is called a *witness path*.

Assume β is in semi-danger at x. Bob can colour another vertex of its witness path to make β in danger at x. With another move, Bob can make β handicapped at x. To prevent this from happening, Alice needs to respond immediately after β becomes in semi-danger at x. Assume Bob's last move coloured a vertex w of the witness path P which makes β in semi-danger at x. If w has an uncoloured ancestor in P, then (the activation part of) the strategy is that Alice will colour an uncoloured ancestor of w in P so that P is no longer a potential bi-coloured path, and hence the danger is gone. If w has no uncoloured ancestor in P, then Alice starts the process of protecting β at x. She colours a son y' of x (which is not on P) with colour β. If Bob makes β in danger at x by colouring another vertex w' of P, then Alice either colours an ancestor of w' in P (if there is such an ancestor) to resolve the danger or colours a son z' of y' with colour $c(x)$ to make β protected at x.

After colouring the vertex y' as described above, Alice needs to remember that y' is coloured for the purpose of protecting the colour of y' at $x = f(y')$. So she keeps record of a set of vertices, called *protectors*. (Once we know y' is a protector, we know it is a protector for colour $c(y')$ at $f(y')$.)

After Alice has chosen and coloured y' as a protector (for colour β at x), Bob may take a move to destroy Alice's plan, by creating a configuration in which no son of y' can be coloured by colour $c(x)$. In this case, we say Bob's move *disables* the protector y'. To disable a protector, Bob needs to use one move, and hence Alice gets one move. Alice will use that move to replace the protector y' by another protector y''. So the set of protectors is dynamic. A protector y' loses its identity as a protector if either y' has no potential to protect the colour $c(y')$ at $f(y')$ (for the reason described above), or the colour $c(y')$ becomes protected at $f(y')$.

The set R of protectors is created (and recorded) by Alice during the play of the game. Alice is unable and it is also unnecessary to protect all colours at a vertex x. To describe in which cases Alice starts protecting a colour β at x, and how to select protectors, we partition the set $S(x)$ of the sons of x into three (possibly empty) parts.

Suppose x and $f(x)$ are both coloured vertices. Then the sons of x are divided into three parts.

$$\begin{aligned} A(x) \;=\; &\{v \in S(x) : v \text{ is activated before both } x, f(x) \text{ are coloured} \\ &\text{or } (x \in R \text{ before } v \text{ is activated and } v \text{ is the first activated son of } x).\} \end{aligned}$$

$$B(x) = \{v \in S(x) \setminus A(x) : \ v \text{ is not a leaf.}\}$$

$$C(x) = S(x) \setminus (A(x) \cup B(x)).$$

In case x and $f(x)$ are not both coloured, the partition is not defined (and no colour protecting will be done at x before both $x, f(x)$ are coloured).

We say Bob's move *attacks colour β at x* if one of the following holds:

- Colour β becomes in semi-danger at a vertex x, $\beta \neq c(f(x))$, the witness path P contains a vertex of $B(x)$, and the vertex coloured by Bob's last move is a vertex of P which has no uncoloured ancestor in P.
- Bob's move disabled a protector $y \in S(x)$ for colour β.

Alice will start protecting colour β at x (by choosing a protector) if and only if colour β is attacked at x. She will always choose a protector from $B(x)$.

We say Bob's move *endangers colour β at x* if the following hold:

- Before Bob's move, β is in semi-danger at x.
- After Bob's move, colour β becomes in danger at x.
- The vertex coloured by Bob's last move is a vertex of the witness path P which has no uncoloured ancestor in P.

Observe that if Bob has made a move, then there is at most one vertex x and at most one colour β so that Bob's move attacks (resp. endangers) colour β at x.

In case $k > 8$, everything is the same except that the definitions of in semi-danger and in danger are different. For a vertex x to be in semi-danger at x, the path P in the definition above would be a potential bi-coloured vertical path on $k - 2$ vertices with two uncoloured vertices. For a vertex x to be in danger at x, the path P would be a potential bi-coloured vertical path on $k - 2$ vertices, with 1 uncoloured vertex.

In the following, A denotes the set of active vertices, and R denotes the set of protectors as defined above. Both sets are dynamic, and whenever they are used, they refer to the sets at that moment of the game.

When playing the game, Alice also needs to be careful that her own moves will not help Bob to produce a potential bi-coloured path on 6 vertices with more than three coloured vertices. For this purpose, when Alice colours a vertex x, she chooses a colour for x carefully. First of all, Alice will not colour x with a colour that is used by its sons or grandsons. Secondly, if x' is a coloured ancestor of x such that the path $P = (x' = v_0, v_1, \ldots, v_{2t} = x\}$ has an odd number of vertices and v_{2j} are uncoloured for $j = 1, 2, \ldots, t - 1$, then except in some very special cases, Alice will not colour x the same colour as x' (to avoid a potential bi-coloured path in which more than three vertices are coloured). Note that in case x' exists, it is unique. Let $F(x) = \{c(v) : v \in S(x) \cup S^2(x) \cup \{f(x), x'\}\}$ (in case x' does not exist, then $F(x) = \{c(v) : v \in S(x) \cup S^2(x) \cup \{f(x)\}\}$).

Definition 5. Suppose the tree T is partially coloured and x is an uncoloured vertex. If x is a non-leaf vertex, then a colour β is *not permissible* for x if $\beta \in F(x)$ or β is handicapped at $f(x)$. If x is a leaf vertex, then a colour β is *not permissible* for x if $\beta \in F(x)$ or β is not legal for x.

Lemma 2. *Suppose T is partially coloured and x is an uncoloured vertex. If β is a permissible colour for x, then β is a legal colour for x.*

Proof. If x is a leaf, then this follows from the definition. Assume x is not a leaf. Since $\beta \notin F(x)$, no neighbour of x is coloured with β. If β is not a legal colour for x, then there must be a colour α and a path P on eight vertices such that $x \in P$ and every other vertex of P is coloured with β and α. By the definition of $F(x)$, we conclude that P does not contain a grandson of x and does not contain $f(f(x))$. Therefore P contains a vertical α-β-path on six vertices with $f(x)$ as the top vertex. Hence β is handicapped at $f(x)$, which is a contradiction. Thus β must be legal. ■

To guarantee that every uncoloured vertex v has a permissible colour, Alice makes sure that $|F(v)|$ is bounded by a small number, and moreover, not many colours are handicapped at v.

In general, Alice colours vertices with permissible colours only. However, when she needs to create a protected colour at a vertex, she may colour a vertex with a legal but not permissible colour. To be precise, there are two exceptional cases: (i) If $f(v) \in R$, then Alice uses colour $c(f(f(v)))$ for v, provided that $c(f(f(v)))$ is a legal colour for v. (ii) If β is handicapped at $f(v)$ but β is legal for v, then Alice may colour v with colour β.

When Alice chooses a colour to colour a vertex x, she follows some simple rules. For brevity, we define a default colour for x, which Alice will use most of the time. When we say that Alice colours an uncoloured vertex x with a *default colour* we mean the following:

Default colour for x: If $f(x) \in R$ then colour x with colour $c(f(f(x)))$ and let $R = R \setminus \{f(x)\}$. Otherwise colour x with any permissible colour.

Suppose $A_1, A_2, \ldots, A_t$ are subsets of vertices of T. When we say that Alice colours a vertex from $A_1, A_2, \ldots, A_t$ we mean that she colours a vertex from $A_1 \cup A_2 \cup \cdots \cup A_t$, with preference order $A_1, A_2, \ldots, A_t$. i.e., if possible, Alice colours an uncoloured vertex of A_1, if not possible, she tries to colour an uncoloured vertex from A_2, and so on. So the order of the sets is important. In case $A_i = \{v\}$ is a singleton, we write v instead of $\{v\}$.

Now we are ready to describe precisely Alice's strategy.

In Alice's first move, she activates u and colours u.

Suppose Bob has just coloured a vertex v with colour α. First Alice does the following:

C1 If v is active, then let y be the ancestor of v such that y is coloured and all the interior vertices of the v-y-path are uncoloured.

C2 If v is inactive, then let y be the ancestor of v such that y is active and all the interior vertices of the v-y-path are inactive, and Alice activates all vertices on the v-y-path.

After choosing vertex y, Alice colours a vertex by applying the following rules in the given order (i.e., first apply R1, if R1 is not applicable, then apply R2, etc.):

R1 If Bob's move attacks colour β at x, then colour a vertex y' from $B(x), A(x), C(x)$ with colour β, provided that β is a permissible colour for y'. If y' is from $B(x)$, then add y' to R.

R2 Assume Bob's move endangers colour β at x. If there is a protector $y' \in B(x)$ for colour β, then Alice colours a son z' of y' with colour $c(x)$. Otherwise Alice colours a vertex y' from $A(x), C(x)$ with colour β, provided that β is a permissible colour for y'.

R3 Let z be the son of y which is an ancestor of v (if $y \neq f(v)$), and let w be a least uncoloured vertex. Then colour a vertex from $y, f(y), z, w$ with a default colour.

If the vertex coloured by Alice is not active, then she activates the vertex. If a vertex $y' \in R$ is disabled or $c(y')$ is protected at $f(y')$, then remove y' from R.

This completes the description of Alice's strategy. Observe that if Bob's move attacks colour β at x, it does not guarantee that R1 is applied. For example, if $B(x) \cup A(x) \cup C(x)$ contains no uncoloured vertex, then R1 is not applicable. In this case, we try R2, R3 in this order.

If R3 is applied, then Alice colours $y, f(y), z, w$ in this order. This means that Alice first tries y, if y is coloured then she tries $f(y)$, if $f(y)$ is coloured then she tries z, and if $y, f(y)$ and z are all coloured, then Alice colours w. Since w is chosen to be an uncoloured vertex, so by following R1–R3, Alice will colour an uncoloured vertex, provided that there are uncoloured vertices, and that every uncoloured vertex does have a permissible colour (which we shall prove).

Now we show that this is a winning strategy for Alice. For this purpose, it suffices to show that at any moment, any uncoloured vertex has a permissible colour.

Lemma 3. *Assume Alice has just finished a move. Then every coloured vertex is active and the set of active vertices induces a subtree.*

Lemma 4. *If Alice colours a vertex x in her current move, then $f(x)$ was active before her current move.*

Both Lemmas 3 and 4 follow easily from the strategy and the proofs are omitted.

Lemma 5. *Assume Alice has just finished a move, colour β is in semi-danger at x, and there is a witness path $P = (x_1, x_2, \ldots, x_6)$ ($x_1 = x$) which contains a vertex of $B(x)$. Then x_1, x_2, x_3 are all coloured.*

Proof. Since $x_2 \in B(x)$, we know that x_2 is activated after x and $f(x)$ are both coloured. It follows that $x = x_1$ is the first coloured vertex of P. The vertex x_2 is activated at the time Bob colours the first vertex in the set $\{w : w \geq x_2\}$ or Alice colours x_2 as a least uncoloured vertex. If the first coloured vertex in $\{w : w \geq x_2\}$ is x_2, then x_2 is the second coloured vertex of P. Otherwise, by Alice's strategy, she first goes to x_1 by $C2$, then colours x_2 by R3. In this case, x_2 is the third coloured vertex of P. After both x_1, x_2 are coloured, if $x_i \in P$ is uncoloured, then the potential colour of x_i with respect to the potential bi-coloured path P is not a permissible colour for x_i anymore. Hence if any of them is coloured in later moves, it is coloured by Bob. In particular, the fourth coloured vertex x_j of P must be coloured by Bob. If x_j has an uncoloured ancestor $x_i \in P$, then it is easy to verify that by Alice's strategy, she goes to an ancestor $x_{i'} \in P$ of x_j by C1 or C2, then she colours an ancestor of x_j in P by R3. Then P would not be a potential bi-coloured path. By our assumption, after Alice's move, P is still a potential bi-coloured path. So the fourth coloured vertex x_j of P has no uncoloured ancestor in P. Therefore x_3 is coloured. ■

Corollary 1. *Assume $v \in R$. If a son w of v is coloured with colour $c(f(v))$, the colour $c(v)$ is protected at $f(v)$.*

Proof. By Alice's strategy, v is added to R because Bob attacked colour $c(v)$ at $f(v)$. This means that there is a witness path $P = (x_1, x_2, \ldots, x_6)$ such that $x_1 = f(v)$ and $x_2 \in B(x_1)$. By Lemma 5, all the vertices x_1, x_2, x_3 are coloured before v is coloured. If a son w of v is coloured with colour $c(f(v))$, then the path $P' = (x_3, x_2, x_1, v, w)$ shows that colour $c(v)$ is protected at $f(v)$. ■

Lemma 6. *Assume Alice has just finished a move. If $v \in R$, then v has no active descendants.*

Proof. Assume that in the kth move, Alice adds v to R. By the strategy, in the $(k-1)$th move, Bob attacks colour β at $x = f(v)$.

First we show that before the kth move, no descendants of v are active. Otherwise v has a coloured descendant. By the rules, Alice never colours a vertex w if $f(w)$ is inactive. Therefore the first coloured descendant of v is coloured by Bob. When Bob colours the first descendant w of v, Alice will go to x by C2 and activate v at this step. By definition of the set R, $v \in B(x)$, so both x and $f(x)$ are coloured when v is activated. Since v is not coloured yet at that time, the move of Bob that colours w does not attack or endanger any colour at a vertex. By Alice's strategy, she will use R3 to colour v. This is a contradiction, because our assumption is that v is coloured at the kth move by Alice, after the $(k-1)$th move by Bob which attacks colour β at x. Therefore at the time v is coloured, it has no active descendants.

Assume in some later move, the first descendant of v is coloured. If the descendant is coloured by Alice, then it must be a son w of v, and w is coloured by the default colour $c(f(v))$. After the colouring, colour β is protected at $f(v)$, and v is removed from R. If the descendant is coloured by Bob, then either Bob's move disabled the protector v, or as a response to Bob's move, Alice colours a son of v with colour $c(f(v))$ and makes the colour of v protected at $f(v)$ (by Corollary 1). In any case, v is removed from R. ■

Lemma 7. *Assume Alice has just finished a move. If $v \in B(x) \cap A$ then v is coloured.*

Proof. Assume the lemma is not true and $v \in B(x) \cap A$ is uncoloured. Then v has a coloured descendant. As Alice never colours a vertex w if $f(w)$ is inactive, the first coloured descendant of v is coloured by Bob.

If $x \in R$ at the time Bob colours the first descendant w of v, then by Lemma 6, v is the first active son of x. This implies that $v \in A(x)$, contrary to our assumption that $v \in B(x)$. Therefore $x \notin R$ at the time v is activated. So in Alice's next move, R1, R2 do not apply. Therefore, Alice colours v by R3 in that move. ■

Lemma 8. *Assume Alice has just finished a move, colour β is in semi-danger at x and not protected at x, and there is a witness path $P = (x_1, x_2, \ldots, x_6)$ ($x_1 = x$) which contains a vertex of $B(x)$. If $\beta \neq c(f(x))$ and $\beta \neq c(w)$ for any $w \in A(x)$, then either there is a protector for β in $B(x)$, or every vertex of $B(x)$ is coloured. If all vertices of $B(x)$ are coloured, then either $A(x) \cup C(x)$ contains a vertex of colour β, or all vertices of $C(x)$ are coloured.*

Proof. As shown in the proof of Lemma 5, the fourth coloured vertex of P is coloured by Bob. Since β is not protected at x, the colour β of x_2 is in semi-danger at x. By definition, Bob's move attacks colour β at x. If $B(x)$ contains an uncoloured vertex y', then by Lemma 7, y' is inactive, and hence has no coloured descendant. This implies that β is a permissible colour for y'. By Alice's strategy, she applies R1 to colour y' with colour β and add y' to R.

In any later move, if y' is disabled, then by R1 again, Alice will colour another $y'' \in B(x)$ with colour β and add y'' to R, provided that $B(x)$ contains uncoloured vertices. If $B(x)$ contains no uncoloured vertices, then Alice will colour a vertex of $A(x) \cup C(x)$ with colour β, provided that $C(x)$ contains uncoloured vertices (note that β is a legal colour for any vertex $w \in C(x)$ and hence is a permissible colour for $w \in C(x)$). ■

Lemma 9. *Assume Alice has just finished a move, colour β is in danger at x and not protected at x, and there is a witness path $P = (x_1, x_2, \ldots, x_6)$ ($x_1 = x$) which contains a vertex of $B(x)$. If β is not used by $f(x)$, then all vertices of $B(x)$ are coloured and moreover, either $A(x)$ has a vertex of colour β or $C(x)$ has a vertex of colour β, or all vertices of $C(x)$ are coloured.*

Proof. Before Bob's last move, colour β was in semi-danger at z. By Lemma 8, if $B(x)$ contains uncoloured vertices, then $B(x)$ contains a protector y' for β.

Now Bob's last move must have coloured the fifth coloured vertex of P. If the fifth coloured vertex of P has an uncoloured ancestor in P, then similarly as above, Alice will colour the ancestor in her next move, and hence P is no longer a witness path and β is not in danger at x. This is contrary to our assumption. So the fifth coloured vertex of P has no uncoloured ancestor in P. i.e., Bob's move endangered colour β at x. By R2, Alice will colour a son of y' and make colour β protected at x, contrary to our assumption. Therefore all vertices of $B(x)$ are coloured by Lemma 8. Then Alice will use R2 to colour a vertex $A(x)$ with β, provided that $A(x)$ contains an uncoloured vertex v' for which β is a permissible colour. If $A(x)$ has no uncoloured vertex v' for which β is a permissible colour, then Alice colours a vertex of $C(x)$ with colour β by R2, provided that $C(x)$ contains an uncoloured vertex (note that if $w \in C(x)$ is uncoloured, then β is a legal colour for w). ■

A colour β is called a *dangerous unprotected colour at x* if the following hold:

- $\beta \neq c(f(x))$.
- β is in danger at x. In particular, β is not a protected colour at x.
- β is not used by any vertex in $A(x)$.

Corollary 2. *Assume Alice has just finished a move. If colour β is dangerous unprotected at x, then all vertices of $B(x)$ are coloured. Moreover, for any $w \in A(x) \cap U$, $\beta \in F(w)$. Furthermore, if β is not used by any vertex in $A(x) \cup C(x)$, then every vertex in $C(x)$ is coloured.*

Proof. Assume colour β is dangerous unprotected at x. By Lemma 9, after all the vertices of $B(x)$ are coloured, Bob attacked colour β at x. Otherwise, by Lemma 8, $B(x)$ always contains a protector for β when the colour is in semi-danger, and hence colour β will become protected at the time it becomes in danger. When Bob attacks β at x and $B(x)$ are all coloured, Alice will try to use R1 to colour a vertex of $A(x)$, provided that there is a vertex $w \in A(x) \cap U$ for which β is a permissible colour. By our assumption, Alice did not colour w with colour β. So we conclude that β is not a permissible colour for w. If $\beta \notin F(w)$, then it must be the case that there is a potential bi-coloured path P on eight vertices in which w is the only uncoloured vertex. An argument as the proof of Lemma 5 shows that the longest vertical bi-coloured path containing x and a vertex in $B(x)$ contains at most five vertices. Since $\beta \notin F(w)$, which implies that no grandson of w is coloured by colour β, we conclude that by colouring w with colour β, any vertical bi-coloured path containing x and w contains at most 3 vertices. This is contrary to the assumption that P has eight vertices (note that x is contained in both vertical subpaths of P). This proves that $\beta \in F(w)$. By R1, when Alice cannot colour a vertex of $A(x)$ by colour β, she will colour a vertex in $C(x)$ with β, provided that $C(x)$ contains an uncoloured vertex. Thus if β is not used by any vertex in $A(x) \cup C(x)$, then every vertex in $C(x)$ is coloured. ■

Lemma 10. *Assume Alice has just finished a move and x is an uncoloured vertex. Then x has at most one active son and at most two active grandsons.*

Proof. When the first son of x is activated, x itself is also activated. When the second son of x is activated, then x is coloured. Since x is uncoloured, x has at most one active son z. All the active grandsons of x are sons of z. When the second son of z is activated, z is coloured. When the third son of z is activated, Alice should have coloured x by R3. So z has at most two active sons. Therefore x has at most two active grandsons. ■

Corollary 3. *For any vertex x, $|A(x)| \leq 3$.*

Proof. If x is a protector at any moment of the game, x is coloured before any of its descendants is activated. Moreover, both $f(x), f(f(x))$ are coloured before x is coloured. So it follows by definition that $|A(x)| = 1$. Assume x has never been a protector. By Lemma 10, if Alice has just finished a move, and $x, f(x)$ are not both coloured, then x has at most two active sons. If Bob has just finished a move, then he can make at most one more son of x active before both x and $f(x)$ are coloured. So $|A(x)| \leq 3$. ■

Lemma 11. *At any moment of the game, any uncoloured vertex x has a permissible colour.*

Proof. We shall prove that after Alice's move, each uncoloured vertex x has at least two permissible colours. This implies that after Bob's move, each uncoloured vertex x has at least one permissible colour, because Bob's last move can change at most one permissible colour for x into a non-permissible colour for x.

Assume Alice has just finished a move and x is an uncoloured vertex. Let $v = f(x)$. By Lemma 10, x has at most one coloured son (because each coloured vertex is active), and at most two coloured grandsons. Therefore $|F(x)| \leq 5$.

If at least one of $v, f(v)$ is not coloured, then by Lemma 10, v has at most two coloured sons. Let $F'(x) = \{c(w) : w \in S(v)\}$ and let $F = F(x) \cup F'(x)$. It is obvious that any colour not in F is permissible for x. As $|F| \leq 7$, x has at least two permissible colours.

Assume both v and $f(v)$ are coloured. Let $F''(x) = \{c(w) : w \in A(v)\}$. By Corollary 3, $|F''(x)| \leq |A(v)| \leq 3$. Let $F = F(x) \cup F''(x)$.

If $x \in B(v)$, then $|F(x)| = 2$ as x has no coloured descendants. First we show that every colour $\beta \notin F$ is a permissible colour for x. Assume to the contrary that β is handicapped at v. Then β was dangerous unprotected at v before, so x is coloured by Corollary 2, which is a contradiction. Hence x has at least four permissible colours. If $x \in A(v)$, then $|F(x)| \leq 5$ and $|F''(x)| \leq 2$ (as $x \in A(v)$ is uncoloured). Thus $|F| \leq 7$. By Corollary 2, any colour $\beta \notin F$ is permissible for x. Therefore x has at least two permissible colours.

If $x \in C(v)$, then $|F(x)| = 2$ and $|F''(x)| \leq 3$. Then by R1, it is easy to see that any colour $\beta \notin F$ is a legal colour for x, because at the time β is attacked at v, a son of v not in the witness path will be coloured by β by R1. This will prevent colour β from becoming illegal at v. As x is a leaf, any colour $\beta \notin F$ which is a legal colour for x is a permissible colour for x. Therefore x has at least four permissible colours. ■

If F is a forest, it is easy to see that the argument presented in this section also applies. Thus we have the following result.

Theorem 5. *If F is a forest and $k \geq 8$, then $\chi_g(\phi_k, F) \leq 9$.*

Theorem 6. *For any positive integer n, there is a forest F_n for which $\chi_g(\phi_6, F_n) \geq n$.*

Proof. For $j = 1, 2, \ldots, 4n$, let $Z_j = \{z_{j,l} : l = 1, 2, \ldots, 12n\}$.

Let T_n be the tree with vertex set $\{u, x_j, y_j : j = 1, 2, \ldots, 4n\} \cup (\cup_{j=1}^{4n} Z_j)$, and edge set $\{ux_j, x_jy_j\} \cup (\cup_{j=1}^{4n}\{y_jz_{j,l} : j = 1, 2, \ldots, 4n, l = 1, 2, \ldots, 12n\})$. We view T_n as a rooted tree with root u. The vertex x_j together with its descendants forms the jth branch of T_n, which is denoted by B_j. We shall show that Bob has a winning strategy for the ϕ_6-colouring game on T_n with n colours.

Bob's first move makes sure that u is coloured. In other words, if Alice does not colour u in her first move, Bob colours u in his first move.

Assume u is coloured with colour α.

If branch B_j contains a vertical path coloured with colours (β, α, β), then B_j is called a *forcing branch for colour* β. A colour β is called *dead* if there is a forcing branch for colour β. A colour $\beta \neq \alpha$ is called an *alive colour* if β is not dead. Bob's goal is to produce a forcing branch for each colour $\beta \neq \alpha$, i.e., make every colour other than α dead. If this goal is achieved and there is a branch B_j such that x_j, y_j are uncoloured, then Bob colours y_j with colour α and wins the game, because no colour is legal for x_j.

If y_j is coloured with α and x_j is coloured with β and β is an alive colour, then B_j is called a *dangerous branch*. If y_j is coloured with a colour distinct from α or Z_j contains a vertex of colour α, then B_j is called a *wasted branch*. If x_j is uncoloured, at least $2n + 1$ vertices of Z_j are uncoloured and y_j coloured with α, then B_j is called a *useful branch*. If x_j is coloured with β, y_j is uncoloured and some vertex of Z_j is coloured with β, then B_j is called a *potential branch for colour* β.

Assume Alice has coloured a vertex. Then Bob colours a vertex using the following rules until all the y_j's are coloured (the rules are applied in the following order: if Rj is the first applicable rule, then apply Rj).

R1 Assume β is an alive colour. If there is a branch B_j in which x_j is coloured with β but no vertex of Z_j is coloured with β and y_j is coloured with α, then Bob colours a vertex of Z_j with colour β.
R2 Assume β is an alive colour. If there is a branch B_j in which x_j is coloured with β and a vertex of Z_j is coloured with β and y_j is uncoloured, then Bob colours y_j with colour α.
R3 Assume β is an alive colour. If there is a branch B_j in which x_j is coloured with β and no vertex of Z_j is coloured with β and y_j is uncoloured, then Bob colours a vertex of Z_j with colour β.
R4 Choose an index j such that neither x_j nor any vertex of Z_j is coloured, colour y_j with colour α.

Lemma 12. *Assume Bob has just finished a move. Then there is no dangerous branch.*

Proof. We prove this lemma by induction on the number of moves. After Bob's first move, there is only one coloured vertex other than u. Thus there is no dangerous branch. Assume after Bob's kth move, there is no dangerous branch. In Alice's $(k + 1)$th move, she can create at most one dangerous branch B_j. If this is the case, then in Bob's $(k + 1)$th move, he colours a vertex of Z_j by R1, and the colour of x_j becomes a dead colour. So there is no dangerous branch anymore. If after Alice's $(k + 1)$th move, there is no dangerous branch, then it is obvious from the rules that Bob's $(k + 1)$th move will not create a dangerous branch. ■

Corollary 4. *Assume Alice has just finished a move. Then there is at most one dangerous branch. If B_j is a dangerous branch, then Bob can colour a vertex of Z_j in his next move to change B_j into a forcing branch.*

Proof. By Lemma 12, before Alice's last move, there is no dangerous branch. Alice's last move can create at most one dangerous branch. Suppose B_j becomes a dangerous branch after Alice's last move. Assume that $c(x_j) = \beta$. We need to show for any uncoloured vertex $z_{j,l} \in Z_j$, β is a legal colour for $z_{j,l}$. Since $c(y_j) = \alpha$, no neighbour of $z_{j,l}$ is coloured with β. If β is not a legal colour for $z_{j,l}$, then there is a potential α-β-coloured path P with six vertices containing $z_{j,l}$ such that all the other vertices of P are coloured. By the structure of T, P contains a segment of the form $(u, x_{j'}, y_{j'})$ for some $j' \neq j$. But then $B_{j'}$ is another dangerous branch, contrary to our conclusion above. ■

Lemma 13. *Within 8n moves, the sum of the number of forcing branches and the number of useful branches is at least n.*

Proof. If Bob applies R1 or R2, then he creates a forcing branch. If Bob applies R4, then he creates a useful branch. It is easy to see that if Bob applied R3 in his current move, then he will not apply R3 in his next move. When Bob applied R1, then he may have changed a previous useful branch into a forcing branch. So after Bob's $4n$ moves, the sum of the number of forcing branches and the number of useful branches is at least n. ■

Assume there are n branches that are either forcing or useful. Bob's next goal is to create $n-1$ forcing branches. Since one dead colour can occur in one forcing branch only, this would imply that all colours other than α are dead colours. Therefore in the remaining useful branch B_j, the vertex x_j has no legal colour, and Bob wins the game.

To change useful branches into forcing branches, Bob does the following: Suppose Alice has just coloured a vertex. If a dangerous branch has been created, then Bob colours a vertex as in R1 to change that branch into a forcing branch. Otherwise, Bob chooses a useful branch, and colours a vertex $z_{j,l} \in Z_j$ with an alive colour β. Alice's next move either produces a dangerous branch which is then changed into a forcing branch by Bob's next move, or Bob will colour x_j with colour β and changes B_j into a forcing branch. In any case, every two moves of Bob will produce at least one forcing branch, and hence produce one dead colour. Thus within $2n-2$ moves of Bob, there will be $n-1$ forcing branches and all colours $\beta \neq \alpha$ are dead colours, and Bob wins the game. Observe that since $|Z_j| = 12n$, and all the above moves are finished within Bob's of the $12n$ moves, so whenever Bob needs to colour a vertex of Z_j, there is at least one uncoloured vertex in Z_j. ■

4. Some open questions

The graph functions f studied in this paper are very special graph functions, and the classes of graphs considered are also very restricted: outerplanar graphs or forests. For general graph functions and for many other well-known classes of graphs, many fundamental questions remain open.

Given a non-empty class $\mathcal{K}$ of graphs, and a graph function f, let

$$\chi(f,\mathcal{K}) = \max\{\chi(f,G) : G \in \mathcal{K}\}$$
$$\chi_g(f,\mathcal{K}) = \max\{\chi_g(f,G) : G \in \mathcal{K}\}.$$

In case $\chi_g(f,G)$ (respectively, $\chi(f,G)$) is not bounded by a constant for $G \in \mathcal{K}$, then $\chi_g(f,\mathcal{K}) = \infty$ (respectively, $\chi(f,\mathcal{K}) = \infty$). Let $\mathcal{F}$ be the class of forests, then Theorem 5 can be stated as $\chi_g(\phi_i,\mathcal{F}) \leq 9$ for every $i \geq 8$.

As observed in Section 1, if f, f' are two graph functions, and f' dominates f, then for any graph G, $\chi(f,G) \leq \chi(f',G)$. However, it is possible that f' dominates f and yet $\chi_g(f,G) > \chi_g(f',G)$. For example, let $f(K_{1,2}) = 2, f'(K_2) = 2$. Then f is dominated by f', but it is easy to verify that $\chi_g(f,K_{n,n}) = n$ and $\chi_g(f',K_{n,n}) = 3$. However, the following question remains open.

Question 1. *Suppose f, f' are two graph functions, and f' dominates f, and $\mathcal{K}$ is a hereditary class of graphs (i.e., $H \in \mathcal{K}$ implies that for any subgraph H', we have $H' \in \mathcal{K}$). Is it true that*

$$\chi_g(f,\mathcal{K}) \leq \chi_g(f',\mathcal{K})?$$

Natural classes of graphs to be considered are proper minor closed classes of graphs. We call a graph function f *game bounded* (with respect to proper minor closed classes) if for each proper minor closed class $\mathcal{K}$ of graphs, there is a constant $C_{\mathcal{K}}$ such that $\chi_g(f,G) \leq C_{\mathcal{K}}$ for any $G \in \mathcal{K}$.

Question 2. *Which graph functions are game bounded?*

It is known [3] that for any proper minor closed class $\mathcal{K}$ of graphs, the acyclic chromatic number $\chi_a(G)$ is bounded by a constant for all $G \in \mathcal{K}$. As $\chi_g(G) \leq \chi_a(G)(\chi_a(G)+1)$ [4], this implies that $\chi_g(G)$ is bounded by a constant for all $G \in \mathcal{K}$. In other words, the graph function Chi is game bounded with respect to proper minor closed classes. The same argument as in [4] can be used to prove the following result:

Theorem 7. *If f is a graph function dominated by* Chi, *then for any graph G with $\chi_a(G) = k$, $\chi_g(f,G) \leq k(k+1)$.*

Corollary 5. *Any graph function dominated by* Chi *is game bounded with respect to proper minor closed classes.*

Let ϕ_i be the graph function defined as in Section 3.

Question 3. *Does there exist an integer k and a constant C such that for any outerplanar graph G, $\chi_g(\phi_k,G) \leq C$? Does there exist an integer k and a constant C such that for any planar graph G, $\chi_g(\phi_k,G) \leq C$? Does there exist an integer k such that ϕ_k is game bounded graph with respect to proper minor closed classes?*

Even if restricted to the class $\mathcal{F}$ of forests, we know very little about $\chi_g(f,\mathcal{F})$ for general graph functions f. In this paper we only considered the game in which a legal colouring requires that a path of certain length uses at least three colours. A natural extension would be to consider games in which long paths need to use more than three colours. For example, the following question is open.

Question 4. *Suppose $i > n$ are positive integers. Let $\phi_{i,n}$ be the graph function defined as $\phi_{i,n}(P_i) = n$ and $\phi_{i,n}(K_2) = 2$. Is there a function $\psi : \mathbb{N} \to \mathbb{N}$ such that for any n, $\chi_g(\phi_{\psi(n),n}, \mathcal{F})$ is bounded by a constant?*

More generally, we have the following question:

Question 5. *Is there a function $\psi : \mathbb{N} \to \mathbb{N}$ such that if $f(H) \leq \min\{p, \psi(\mathrm{td}(H))\}$ for some positive integer p, then $\chi_g(f, \mathcal{F})$ is bounded by a constant?*

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