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Complementary cycles in regular multipartite tournaments, where one cycle has length five

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ABSTRACT

The vertex set of a digraph D is denoted by $V(D)$. A c-partite tournament is an orientation of a complete c-partite graph.

In 1999, Yeo conjectured that each regular c-partite tournament D with $c \geq 4$ and $|V(D)| \geq 10$ contains a pair of vertex disjoint directed cycles of lengths 5 and $|V(D)| - 5$. In this paper we shall confirm this conjecture using a computer program for some cases.

1. Terminology

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. If x is a vertex of multipartite tournament D, then $V(x)$ is the partite set of D such that $x \in V(x)$. A *tournament* is a c-partite tournament with exactly c vertices. By a *cycle* or *path* we mean a directed cycle or directed path.

In this paper all digraphs are finite without loops or multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X.

If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y. If X and Y are two disjoint subsets of $V(D)$ or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \Rightarrow Y$ denotes the property that there is no arc from Y to X. By $d^+(X, Y)$ we define the number of arcs going from X to Y.

The *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x, and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are the *outdegree* and *indegree* of x, respectively. The *minimum outdegree* and the *minimum indegree* of D are denoted by $\delta^+(D)$ and $\delta^-(D)$, and the *maximum outdegree* and the *maximum indegree* of D are denoted by $\Delta^+(D)$ and $\Delta^-(D)$, respectively.

The *global irregularity* of a digraph D is defined by

$$i_g(D) = \max\{\max(d^+(x), d^-(x)) - \min(d^+(y), d^-(y)) | x, y \in V(D)\},$$

and the *local irregularity* by $i_l(D) = \max |d^+(x) - d^-(x)|$ over all vertices x of D. If $i_g(D) \leq 1$, then D is called *almost regular*, and if $i_g(D) = 0$, then D is *regular*.

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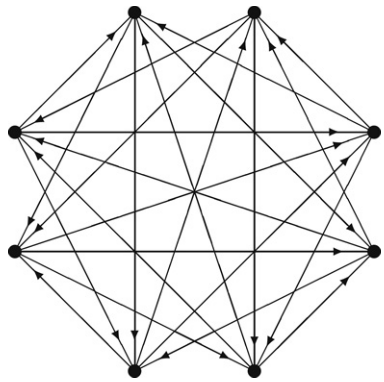

Fig. 1. The 3-regular 4-partite tournament $D^*_{4,2}$.

A cycle of length m is an *m-cycle*. A cycle or a path in a digraph D is *Hamiltonian* if it includes all the vertices of D. A set $X \subseteq V(D)$ of vertices is *independent* if the induced subdigraph $D[X]$ has no arcs. The *independence number* $\alpha(D) = \alpha$ is the maximum size among the independent sets of vertices of D.

A digraph D is *strongly connected* or *strong* if, for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D-A$ obtained by deleting A is strong. The *connectivity* of D, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected.

A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. A cycle-factor with the minimum number of cycles is called a *minimal cycle-factor*. If x is a vertex of a cycle C, then the *predecessor* and the *successor* of x on C are denoted by x^- and x^+, respectively. If we replace every arc xy of D by yx, then we call the resulting digraph, denoted by D^{-1}, the *converse digraph* of D.

2. Introduction and Preliminary Results

A digraph D is called *cycle complementary* if there exist two vertex disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$. The problem of complementary cycles in tournaments was almost completely solved by Reid [4] in 1985 and by Z. Song [5] in 1993. These authors proved that every 2-connected tournament T on at least 8 vertices has complementary cycles of length t and $|V(T)| - t$ for all $t \in \{3, 4, \ldots, |V(T)| - 3\}$. For c-partite tournaments with $c \geq 3$, there exist the following two conjectures.

Conjecture 2.1 (*[14]*). *A regular c-partite tournament D with $c \geq 4$ and $|V(D)| \geq 8$ has a pair of vertex disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \ldots, |V(D)| - 3\}$.*

Conjecture 2.2 (*[6]*). *Let D be a multipartite tournament. If $\kappa(D) \geq \alpha(D) + 1$, then D is cycle complementary, unless D is a member of a finite family of multipartite tournaments.*

In 2005, Volkmann [8] confirmed the first conjecture for $t = 3$, unless D is isomorphic to two fixed regular 4-partite tournament with two vertices in each partite set. In addition, Volkmann [7] showed that Conjecture 2.1 is also valid for $t = 4$ when $c \geq 5$ or $c \geq 4$ and $\alpha(D) \geq 4$. Example 2.3 below by Volkmann [7] demonstrates that Yeo's conjecture is not true in general for $t = 4$ when $c = 4$ and $\alpha(D) = 2$. In this paper we will show that Conjecture 2.1 is valid for $t = 5$, where we use a computer program for some cases.

Example 2.3 (*[7]*). Let $D^*_{4,2}$ be the 3-regular 4-partite tournament presented in Fig. 1. Then it is straightforward to verify that $D^*_{4,2}$ does not contain two 4-cycles C_4 and C^*_4 such that $V(D^*_{4,2}) = V(C_4) \cup V(C^*_4)$.

A computer program (cf. the Appendix) has shown that $D^*_{4,2}$ is the only regular 4-partite tournament with two vertices in each partite sets that does not contain two complementary cycles of length 4. Hence one can conclude from Volkmann's paper [8] that Conjecture 2.1 is valid for $t = 4$ with exception of $D^*_{4,2}$.

The following results play an important role in our investigations. We start with a well-known fact about regular multipartite tournaments.

Lemma 2.4. *If D is a regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$, then $\alpha(D) = |V_1| = |V_2| = \cdots = |V_c|$.*

Theorem 2.5 (*[3]*). *Let T be a strongly connected tournament. Then, every vertex of T is contained in an m-cycle for each m between 3 and $|V(T)|$.*

Theorem 2.6 ([1]). *Each strongly connected c-partite tournament contains an m-cycle for each $m \in \{3, 4, \ldots, c\}$.*

Theorem 2.7 ([4,5]). *If T is a 2-connected tournament with $|V(T)| \geq 8$, then T contains two complementary cycles of length t and $|V(T)| - t$ for all $3 \leq t \leq |V(T)|/2$.*

Theorem 2.8 ([13]). *If D is a multipartite tournament, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D)}{3} \right\rceil.$$

Theorem 2.9 ([10]). *Let D be a multipartite tournament. If $\alpha(D)$ is odd, then*

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - 2i_l(D) - \alpha(D) + 1}{3} \right\rceil.$$

Theorem 2.10 ([12]). *Let D be a $(\lfloor q/2 \rfloor + 1)$-connected multipartite tournament such that $\alpha(D) \leq q$. If D has a cycle-factor, then D is Hamiltonian.*

Theorem 2.11 ([15]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. If*

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 2}{2},$$

then D is Hamiltonian.

Lemma 2.12 ([15,2]). *A digraph D has no cycle-factor if and only if its vertex set $V(D)$ can be partitioned into four subsets Y, Z, R_1, and R_2 such that*

$$R_1 \Rightarrow Y, \qquad (R_1 \cup Y) \Rightarrow R_2, \quad \text{and} \quad |Y| > |Z|, \tag{1}$$

where Y is an independent set.

Theorem 2.13 ([12]). *Let D be a multipartite tournament having a cycle-factor but no Hamiltonian cycle. Then there exists a partite set V^* of D and an indexing $C_1, C_2, \ldots, C_t$ of the cycles of some minimal cycle-factor of D such that for all arcs yx from C_j to C_1 for $2 \leq j \leq t$, it holds $\{y^+, x^-\} \subseteq V^*$.*

Theorem 2.14 ([11]). *Let D be an almost regular c-partite tournament with $c \geq 5$. Then D contains a strongly connected subtournament of order p for every $p \in \{3, 4, \ldots, c\}$.*

Theorem 2.15 ([9]). *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a c-partite tournament D with no cycle-factor such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$. According to Lemma 2.12, the vertex set $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with integers $k, t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. If $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$, and $Q_2 = Q \cap R_2$, then*

$$i_l(D) \geq |V(D)| - 3|V_c| + 2t + 2k + 2 \quad \text{and}$$
$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3}{2}$$

if $Q_1 = \emptyset$ or $Q_2 = \emptyset$ and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2}$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

Lemma 2.16. *Each regular 4-partite tournament contains a 5-cycle through all partite sets.*

Proof. Let D be a regular 4-partite tournament with the partite sets V_1, V_2, V_3, V_4. In view of Lemma 2.4, we have $|V_1| = |V_2| = |V_3| = |V_4| = r$. Since D is regular, we note that $r \geq 2$. Suppose that D does not contain any 5-cycle through all partite sets. To derive a contradiction we distinguish two cases.

Case 1. Assume that D contains a strongly connected subtournament T_4 of order 4. If $V(T_4) = \{v_1, v_2, v_3, v_4\}$ such that, without loss of generality, $v_1 \to v_2 \to v_3 \to v_4 \to v_1$, $v_3 \to v_1$ and $v_4 \to v_2$, then we assume, without loss of generality, that $v_i \in V_i$ for $i \in \{1, 2, 3, 4\}$. Let us define the sets $A = N^+(v_1) - V(T_4)$, $B = N^-(v_1) - V(T_4)$, $V_i' = V_i \cap A$ and $V_i'' = V_i \cap B$ for $i = 2, 3, 4$.

If there is a vertex $a \in A$ such that $a \to v_2$, then $v_1 a v_2 v_3 v_4 v_1$ is a 5-cycle containing vertices of all partite sets, a contradiction. Hence we assume in the following that $v_2 \Rightarrow A$.

If there is a vertex $a \in A$ such that $a \to v_4$, then $v_1 a v_4 v_2 v_3 v_1$ is a 5-cycle through all partite sets, a contradiction. Thus we assume in the following that $v_4 \Rightarrow A$. This implies that $V_4' \neq \emptyset$, since otherwise we obtain the contradiction

$$d^+(v_1) = d^+(v_4) \geq |A| + 2 = d^+(v_1) + 1.$$

If there is a vertex $v_4' \in V_4'$ such that $v_4' \to v_3$, then $v_1 v_2 v_4' v_3 v_4 v_1$ is a 5-cycle containing vertices of 4 partite sets, a contradiction. Hence we assume that $v_3 \to V_4'$.

If there is a vertex $b \in B$ with the property that $v_4 \to b$, then the 5-cycle $v_1 v_2 v_3 v_4 b v_1$ leads to a contradiction. Hence we assume that $B \Rightarrow v_4$.

If there is a vertex $v_2'' \in V_2''$ such that $v_3 \to v_2''$, then the cycle $v_1 v_2 v_3 v_2'' v_4 v_1$ yields a contradiction. It remains the case that $V_2'' \to v_3$.

This yields $V_2'' \to V_4'$, since otherwise we arrive at the contradiction that $v_1 v_4' v_2'' v_3 v_4 v_1$ is a 5-cycle through all partite sets, where $v_4' \in V_4'$ and $v_2'' \in V_2''$ such that $v_4' \to v_2''$.

If there are vertices $v_4' \in V_4'$ and $v_3'' \in V_3''$ such that $v_4' \to v_3''$, then we find the 5-cycle $v_1 v_2 v_3 v_4' v_3'' v_1$, a contradiction. Thus assume in the following that $V_3'' \to V_4'$.

Summarizing some of our results we deduce that

$$(V_2'' \cup V_3'' \cup \{v_1, v_2, v_3\}) \to V_4' \neq \emptyset.$$

This implies that $V_4'' \neq \emptyset$, since otherwise, for each $v_4' \in V_4'$ we arrive at the contradiction

$$d^-(v_1) = d^-(v_4') \geq |B| + 3 = d^-(v_1) + 1.$$

Similarly, we conclude that for each vertex $v_4' \in V_4'$ there is a vertex $v_1' \in V_1 - \{v_1\}$ such that $v_4' \to v_1'$. Now we choose two fix vertices $v_4' \in V_4'$ and $v_1' \in V_1 - \{v_1\}$ such that $v_4' \to v_1'$.

If there is a vertex $v_3'' \in V_3''$ such that $v_1' \to v_3''$, then $v_1 v_2 v_4' v_1' v_3'' v_1$ is a 5-cycle, a contradiction. Hence assume that $V_3'' \to v_1'$.

If there is a vertex $v_2'' \in V_2''$ with the property that $v_1' \to v_2''$, then $v_1 v_4' v_1' v_2'' v_3 v_1$ is a cycle through all partite sets, a contradiction. Hence assume that $V_2'' \to v_1'$.

If $v_1' \to v_2$, then $v_1 v_4' v_1' v_2 v_3 v_1$ is a 5-cycle containing vertices of all partite sets, a contradiction. Thus assume in the following that $v_2 \to v_1'$.

Furthermore we conclude that $v_3 \to v_1'$, since otherwise $v_1 v_2 v_1' v_3 v_4 v_1$ is a cycle through all partite sets, a contradiction.

If there is a vertex $v_4'' \in V_4''$ such that $v_1' \to v_4''$, then $v_1 v_2 v_3 v_1' v_4'' v_1$ is a 5-cycle, a contradiction. It remains the case that $V_4'' \to v_1'$.

Altogether, we obtain the contradiction

$$d^-(v_1) = d^-(v_1') \geq |B| + |\{v_2, v_3, v_4'\}| = d^-(v_1) + 1.$$

Case 2. Suppose that D does not contain any strong subtournament of order 4. By the hypothesis that D is regular, Theorem 2.8 yields that D is strongly connected. Hence, according to Theorem 2.6, there exists a 3-cycle $C = v_1 v_2 v_3 v_1$ in D. Assume, without loss of generality, that $v_i \in V_i$ for $i \in \{1, 2, 3\}$.

If there exists a vertex $v_4 \in V_4$ having an in- and an out-neighbor in $V(C)$, then $D[\{v_1, v_2, v_3, v_4\}]$ is a strong subtournament of order 4, a contradiction to our assumption. Hence we can decompose V_4 into two subsets V_4' and V_4'' such that $V_4'' \to V(C) \to V_4'$. Assume, without loss of generality, that $V_4' \neq \emptyset$. In addition, let $v_4' \in V_4'$ and define $U = N^+(v_4')$.

Subcase 2.1. Assume that $V_4'' \neq \emptyset$ and let $v_4'' \in V_4''$. Suppose that $v_4'' \to U = N^+(v_4')$. Using the fact that $V_4'' \to V(C)$, we arrive at the contradiction

$$d^+(v_4'') \geq |N^+(v_4')| + |V(C)| = d^+(v_4') + 3.$$

Thus there is a vertex $u \in U \cap (V_1 \cup V_2 \cup V_3)$ such that $u \to v_4''$. If $u \in V_i$, then $D[(V(C) - \{v_i\}) \cup \{v_4', v_4'', u\}]$ contains a 5-cycle through all partite sets, a contradiction.

Subcase 2.2. Assume that $V_4'' = \emptyset$ and thus $V_4' = V_4$. If there are vertices $u \in U$ and $v_j \in V(C)$ such that $u \to v_j$, then $u v_j v_{j+1} v_{j+2} v_4' u$ is a 5-cycle through all partite sets, a contradiction. Thus it remains the case that $V(C) \Rightarrow U$. But now we arrive at the contradiction

$$\begin{aligned} d^+(v_4') = d^+(v_1) &\geq |\{v_2\}| + |V_4| + |U - V(v_1)| \\ &\geq 1 + r + d^+(v_4') - (r - 1) = d^+(v_4') + 2, \end{aligned}$$

and the proof of this lemma is complete. $\square$

3. Main result

Theorem 3.1. *If D is a regular c-partite tournament with $c \geq 4$ and $|V(D)| \geq 10$, then D contains two complementary cycles of length 5 and $|V(D)| - 5$.*

Proof. Let $V_1, V_2, \ldots, V_c$ be the partite sets of D and let $r = \alpha(D)$. Then it follows from Lemma 2.4 that $|V_1| = |V_2| = \cdots = |V_c| = \alpha(D) = r$ and $|V(D)| = cr$. According to Theorem 2.8, we have

$$\kappa(D) \geq \left\lceil \frac{|V(D)| - \alpha(D)}{3} \right\rceil = \left\lceil \frac{(c-1)r}{3} \right\rceil. \tag{2}$$

If $r = 1$, that means that D is a tournament, then $|V(D)| = c \geq 10$ and (2) yield $\kappa(D) \geq 3$. The desired result follows from Theorem 2.7.

Therefore, it remains the case that $r \geq 2$. In view of Lemma 2.16 and Theorem 2.6, there exists a 5-cycle C_5 through exactly 4 partite sets when $c = 4$. According to Theorem 2.14, there is a 5-cycle C_5 through exactly 5 partite sets when $c \geq 5$. If we define the c-partite tournament H by $H = D - V(C_5)$, then $|V(H)| = cr - 5$. Let $V_1', V_2', \ldots, V_c'$ be the partite sets of H such that $|V_1'| \leq |V_2'| \leq \cdots \leq |V_c'|$.

A. Assume that $c = 4$. As D is regular and $|V(D)| \geq 10$, it follows that $r \geq 4$ is even $|V_3'| \leq r - 1$, and $|V_4'| \leq r - 1$ and $i_g(H) \leq 4$. If $r \geq 8$, then we deduce that

$$i_g(H) \leq 4 \leq \frac{r}{2} = \frac{4r - 5 - (r-1) - 2(r-1) + 2}{2} \leq \frac{|V(H)| - |V_3'| - 2|V_4'| + 2}{2}.$$

Applying Theorem 2.11, we conclude that H has a Hamiltonian cycle C, and so we have found two complementary cycles C and C_5, where C_5 has length five. If $c = 4$, there remain the cases $r = 4, 6$.

B. Assume that $c = 5$. Since C_5 contains vertices from 5 partite sets, we deduce that $i_g(H) \leq 4$ and $|V_i'| = r - 1$ for $1 \leq i \leq 5$. If $r \geq 4$, then we deduce that

$$i_g(H) \leq 4 \leq \frac{2r}{2} = \frac{5r - 5 - (r-1) - 2(r-1) + 2}{2} = \frac{|V(H)| - |V_4'| - 2|V_5'| + 2}{2}.$$

Applying Theorem 2.11, we conclude that H has a Hamiltonian cycle C, and we obtain the desired complementary cycles. Thus there remain the cases $c = 5$ and $r = 2, 3$.

C. Assume that $c = 6$. It follows that r is even. Since C_5 contains vertices from 5 partite sets, we observe that $|V_5'| \leq r - 1$ and $i_g(H) \leq 5$. If $r \geq 4$, then we deduce that

$$i_g(H) \leq 5 \leq \frac{3r - 2}{2} = \frac{6r - 5 - (r-1) - 2r + 2}{2} = \frac{|V(H)| - |V_5'| - 2|V_6'| + 2}{2}.$$

Applying Theorem 2.11, we obtain the desired complementary cycles. Thus there remains the case that $c = 6$ and $r = 2$.

D. Assume that $c \geq 7$. With exception of the four cases $c = 7$ and $r = 2, 3$, $c = 8$, and $r = 2$, as well as $c = 9$ and $r = 2$, we have

$$i_g(H) \leq 5 \leq \frac{(c-3)r - 3}{2} = \frac{cr - 5 - r - 2r + 2}{2} = \frac{|V(H)| - |V_{c-1}'| - 2|V_c'| + 2}{2}.$$

Again, Theorem 2.11 leads to the desired complementary cycles.

Case 1. Assume that $c = 9$ and $r = 2$. Then D is 8-regular and $\alpha(H) = 2$. In addition, Theorem 2.8 yields $\kappa(D) \geq 6$ and thus $\kappa(H) \geq 1$.

Subcase 1.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor of H with the properties described in Theorem 2.13. Because of $|V^*| \leq 2$, it follows from Theorem 2.13 that there is at most one arc from $H - V(C_1')$ to C_1'. As $\kappa(H) \geq 1$, we see that there is exactly one arc from $H - V(C_1')$ to C_1'. Since $|V(H)| = 13$, we can assume, without loss of generality, that $|V(C_1')| \leq 6$, because we consider the inverse digraph D^{-1} when $|V(C_1')| \geq 7$. This implies that there are at least two vertices $x_1, x_2 \in V(C_1')$ such that $d^-_{D[V(C_1')]}(x_i) \leq 2$ for $i = 1, 2$ and thus $d^-_D(x_1) \leq 7$ or $d^-_D(x_2) \leq 7$, a contradiction to the 8-regularity of D.

Subcase 1.2. Assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 1$ and $\alpha(H) = 2$, we see that $1 = |Z| < |Y| = 2$. Let, without loss of generality, $Y = V_9'$ and $|R_1| \leq |R_2|$. Since D is 8-regular, we see that $d^+_H(x), d^-_H(x) \geq 3$ for every $x \in V(H)$ and $d^+_H(x), d^-_H(x) \geq 4$ for $x \in (V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5')$.

If $R_1 = \emptyset$, then $V_9' = Y \rightarrow R_2$ leads to the contradiction $d^-_H(y) \leq 1$ for $y \in Y$. If $1 \leq |R_1| \leq 4$, then there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 2$, a contradiction. In the remaining case $|R_1| = 5$, we arrive at the contradiction that there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 2$ or the induced subdigraph $H[R_1]$ is a 2-regular tournament. In the second case, we obtain the contradiction $d^-_H(x) \leq 3$ for some vertex $x \in R_1 \cap (V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5')$.

Case 2. Assume that $c = 8$ and $r = 2$. Then D is 7-regular and $\alpha(H) = 2$. Let $V_1' = \{a\}$, $V_2' = \{b\}$, $V_3' = \{c\}$, $V_4' = \{d\}$, $V_5' = \{z\}$, $V_6' = \{u_1, u_2\}$, $V_7' = \{v_1, v_2\}$, and $V_8' = \{w_1, w_2\}$ be the partite sets of H and $W = \{a, b, c, d, z\}$

Subcase 2.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor with the properties described in Theorem 2.13. Because of $|V^*| \leq 2$, it follows from Theorem 2.13 that there is at most one arc from $H - V(C_1')$ to C_1'. Since $|V(H)| = 11$, we can assume, without loss of generality, that $|V(C_1')| \leq 5$.

If $|V(C_1')| \leq 4$, then there are at least two vertices $x_1, x_2 \in V(C_1')$ such that $d^-_{D[V(C_1')]}(x_i) = 1$ for $i = 1, 2$. This implies $d^-_D(x_1) \leq 6$ or $d^-_D(x_2) \leq 6$, a contradiction to the 7-regularity of D.

Assume now that $|V(C_1')| = 5$. If there exist at least two vertices $x_1, x_2 \in V(C_1')$ such that $d^-_{D[V(C_1')]}(x_i) = 1$ for $i = 1, 2$, then we arrive at a contradiction as in the case $|V(C_1')| \leq 4$. Otherwise, the digraph $D[V(C_1')]$ is 4- or 5-partite. If $D[V(C_1')]$ is 4-partite, then there exists a vertex $x_1 \in V(C_1')$ such that $d^-_{D[V(C_1')]}(x_1) = 1$, and there are vertices $x \in V(C_5)$ and $y \in V(C_1')$ which are not adjacent. This leads to the contradiction $d^-_D(x_1) \leq 6$ or $d^-_D(y) \leq 6$. If $D[V(C_1')]$ is 5-partite, then there exist $x_1, x_2 \in V(C_5)$ and $y_1, y_2 \in V(C_1')$ such that x_i and y_i are not adjacent for $i = 1, 2$, and we arrive analogously at a contradiction to the 7-regularity of D.

Subcase 2.2. Assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since D is 7-regular, we see that $d^+_H(x), d^-_H(x) \geq 2$ for every $x \in V(H)$ and $d^+_H(x), d^-_H(x) \geq 3$ for every $x \in W$. This easily implies that $Z = \emptyset$ is not possible. Thus let now $1 = |Z| < |Y| = 2$ and let, without loss of generality, $Y = V_8' = \{w_1, w_2\}$ and $|R_1| \leq |R_2|$.

If $R_1 = \emptyset$, then $Y \Rightarrow R_2$ leads to the contradiction $d^-_H(y) \leq 1$ for $y \in Y$. If $1 \leq |R_1| \leq 2$, then there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 1$, a contradiction. If $|R_1| = 3$, we arrive at the contradiction that there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 1$ or the induced subdigraph $H[R_1]$ is a 3-cycle. In the second case, we obtain the contradiction $d^-_H(x) \leq 3$ for some vertex $x \in R_1 \cap W$.

In the remaining case that $|R_1| = 4$, we deduce that $|R_2| = 4$. If there is a vertex $y \in R_1$ with $d^-_{H[R_1]}(y) = 0$ or a vertex $y \in R_2$ with $d^+_{H[R_2]}(y) = 0$, then we obtain a contradiction to $d^+_H(x), d^-_H(x) \geq 2$ for every $x \in V(H)$. Thus we assume in the following that $d^-_{H[R_1]}(x) \geq 1$ for every $x \in R_1$ and $d^+_{H[R_2]}(x) \geq 1$ for every $x \in R_2$. Now we distinguish 3 cases.

Assume that $H[R_1]$ is a bipartite tournament. It follows that $R_1 = V_6' \cup V_7'$. Hence there exists at least one vertex $x \in R_2 \cap W$ such that $d^+_H(x) \leq 2$, a contradiction.

Assume that $H[R_1]$ is a 3-partite tournament but not bipartite. Let, without loss of generality, $V_7' \subset R_1$. In the case that $R_1 \cap V_6' = \emptyset$, we arrive at the contradiction that there exists a vertex $x \in R_1 \cap W$ such that $d^-_H(x) \leq 2$. In the remaining case that $R_1 \cap V_6' \neq \emptyset$, we arrive at the contradiction that there exists at least one vertex $x \in R_2 \cap W$ such that $d^+_H(x) \leq 2$.

Assume that $H[R_1]$ is a tournament. If $H[R_2]$ is not a tournament, then we arrive at a contradiction similar to the two cases above. Furthermore, we obtain a contradiction or we deduce that, without loss of generality, $R_1 = \{u_1, v_1, a, b\}$, $R_2 = \{u_2, v_2, c, d\}$ and $Z = \{z\}$ such that $Z \to R_1 \to Y \to R_2 \to Z$ and $R_1 \Rightarrow R_2$ so that $d^+_H(x) = 7$ for every $x \in R_1$ and $d^-_H(y) = 7$ for every $y \in R_2$. If $C_5 = x_1x_2x_3x_4x_5x_1$, then the 7-regularity of D implies that $R_2 \Rightarrow C_5 \Rightarrow R_1$. Hence there exists the new 5-cycle $C_5^* = v_1w_1u_2x_1x_2v_1$. If we assume, without loss of generality, that $a \to b$ and $c \to d$, then there exists the complementary cycle $x_3x_4x_5u_1w_2cdzabv_2x_3$.

Case 3. Assume that $c = 7$ and $r = 3$. Then D is 9-regular and $\alpha(H) = 3$. In addition, Theorem 2.9 yields $\kappa(D) \geq 7$ and thus $\kappa(H) \geq 2$. If H has a cycle factor, then Theorem 2.10 shows that H is Hamiltonian, and we are done.

Assume next that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 2$ and $\alpha(H) = 3$, we see that $2 = |Z| < |Y| = 3$. Let, without loss of generality, $Y = V_7'$ and $|R_1| \leq |R_2|$. Since D is 9-regular, we see that $d^+_H(x), d^-_H(x) \geq 4$ for every $x \in V(H)$ and $d^+_H(x), d^-_H(x) \geq 5$ for $x \in (V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5')$.

If $R_1 = \emptyset$, then $Y \to R_2$ leads to the contradiction $d^-_H(y) \leq 2$ for $y \in Y$. If $1 \leq |R_1| \leq 4$, then there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 3$, a contradiction. In the remaining case $|R_1| = 5$, we arrive at the contradiction that there exists a vertex $x \in R_1$ such that $d^-_H(x) \leq 3$ or the induced subdigraph $H[R_1]$ is a 2-regular tournament. In the second case, we obtain the contradiction $d^-_H(x) \leq 4$ for some vertex $x \in R_1 \cap (V_1' \cup V_2' \cup V_3' \cup V_4' \cup V_5')$.

Case 4. Assume that $c = 7$ and $r = 2$. Then D is 6-regular and $\alpha(H) = 2$. Let $V_1' = \{a\}$, $V_2' = \{b\}$, $V_3' = \{c\}$, $V_4' = \{d\}$, $V_5' = \{z\}$, $V_6' = \{u_1, u_2\}$, and $V_7' = \{v_1, v_2\}$ be the partite sets of H and $W = \{a, b, c, d, z\}$. Since D is 6-regular, we observe that $d^+_H(x), d^-_H(x) \geq 1$ for every $x \in V(H)$ and $d^+_H(x), d^-_H(x) \geq 2$ for every $x \in W$. In addition, let $C_5 = x_1x_2x_3x_4x_5x_1$.

Subcase 4.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor with the properties described in Theorem 2.13. Because of $|V^*| \leq 2$, it follows from Theorem 2.13 that there is at most one arc from $H - V(C_1')$ to C_1'.

If C_1' is a 3-cycle, then we arrive at a contradiction with exception of the case that C_1' has, without loss of generality, the form $C_1' = au_1v_1a$, and there is an arc from $H - V(C_1')$ to a. In addition, we deduce that $T_6 = H - V(C_1')$ is a strong tournament and $C_5 \to v_1$. According to Theorem 2.5, there exists a 5-cycle C_5^* containing u_2 in T_6. Now let $y \in (V(T_6) - V(C_5^*))$. Since D is 6-regular, there exists an arc from y to C_5, say $y \to x_1$. This implies that $x_1x_2x_3x_4x_5v_1au_1yx_1$ is a complementary cycle of C_5^*.

Subcase 4.1.1. Assume that C_1' is a 4-cycle and that there is no arc from the 5-cycle C_2' to C_1'. It follows that $V(C_1') \cap V_6' \neq \emptyset$ and $V(C_1') \cap V_7' \neq \emptyset$. We distinguish the three cases that $H[V(C_1')]$ is 4-partite, 3-partite or bipartite.

Subcase 4.1.1.1. Assume that $H[V(C_1')]$ is 4-partite. This implies, without loss of generality, that $C_1' = av_1u_1ba$ such that $v_1 \to b$ and $u_1 \to a$. It follows that $C_5 \Rightarrow C_1'$. Next let, without loss of generality, $C_2' = v_2y_2y_3y_4y_5v_2$. Since $T_5 = D[V(C_5)]$ is a strong tournament, we conclude from Theorem 2.5 that either there are at least three distinct vertices w_1, w_2, w_3 in T_5 such that $T_5 - w_i$ is strong for $i = 1, 2, 3$ or we suppose that $x_j \to x_i$ for $1 \leq i < j \leq 5$ and $j - i \geq 2$.

If $T_5 - w_i$ is strong for $i = 1, 2, 3$, then it follows that $v_2 \to w_1$ or $v_2 \to w_2$ or $v_2 \to w_3$, say $v_2 \to w_1 = x_1$. Since y_5 dominates at least one vertex of $T_5 - x_1$, say $y_5 \to x_2$, we arrive at the complementary cycles $x_1u_1bav_2x_1$ and $x_2x_3x_4x_5v_1y_2y_3y_4y_5x_2$.

If $x_j \to x_i$ for $1 \leq i < j \leq 5$ and $j - i \geq 2$, then $C_2' \to x_5$ and y_5 dominates at least one vertex of $T_5 - x_5$, say $y_5 \to x_1$. Now we arrive at the complementary cycles $x_5u_1bav_2x_5$ and $x_1x_2x_3x_4v_1y_2y_3y_4y_5x_1$.

Subcase 4.1.1.2. Assume that $H[V(C_1')]$ is 3-partite. This implies, without loss of generality, that $C_1' = au_1v_1u_2a$ such that $v_1 \to a$. It follows that $C_5 \Rightarrow C_1'$. Next let, without loss of generality, $C_2' = v_2y_2y_3y_4y_5v_2$. As above, the strong connectivity of $T_5 = D[V(C_5)]$ implies that either there are at least three distinct vertices w_1, w_2, w_3 in T_5 such that $T_5 - w_i$ is strong for $i = 1, 2, 3$ or we suppose that $x_j \to x_i$ for $1 \leq i < j \leq 5$ and $j - i \geq 2$.

If $T_5 - w_i$ is strong for $i = 1, 2, 3$, then it follows that $v_2 \to w_1$ or $v_2 \to w_2$ or $v_2 \to w_3$, say $v_2 \to w_1 = x_1$. Since y_5 dominates at least one vertex of $T_5 - x_1$, say $y_5 \to x_2$, we arrive at the complementary cycles $x_1v_1au_1v_2x_1$ and $x_2x_3x_4x_5u_2y_2y_3y_4y_5x_2$.

If $x_j \to x_i$ for $1 \leq i < j \leq 5$ and $j - i \geq 2$, then $C_2' \to x_5$ and y_5 dominates at least one vertex of $T_5 - x_5$, say $y_5 \to x_1$. Now we have the complementary cycles $x_5v_1au_1v_2x_5$ and $x_1x_2x_3x_4u_2y_2y_3y_4y_5x_1$.

Subcase 4.1.1.3. Assume that $H[V(C_1')]$ is bipartite. This implies, without loss of generality, that $C_1' = u_1v_1u_2v_2u_1$ and that $C_5 \to C_1'$. Next let $C_2' = y_1y_2y_3y_4y_5y_1$. It follows that $D[V(C_2')]$ as well as $D[V(C_5)]$ are 2-regular tournaments and that $D[V(C_5)] - x_i$ is strong for each $1 \leq i \leq 5$. If we assume, without loss of generality, that $y_1 \to x_1$, then we can assume, without loss of generality, that $y_5 \to x_2$. Now we have the complementary cycles $x_1u_2v_2u_1y_1x_1$ and $x_2x_3x_4x_5v_1y_2y_3y_4y_5x_2$.

Subcase 4.1.2. Assume that $C_1' = p_1p_2p_3p_4p_1$ is a 4-cycle such that there is an arc, say y_1p_1 from the 5-cycle $C_2' = y_1y_2y_3y_4y_5y_1$ to C_1'. If, without loss of generality, $V^* = V_7' = \{v_1, v_2\}$, then Theorem 2.13 shows that, without loss of generality, $p_4 = v_1$ and $y_2 = v_2$.

Subcase 4.1.2.1. Assume that $V(C_1') \cap V_6' = \emptyset$. The 6-regularity of D leads to $p_1 \to p_3$ and $v_1 \to p_2$. Thus $y_1p_1p_3v_1p_2v_2y_3y_4y_5y_1$ is a complementary cycle of C_5.

Subcase 4.1.2.2. Assume that $V(C_1') \cap V_6' = \{u_1\}$ and $u_1 = p_1$. The 6-regularity of D leads to $u_1 \to p_3$ and $v_1 \to p_2$. Thus $y_1u_1p_3v_1p_2v_2y_3y_4y_5y_1$ is a complementary cycle of C_5.

Subcase 4.1.2.3. Assume that $V(C_1') \cap V_6' = \{u_1\}$ and $u_1 = p_2$. The 6-regularity of D leads to $p_1 \to p_3$. If $v_1 \to u_1$, then $y_1p_1p_3v_1u_1v_2y_3y_4y_5y_1$ is a complementary cycle of C_5.

Assume next that $u_1 \to v_1$. The 6-regularity of D leads to $C_5 \Rightarrow \{p_1, p_3\}$ and $C_5 \to u_1$. We assume, without loss of generality, that $v_1 \to x_1$ and thus $\{x_2, x_3, x_4, x_5\} \to v_1$. If $x_5 \to p_1$, then we have the two complementary cycles C_2' and $v_1x_1x_2x_3x_4x_5p_1u_1p_3v_1$. Hence there remains the case that $V(x_5) = V(p_1)$. Because of $\sum_{i=1}^{5} d_H^-(y_i) = 27$, there are at least two vertices y_i and y_j in C_2' such that $\{y_i, y_j\} \Rightarrow C_5$. We distinguish the following subcases where the subscripts are taken modulo 5.

Subcase 4.1.2.3.1. Assume that $y_j = y_{i+1}$.

If $y_{i+1} \neq v_2$ and $V(y_i) \neq V(x_4)$, then $x_5x_1x_2v_1y_{i+1}x_5$ and $x_3x_4p_1u_1p_3y_{i-3}y_{i-2}y_{i-1}y_ix_4$ are complementary cycles.
If $y_{i+1} = v_2$ and $V(y_i) \neq V(x_4)$, then $x_2x_3v_1p_1y_{i+1}x_2$ and $x_4x_5x_1u_1p_3y_{i-3}y_{i-2}y_{i-1}y_ix_4$ are complementary cycles.
If $y_{i+1} \neq v_2$ and $V(y_i) = V(x_4)$, then $x_4x_5u_1v_1y_{i+1}x_4$ and $x_1x_2x_3p_1p_3y_{i-3}y_{i-2}y_{i-1}y_ix_1$ are complementary cycles.
If $y_{i+1} = v_2$ and $V(y_i) = V(x_4)$, then $x_4x_5v_1p_1y_{i+1}x_4$ and $x_1x_2x_3u_1p_3y_{i-3}y_{i-2}y_{i-1}y_ix_1$ are complementary cycles.

Subcase 4.1.2.3.2. Assume that $y_j = y_{i+2}$.

If $y_{i+1} \to y_{i+3}$, then we are in the same situation as in Subcase 4.1.2.3.1 when we use y_{i+2} instead of y_{i+1} and $y_{i+1}y_{i+3}y_{i+4}y_i$ instead of $y_{i-3}y_{i-2}y_{i-1}y_i$.

If $y_{i+4} \to y_{i+1}$, then we are in the same situation as in Subcase 4.1.2.3.1 when we use y_i instead of y_{i+1} and $y_{i+3}y_{i+4}y_{i+1}y_{i+2}$ instead of $y_{i-3}y_{i-2}y_{i-1}y_i$.

In the remaining case that $y_{i+3} \to y_{i+1}$ and $y_{i+1} \to y_{i+4}$, we use in Subcase 4.1.2.3.1 y_{i+2} instead of y_{i+1} and $y_{i+3}y_{i+1}y_{i+4}y_i$ instead of $y_{i-3}y_{i-2}y_{i-1}y_i$.

Subcase 4.1.2.4. Assume that $V(C_1') \cap V_6' = \{u_1\}$ and $u_1 = p_3$. The 6-regularity of D leads to $v_1 \to p_2$. If $p_1 \to u_1$, then $y_1p_1u_1v_1p_2v_2y_3y_4y_5y_1$ is a complementary cycle of C_5. Otherwise we have $u_1 \to p_1$, and the 6-regularity of D leads to $C_5 \to v_1$. If, without loss of generality, $p_1 \to x_1$, then there exist the complementary cycles C_2' and $p_1x_1x_2x_3x_4x_5v_1p_2u_1p_1$.

Subcase 4.1.2.5. Assume that $V(C_1') \cap V_6' = \{u_1, u_2\}$. This implies, without loss of generality, that $C_1' = u_1p_2u_2v_1u_1$. The 6-regularity of D leads to $v_1 \to p_2$ and thus $C_5 \Rightarrow p_2$ and $C_5 \to \{v_1, u_2\}$. We assume, without loss of generality, that $u_1 \to x_1$ and thus $\{x_2, x_3, x_4, x_5\} \to u_1$. If $x_5 \to p_2$, then we have the two complementary cycles C_2' and $u_1x_1x_2x_3x_4x_5p_2u_2v_1u_1$. Hence there remains the case that $V(x_5) = V(p_2)$. Because of $\sum_{i=1}^{5} d_H^-(y_i) = 28$, there are at least two vertices y_i and y_{i+1} in C_2' such that $\{y_i, y_{i+1}\} \Rightarrow C_5$.

If $y_{i+1} \neq v_2$ and $V(y_i) \neq V(x_4)$, then $x_5x_1x_2v_1y_{i+1}x_5$ and $x_3x_4u_1p_2u_2y_{i-3}y_{i-2}y_{i-1}y_ix_4$ are complementary cycles.
If $y_{i+1} = v_2$ and $V(y_i) \neq V(x_4)$, then $x_2x_3v_1u_1y_{i+1}x_2$ and $x_4x_5x_1p_2u_2y_{i-3}y_{i-2}y_{i-1}y_ix_4$ are complementary cycles.
If $y_{i+1} \neq v_2$ and $V(y_i) = V(x_4)$, then $x_4x_5u_2v_1y_{i+1}x_4$ and $x_1x_2x_3u_1p_2y_{i-3}y_{i-2}y_{i-1}y_ix_1$ are complementary cycles.
If $y_{i+1} = v_2$ and $V(y_i) = V(x_4)$, then $x_4x_5v_1u_1y_{i+1}x_4$ and $x_1x_2x_3p_2u_2y_{i-3}y_{i-2}y_{i-1}y_ix_1$ are complementary cycles.

Subcase 4.1.3. Assume that C_1' is a 5- or 6-cycle. Using the converse D^{-1} of D, we obtain the desired results by the cases discussed above.

Subcase 4.2. Assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Assume, without loss of generality, that $|R_1| \leq |R_2|$.

Subcase 4.2.1. Assume that $Z = \emptyset$. If $R_1 = \emptyset$, then we arrive at the contradiction $d_H^+(y) \geq 7$ for every $y \in Y$. If $1 \leq |R_1| \leq 3$, then we obtain the contradiction $d_H^-(x) = 0$ for a vertex $x \in R_1$ or $d_H^-(x) \leq 1$ for a vertex $x \in R_1 \cap W$. In the remaining case that $|R_1| = |R_2| = 4$ and $|Y| = 1$, we receive at a contradiction or, without loss of generality, the subdigraph $H[R_1]$ consists of the 4-cycle $u_1v_1abu_1$ such that $u_1 \to a$ and $v_1 \to b$ and the subdigraph $H[R_2]$ consists of the 4-cycle $u_2v_2cdu_2$ such that $c \to u_2$ and $d \to v_2$ or $H[R_2]$ consists of the 4-cycle $v_2u_2cdv_2$ such that $c \to v_2$ and $d \to u_2$ and $Y = \{z\}$. In addition, we deduce that $R_2 \Rightarrow C_5 \Rightarrow R_1$. In the first case, we obtain the complementary cycles $x_1v_1abu_2x_1$ and $x_2x_3x_4x_5u_1zcdv_2x_2$ and in the second case $x_1v_1abv_2x_1$ and $x_2x_3x_4x_5u_1zcdu_2x_2$.

Subcase 4.2.2. Assume that $Z \neq \emptyset$. It follows that $1 = |Z| < |Y| = 2$. Assume, without loss of generality, that $Y = V_7' = \{v_1, v_2\}$. It follows that $R_1 \to Y \to R_2$.

Subcase 4.2.2.1. Assume that $R_1 = \emptyset$. This implies that $Z \to Y$ and $C_5 \to Y$. Let $R_2 = \{y_1, y_2, y_3, y_4, y_5, y_6\}$.

Subcase 4.2.2.1.1. Assume that $Z \subset W$, say $Z = \{z\}$. Since D is 6-regular, we see that there are at least two vertices in R_2, say y_1 and y_2, such that $\{y_1, y_2\} \to z$.

Subcase 4.2.2.1.1.1. Assume that $\{y_1, y_2\} = \{u_1, u_2\}$. Let, without loss of generality, $y_3y_4y_5y_6$ be a Hamiltonian path of the tournament induced by this vertex set. If $y_i \to u_j$ for $i = 4, 5, 6$ and $j = 1, 2$, then there exists the 5-cycle $y_{i-1}y_iu_jzv_1y_{i-1}$. If $p_1p_2p_3$ is a Hamiltonian path of the remaining vertices in R_2 such that, without loss of generality, $p_3 \to x_1$, then $x_1x_2x_3x_4x_5v_2p_1p_2p_3x_1$ is a complementary cycle. Therefore we can assume in the following that $\{u_1, u_2\} \to \{y_4, y_5, y_6\}$. If $y_i \to z$ for $i = 5, 6$, then there exists the 5-cycle $u_1y_{i-1}y_izv_1u_1$, and analogously to above also a complementary cycle. So we assume now that $z \to \{y_5, y_6\}$. This implies that $\{y_5, y_6\} \to y_3$. As above we receive to two desired complementary cycles or $\{u_1, u_2\} \to y_3$. It follows that $y_3 \to z$ and as before we obtain the desired complementary cycles.

Subcase 4.2.2.1.1.2. Assume that $\{y_1, y_2\} \neq \{u_1, u_2\}$. Assume, without loss of generality, that $y_1 \to y_2$.

If there is a vertex in R_2, say y_3, such that $y_3 \to y_1$, then there is the 5-cycle $y_3y_1y_2zv_1y_1$ and a complementary 9-cycle, with exception of the cases that $\{y_4, y_5, y_6\} = \{u_1, u_2, y_6\}$ such that $\{u_1, u_2\} \to y_6$ or $y_6 \to \{u_1, u_2\}$. Assume first that $\{u_1, u_2\} \to y_6$. If $y_6 \to y_i$ for an $i = 1, 2$, then there exists the 5-cycle $u_1y_6y_izv_1u_1$ and a complementary 9-cycle. Otherwise, we have $\{y_1, y_2\} \to y_6$ and thus $y_6 \to z$. If $y_1 \to u_1$, then there is the 5-cycle $y_1u_1y_6zv_1y_1$ and also a complementary cycle. In the other case $u_1 \to y_1$, there is the 5-cycle $u_1y_1y_2zv_1u_1$ and a complementary cycle. The second case that $y_6 \to \{u_1, u_2\}$ is similar. It remains the case that $y_1 \Rightarrow \{y_3, y_4, y_5, y_6\}$.

Subcase 4.2.2.1.1.2.1. Assume that $y_1 = u_1$ and let, without loss of generality, $y_3 = u_2$. If $y_i \to y_2$ for an $i = 4, 5, 6$, then there is the 5-cycle $u_1y_iy_2zv_1u_1$ and a complementary cycle. It remains the case that $y_2 \to \{y_4, y_5, y_6\}$. Because of Subcase 4.2.2.1.1.1, we can assume that $z \to u_2$. If $y_i \to z$ for an $i = 4, 5, 6$, then there is the 5-cycle $u_1y_2y_izv_1u_1$ and a complementary cycle. Otherwise we have $z \to \{y_4, y_5, y_6\}$, and the 6-regularity shows, without loss of generality, that $y_4 \to y_5 \to y_6 \to y_4$. Thus we obtain $\{y_4, y_5, y_6\} \to u_2 \to C_5$, If, without loss of generality, $y_6 \to x_4$, then we arrive at the complementary cycles $x_1x_2x_3v_2u_2x_1$ and $x_4x_5v_1u_1y_2zy_4y_5y_6x_4$.

Subcase 4.2.2.1.1.2.2. Assume that $y_2 = u_1$ and let, without loss of generality, $y_3 = u_2$. If $y_i \to u_1$ for an $i = 4, 5, 6$, then there is the 5-cycle $y_1y_iu_1zv_1y_1$ and a complementary cycle. It remains the case that $u_1 \to \{y_4, y_5, y_6\}$. Because of Subcase 4.2.2.1.1.1, we can assume that $z \to u_2$. If $y_i \to z$ for an $i = 4, 5, 6$, then there is the 5-cycle $y_1u_1y_izv_1y_1$ and a complementary cycle. Otherwise we have $z \to \{y_4, y_5, y_6\}$, and the 6-regularity shows, without loss of generality, that $y_4 \to y_5 \to y_6 \to y_4$. Thus we obtain $\{y_4, y_5, y_6\} \to u_2$, and hence the contradiction $d^-(u_2) \geq 7$.

Subcase 4.2.2.1.1.2.3. Assume that $\{y_1, y_2\} \subset W$ and let, without loss of generality, $y_3 = u_1$ and $y_4 = u_2$. In addition, we assume, without loss of generality, that $z \to u_1$ and $y_5 \to y_6$. If $u_i \to y_2$ for an $i = 1, 2$, then there is the 5-cycle $y_1u_iy_2zv_1y_1$ and a complementary cycle. Otherwise we have $y_2 \to \{u_1, u_2\}$. If $u_2 \to z$, then there is the 5-cycle $y_1y_2u_2zv_1y_1$ and a complementary cycle. It remains the case that $z \to u_2$.

Assume next that $y_6 \to y_2$. If $u_i \to y_6$ for an $i = 1, 2$, then there is the 5-cycle $u_iy_6y_2zv_1u_i$ and a complementary cycle. Otherwise we have $y_6 \to \{u_1, u_2\}$, and this implies that $u_1 \to C_5$ and $u_2 \to y_5$. If, without loss of generality, $y_6 \to x_4$, then we arrive at the complementary cycles $x_1x_2x_3v_2u_1x_1$ and $x_4x_5v_1y_1y_2zu_2y_5y_6x_4$.

Assume now that $y_2 \to y_6$ and $y_5 \to y_2$. If $u_i \to y_5$ for an $i = 1, 2$, then there is the 5-cycle $u_iy_5y_2zv_1u_i$ and a complementary cycle. Otherwise we have $y_5 \to \{u_1, u_2\}$, and this implies that $\{u_1, u_2\} \to y_6$. This yields the contradiction $d^-(y_6) \geq 7$.

Finally, assume that $y_2 \to y_6$ and $y_2 \to y_5$. The 6-regularity of D shows that $y_6 \to u_1$ or $y_6 \to u_2$, say $y_6 \to u_1$. This implies $u_1 \to y_5$. If $y_5 \to z$, then there is the 5-cycle $y_1u_1y_5zv_1y_1$ and a complementary cycle. Otherwise we have $z \to y_5$.

It follows that $y_5 \to u_2$ and thus $u_2 \to y_6$ and so $y_6 \to z$. This finally leads to the complementary cycles $u_1y_5y_6zv_1u_1$ and $x_1x_2x_3x_4x_5v_2y_1y_2u_2x_1$.

Subcase 4.2.2.1.2. Assume that $Z \subset V_6'$, say $Z = \{u_1\}$. Since D is 6-regular, there is at least one vertex in R_2, say y_1, such that $y_1 \to u_1$.

Subcase 4.2.2.1.2.1. Assume that there exists a vertex in R_2, say y_2, such that $y_2 \to y_1$. If $y_i \to y_2$ for an $i = 3, 4, 5, 6$, then there is the 5-cycle $y_iy_2y_1u_1v_1y_i$ and a complementary cycle. Hence it remains the case that $y_2 \to \{y_3, y_4, y_5, y_6\}$. If $y_i \to y_1$ for an $i = 3, 4, 5, 6$, then there is the 5-cycle $y_2y_iy_1u_1v_1y_i$ and a complementary cycle. Hence it remains the case that $y_1 \to \{y_3, y_4, y_5, y_6\}$.

Assume that $y_2 = u_2$. If $y_i \to u_1$ for an $i = 3, 4, 5, 6$, then the same arguments as above lead to two desired complementary cycles. It remains the case that $u_1 \to \{y_3, y_4, y_5, y_6\}$. This leads to the contradiction $d^-(y_i) \geq 7$ for at least two $i \in \{3, 4, 5, 6\}$.

Assume that $y_2 \neq u_2$ and, without loss of generality, that $y_6 = u_2$. If $y_i \to u_1$ for an $i = 3, 4, 5$, then the same arguments as above lead to two desired complementary cycles. It remains the case that $u_1 \to \{y_3, y_4, y_5\}$. It follows, without loss of generality, that $y_3 \to y_4 \to y_5 \to y_3$ and thus $\{y_3, y_4, y_5\} \to u_2$, and we arrive at the contradiction $d^-(u_2) \geq 7$.

Subcase 4.2.2.1.2.2. Assume that $y_1 \to \{y_2, y_3, y_4, y_5, y_6\}$ and let, without loss of generality, $y_6 = u_2$. With respect to Subcase 4.2.2.1.2.1, we can assume that $u_1 \to \{y_2, y_3, y_4, y_5\}$. This implies that $C_5 \to u_1$. For the tournament induced by $\{y_2, y_3, y_4, y_5\}$, we have, without loss of generality, the following two possibilities.

1. *Possibility*. Assume that $y_2 \to \{y_3, y_4, y_5\}$ and $y_3 \to y_4 \to y_5 \to y_3$. This implies $\{y_3, y_4, y_5\} \to u_2$ and thus $u_2 \to y_2$, $u_2 \to C_5$ and $\{y_3, y_4, y_5\} \Rightarrow C_5$. If, without loss of generality, $y_5 \to x_4$, then we arrive at the complementary cycles $x_1x_2x_3v_1u_2x_1$ and $x_4x_5v_2y_1u_1y_2y_3y_4y_5x_4$.

2. *Possibility*. Assume that $y_2 \to y_3 \to y_4 \to y_5 \to y_2$, $y_2 \to y_4$, and $y_3 \to y_5$. This implies $\{y_4, y_5\} \to u_2$ and $\{y_4, y_5\} \Rightarrow C_5$. If, without loss of generality, $u_2 \to x_1$, then we arrive at the complementary cycles $x_1x_2x_3v_1u_2x_1$ and $x_4x_5v_2y_1u_1y_2y_3y_4y_5x_4$ when $V(y_5) \neq V(x_4)$ or $x_4x_5v_2y_1u_1y_5y_2y_3y_4x_4$ when $V(y_5) = V(x_4)$.

Subcase 4.2.2.2. Assume that $|R_1| = 1$. We deduce, without loss of generality, that $R_1 = \{u_1\}$ and $Z = \{z\}$. This implies that $z \to u_1$ and $C_5 \to u_1$. Let $R_2 = \{y_1, y_2, y_3, y_4, y_5\}$.

Subcase 4.2.2.2.1. Assume that there is an arc from Z to Y, say $z \to v_1$. This implies that there exists an arc from R_2 to Z, say $y_1 \to z$.

Assume that there is an arc $y_i \to y_1$, say $y_2 \to y_1$. If there is a further arc, say $y_3 \to y_2$, then there is the 5-cycle $y_3y_2y_1zv_1y_3$. If, without loss of generality, $y_4 \to y_5 \to x_1$, then there exists the complementary cycle $x_1x_2x_3x_4x_5u_1v_2y_4y_5x_1$. Otherwise we have $y_2 \to \{y_3, y_4, y_5\}$. If $y_i \to y_1$ for an $i = 3, 4, 5$, then we find the desired complementary cycles as in the last case. Thus assume that $y_1 \to \{y_3, y_4, y_5\}$. If $y_i \to z$ for an $i = 3, 4, 5$, then we obtain our complementary cycles as above. However, if $z \to \{y_3, y_4, y_5\}$, then we arrive at a contradiction to the 6-regularity of D.

Next assume that $y_1 \to \{y_2, y_3, y_4, y_5\}$. If $y_i \to z$ for an $i = 2, 3, 4, 5$, then we are in a situation as discussed before. However, the case $z \to \{y_2, y_3, y_4, y_5\}$, leads to a contradiction to the 6-regularity of D.

Subcase 4.2.2.2.2. Assume that $Y \to z$. It follows that $C_5 \to Y$. If there is an arc from R_2 to Z, say $y_1 \to z$ and an arc, say $y_2 \to y_1$, then there is the 5-cycle $y_2y_1zu_1v_1y_2$. Since $C_5 \to v_2$ it is easy to find a complementary cycle. If $y_1 \to z$ and $y_1 \to \{y_2, y_3, y_4, y_5\}$, then we arrive at a contradiction to the 6-regularity as above. Therefore it remains the case that $z \to R_2$. Let, without loss of generality, $y_5 = u_2$. If the tournament induced by the vertices y_1, y_2, y_3, y_4 is transitive, then we obtain a contradiction to the 6-regularity of D. Hence there exists a 3-cycle, say $y_1y_2y_3y_1$. If we assume, without loss of generality, that $y_4 \to y_5 \to x_1$, then there is the 5-cycle $x_1x_2v_1y_4y_5x_1$. In addition, we observe that $y_1 \to x_3$ or $y_2 \to x_3$ or $y_3 \to x_3$. If not, then we arrive at the contradiction $x_3 \to \{x_4, v_1, v_2, u_1, y_1, y_2, y_3\}$ or $x_3 \to \{x_4, v_1, v_2, u_1, z, y_2, y_3\}$ when $V(x_3) = V(y_1)$ for example. If, without loss of generality, $y_1 \to x_3$, then there is the complementary cycle $x_3x_4x_5v_2zu_1y_2y_3y_1x_3$.

Subcase 4.2.2.3. Assume that $|R_1| = 2$. In this case we distinguish two cases.

Subcase 4.2.2.3.1. Assume that $R_1 = \{u_1, u_2\}$. This implies, without loss of generality, that $Z = \{a\}$ and $Z \to R_1$ and thus $C_5 \to R_1$. Since D is 6-regular, there are at least two vertices, say d and z, in R_2 such that $\{d, z\} \to a$ and $\{d, z\} \Rightarrow C_5$. If we assume, without loss of generality, that $d \to z$, then we deduce that $\{b, c\} \to d$ and $c \to z$. It follows that $z \to b$. Next we assume, without loss of generality, that $V(z) \neq V(x_1)$ and $x_2 \to v_1$.

If $b \to a$ and $V(d) \neq V(x_3)$, then there are the complementary cycles $x_1x_2v_1czx_1$ and $x_3x_4x_5u_1v_2bau_2dx_3$.

If $b \to a$ and $V(d) = V(x_3)$, then there are the complementary cycles $x_1x_2v_1cdx_1$ and $x_3x_4x_5u_1v_2bau_2zx_3$.

If $a \to b$, then we observe that $b \Rightarrow C_5$. If $V(b) \neq V(x_3)$, then there are the complementary cycles $x_1x_2v_1czx_1$ and $x_3x_4x_5u_1v_2dau_2bx_3$.

It remains the case that $V(b) = V(x_3)$. If $x_3 \to v_1$ and $V(d) \neq V(x_1)$, then there are the complementary cycles $x_1x_2x_3v_1dx_1$ and $x_4x_5u_1v_2czau_2bx_4$. If $x_3 \to v_1$ and $V(d) = V(x_1)$, then there are the complementary cycles $x_2x_3v_1cdx_2$ and $x_4x_5x_1u_1v_2zau_2bx_4$. Otherwise $v_1 \to x_3$, and we arrive at the complementary cycles $v_1x_3x_4x_5u_1v_1$ and $x_1x_2u_2v_2cdzabx_1$.

Subcase 4.2.2.3.2. Assume, without loss of generality, that $R_1 = \{b, u_1\}$ and $Z = \{a\}$ such that $u_1 \to b$ and $a \to \{b, u_1\}$. This implies that $C_5 \Rightarrow b$ and $C_5 \to u_1$. Let $R_2 = \{y_1, y_2, y_3, y_4\}$.

Assume that there exists an arc from R_2 to Z, say $y_1 \to a$. If there is an arc $y_i \to y_1$, say $y_2 \to y_1$, then we have the 5-cycle $y_2y_1abv_1y_2$. If, without loss of generality, $y_3 \to y_4 \to x_1$, then there exists the complementary cycle $x_1x_2x_3x_4x_5u_1v_2y_3y_4x_1$.

Otherwise we have $y_1 \to \{y_2, y_3, y_4\}$. If $y_i \to a$ for an $i = 2, 3, 4$, then we obtain our complementary cycles as in the case before. However, if $a \to \{y_2, y_3, y_4\}$, then we arrive at a contradiction to the 6-regularity of D. In the remaining case that $a \to \{y_1, y_2, y_3, y_4\}$, there is a vertex $y_i \in R_2$ with $d^-(y_i) \geq 7$, a contradiction.

Subcase 4.2.2.4. Assume that $|R_1| = |R_2| = 3$. Under this condition we discuss three cases. In the remaining cases we obtain the desired result by using the converse D^{-1} of D.

Subcase 4.2.2.4.1. Assume that $R_1 = \{b, u_1, u_2\}$. This implies, without loss of generality, that $Z = \{a\}$, $u_1 \to b$, $c \to d \to z \to c$, $R_2 \to a$, and $R_2 \Rightarrow C_5$. Now we distinguish two cases.

Assume first that $u_2 \to b$. It follows that $C_5 \to \{u_1, u_2\}$ and $a \to \{u_1, u_2\}$. If we assume, without loss of generality, that $x_3 \to v_2$ and $V(x_1) \neq V(c)$, then there are the complementary cycles $x_1x_2x_3v_2cx_1$ and $x_4x_5u_2bv_1dau_1zx_4$ when $V(z) \neq V(x_4)$ or $x_4x_5u_2bv_1zau_1dx_4$ when $V(z) = V(x_4)$.

Assume second that $b \to u_2$. It follows that $C_5 \to u_1$ and $a \to \{u_1, b\}$. If we assume, without loss of generality, that $x_3 \to v_2$ and $V(x_1) \neq V(c)$, then there are the complementary cycles $x_1x_2x_3v_2cx_1$ and $x_4x_5u_1v_1dabu_2zx_4$ when $V(z) \neq V(x_4)$ or $x_4x_5u_1v_1zabu_2dx_4$ when $V(z) = V(x_4)$.

Subcase 4.2.2.4.2. Assume that $R_1 = \{b, c, u_1\}$ and $Z = \{u_2\}$. It follows, without loss of generality, that $a \to d \to z \to a$, $R_2 \to u_2$, and $R_2 \Rightarrow C_5$. If, without loss of generality, $b \to c$, then we conclude that $u_2 \to \{b, c\}$, $c \to u_1 \to b$, and $C_5 \to u_1$. If we assume, without loss of generality, that $x_3 \to v_2$ and $V(x_1) \neq V(a)$, then there are the complementary cycles $x_1x_2x_3v_2ax_1$ and $x_4x_5u_1v_1du_2bczx_4$ when $V(z) \neq V(x_4)$ or $x_4x_5u_1v_1zu_2bcdx_4$ when $V(z) = V(x_4)$.

Subcase 4.2.2.4.3. Assume that $R_1 = \{b, c, u_1\}$ and, without loss of generality, $Z = \{a\}$. If, without loss of generality, $b \to c$ and $d \to z$, then we deduce that $\{a, u_1\} \to b$ and $z \to \{a, u_2\}$. It follows that $z \Rightarrow C_5 \Rightarrow b$. Now there remain the four cases that $c \to u_1$ and $d \to u_2$, $c \to u_1$ and $u_2 \to d$, $u_1 \to c$ and $d \to u_2$, as well as $u_1 \to c$ and $u_2 \to d$.

We only discuss the case that $c \to u_1$ and $u_2 \to d$, the other cases are similar. It follows that $d \Rightarrow C_5 \Rightarrow c$ and $d \to a \to c$. Since u_2 has at least 4 out-neighbors in C_5, and u_1 has at least 4 in-neighbors in C_5, there exists an index i such that $x_i \to u_1$ and $u_2 \to x_{i+1}$ for $1 \leq i \leq 5$. This leads to the complementary cycles $dabcv_1d$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iu_1v_2zu_2x_{i+1}$.

Case 5. Assume that $c = 6$ and $r = 2$. This case was solved with the help of an algorithm programmed in GAP [16] (cf. the Appendix).

Case 6. Assume that $c = 5$ and $r = 3$. Then D is 6-regular and $\alpha(H) = 2$. Let $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, b_3\}$, $V_3 = \{c_1, c_2, c_3\}$, $V_4 = \{u_1, u_2, u_3\}$, $V_5 = \{v_1, v_2, v_3\}$ and, without loss of generality, $C_5 = a_1b_1c_1u_1v_1a_1$. Since D is 6-regular, we observe that $d_H^+(x), d_H^-(x) \geq 2$ for every $x \in V(H)$.

Subcase 6.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \ldots, C_t'$ be a minimal cycle-factor of H with the properties described in Theorem 2.13. Because of $|V^*| \leq 2$, it follows from Theorem 2.13 that there is at most one arc from $H - V(C_1')$ to C_1'. If $|V(C_1')| \leq 4$, then we conclude that there exists a vertex $x \in V(C_1')$ with $d_H^-(x) \leq 1$, a contradiction. If $|V(C_1')| \geq 6$, then we obtain similarly the contradiction that there exists a vertex $x \in V(C_2')$ with $d_H^+(x) \leq 1$. It remains the case $t = 2$ such that C_1' and C_2' are 5-cycles.

Subcase 6.1.1. Assume that there does not exist an arc from C_2' to C_1'. This leads to a contradiction, with exception of the case that C_1' and C_2' induce 2-regular tournaments T_1 and T_2 such that $C_5 \Rightarrow C_1' \Rightarrow C_2' \Rightarrow C_5$. Now let $C_5^* = a_1b_1c_1xya_1$ be a new 5-cycle of D such that $x \in (V(C_1') \cap V_5)$ and $y \in (V(C_2') \cap V_4)$. Since T_1 and T_2 are regular tournaments, we observe that $T_1 - x$ and $T_2 - y$ contain Hamiltonian cycles $x_1x_2x_3x_4x_1$ and $y_1y_2y_3y_4y_1$, respectively. If, without loss of generality, x_4 and y_1 belong to different partite sets, then $u_1v_1x_1x_2x_3x_4y_1y_2y_3y_4u_1$ is a complementary cycle of C_5^*, and we are done.

Subcase 6.1.2. Assume that there exists an arc from C_2' to C_1'. If $H[V(C_1')]$ is 3-partite, then it follows that there exists a vertex $x \in V(C_1')$ with $d_H^-(x) \leq 1$, a contradiction.

Subcase 6.1.2.1. Assume that $H[V(C_1')]$ is exactly 5-partite. This implies that $H[V(C_2')]$ is also 5-partite. Let $C_1' = x_1x_2x_3x_4x_5x_1$ and $C_2' = y_1y_2y_3y_4y_5y_1$ such that $y_1 \to x_1$. Because of Theorem 2.13, we see that y_2 and x_5 belong to the same partite set V^*.

If $x_5 \to x_2$, then the 6-regularity implies that $x_3 \to x_5$ and so $x_1 \to x_3$. This yields the complementary cycle $y_1x_1x_3x_4x_5x_2y_2y_3y_4y_5y_1$.

If $x_2 \to x_5$, then we deduce that $x_4 \to x_2$ and thus $x_1 \to x_4$. If $x_5 \to x_3$, then we receive at the complementary cycle $y_1x_1x_2x_5x_3x_4y_2y_3y_4y_5y_1$. If $x_3 \to x_5$, then it follows that $x_1 \to x_3$. Thus it remains the situation that

$$x_1 \to x_4 \to x_2 \to x_5 \quad \text{and} \quad x_1 \to x_3 \to x_5.$$

Analogously one can show that there remains the case that

$$y_2 \to y_5 \to y_3 \to y_1 \text{and} y_2 \to y_4 \to y_1.$$

The 6-regularity of D implies that

$$(C_2' - y_2) \Rightarrow C_5 \Rightarrow (C_1' - x_5).$$

Let in the following, without loss of generality, $v_1 \in V^*$. If $x_5 \to a_1$, then we arrive at the complementary cycles C_2' and $x_5a_1b_1c_1u_1v_1x_1x_2x_3x_4x_5$.

If $a_1 \to x_5$, then we distinguish different cases.

Assume that $V(x_1) \neq V(y_3)$. We deduce that there exists the 5-cycle $a_1x_5x_1y_3v_1a_1$.
If $V(u_1) \neq V(x_2)$ and $V(y_1) \neq V(b_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_2y_4y_5y_1b_1$.
If $V(u_1) \neq V(x_2)$, $V(y_1) = V(b_1)$ and $V(y_1) \neq V(x_4)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_1y_2y_4y_5b_1$.
If $V(u_1) \neq V(x_2)$, $V(y_1) = V(b_1)$ and $V(y_1) = V(x_4)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_5y_1y_2y_4b_1$.
If $V(u_1) = V(x_2)$ and $V(y_1) \neq V(b_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_2y_4y_5y_1b_1$.
If $V(u_1) = V(x_2)$, $V(y_1) = V(b_1)$ and $V(x_2) \neq V(y_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_1y_2y_4y_5b_1$.
If $V(u_1) = V(x_2)$, $V(y_1) = V(b_1)$ and $V(x_2) = V(y_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_5y_1y_2y_4b_1$.
Assume that $V(x_1) = V(y_3)$. We deduce that there exists the 5-cycle $a_1x_5x_1y_4v_1a_1$.
If $V(u_1) \neq V(x_2)$ and $V(y_1) \neq V(b_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_2y_5y_3y_1b_1$.
If $V(u_1) \neq V(x_2)$, $V(y_1) = V(b_1)$, and $V(x_4) \neq V(y_5)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_5y_1y_2y_3b_1$.
If $V(u_1) \neq V(x_2)$, $V(y_1) = V(b_1)$, and $V(x_4) = V(y_5)$, then we arrive at the complementary cycle $b_1c_1u_1x_2x_3x_4y_1y_2y_5y_3b_1$.
If $V(u_1) = V(x_2)$ and $V(y_1) \neq V(b_1)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_2y_5y_3y_1b_1$.
If $V(u_1) = V(x_2)$, $V(y_1) = V(b_1)$ and $V(x_2) \neq V(y_3)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_3y_1y_2y_5b_1$.
If $V(u_1) = V(x_2)$, $V(y_1) = V(b_1)$ and $V(x_2) = V(y_3)$, then we arrive at the complementary cycle $b_1c_1u_1x_3x_4x_2y_1y_2y_5y_3b_1$.

Subcase 6.1.2.2. Assume that $H[V(C_1')]$ is exactly 4-partite. This implies that $H[V(C_2')]$ is also 4-partite. If there does not exist an arc from C_2' to C_1', then there exists at least one vertex $x \in V(C_1')$ with $d_H^-(x) \leq 1$, a contradiction. Let now $C_1' = x_1x_2x_3x_4x_5x_1$ and $C_2' = y_1y_2y_3y_4y_5y_1$ such that $y_1 \to x_1$. Because of Theorem 2.13, we see that y_2 and x_5 belong to the same partite set V^*.

If $V(y_1) = V(y_3)$, then it follows that $y_3 \to y_5 \to y_2 \to y_4 \to y_1$ and we obtain the complementary cycle $y_1x_1x_2x_3x_4x_5y_3y_5y_2y_4y_1$.

If $V(y_1) = V(y_4)$, then it follows that $y_4 \to y_2 \to y_5 \to y_3 \to y_1$ and we obtain the complementary cycle $y_1x_1x_2x_3x_4x_5y_4y_2y_5y_3y_1$.

Next assume that $V(y_1) \neq V(y_i)$ for $i = 3, 4$. Because of $y_2 \in V^*$, it remains the case that $V(y_3) = V(y_5)$. This implies that $y_5 \to y_2 \to y_4 \to y_1$ and $y_3 \to y_1$, and we arrive at the complementary cycle $y_1x_1x_2x_3x_4x_5y_4y_5y_2y_3y_1$.

Subcase 6.2. Assume that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Let, without loss of generality, $|R_1| \leq |R_2|$.

Assume first that $Z = \emptyset$. If $R_1 = \emptyset$, then we obtain the contradiction $d_H^+(y) \geq 8$ for every $y \in Y$. In the remaining case that $1 \leq |R_1| \leq 4$, we see that there exists a vertex $x \in R_1$ with $d_H^-(x) \leq 1$, a contradiction.

Next assume that $1 = |Z| < |Y| = 2$. If $R_1 = \emptyset$, then we obtain the contradiction $d_H^+(y) \geq 7$ for every $y \in Y$. If $1 \leq |R_1| \leq 2$, then there exists a vertex $x \in R_1$ such that $d_H^+(x) \leq 1$, a contradiction.

In the remaining case that $|R_1| = 3$, we arrive at a contradiction or $H[R_1]$ is a 3-cycle, $Z \to R_1$, and $C_5 \Rightarrow R_1$. Let, without loss of generality, $Y = \{v_2, v_3\}$. We discuss the case that $R_1 = \{a_2, b_2, c_2\}$, $Z = \{u_2\}$, and thus $R_2 = \{a_3, b_3, c_3, u_3\}$. The proofs of the other cases are analogously:

Subcase 6.2.1. Assume that $u_3 \to \{a_3, b_3, c_3\}$. This implies that $\{a_3, b_3, c_3\} \to u_2$ and $\{a_3, b_3, c_3\} \Rightarrow C_5$, and we have found the two complementary cycles $a_1b_1c_2v_2c_3a_1$ and $c_1u_1v_1b_2v_3u_3a_3u_2a_2b_3c_1$.

Subcase 6.2.2. Assume that u_3 has exactly two out-neighbors in R_2. We only discuss the case that $u_3 \to \{a_3, b_3\}$ and $c_3 \to u_3$ completely, because the other cases are similar. This leads to $u_3 \Rightarrow C_5$.

Subcase 6.2.2.1. Assume that $a_3 \to b_3$. This implies $b_3 \to \{c_3, u_2\}$ and $b_3 \Rightarrow C_5$, and there is the 5-cycle $C_5^* = a_1b_1c_2v_2u_3a_1$.

If $a_3 \to c_3$, then we observe that $c_3 \to u_2$ and $c_3 \Rightarrow C_5$, and we arrive at the complementary cycles C_5^* and $c_1u_1v_1b_2v_3a_3c_3u_2a_2b_3c_1$.

If $c_3 \to a_3$, then we obtain $a_3 \to u_2$ and $a_3 \Rightarrow C_5$, and we arrive at the complementary cycles C_5^* and $c_1u_1v_1b_2v_3c_3a_3u_2a_2b_3c_1$.

Subcase 6.2.2.2. Assume that $b_3 \to a_3$. This implies $a_3 \to \{c_3, u_2\}$ and $a_3 \Rightarrow C_5$, and there is the 5-cycle $C_5^* = a_1b_1c_2v_2u_3a_1$.

If $b_3 \to c_3$, then we observe that $c_3 \to u_2$ and $c_3 \Rightarrow C_5$, and we arrive at the complementary cycles C_5^* and $c_1u_1v_1a_2v_3b_3c_3u_2b_2a_3c_1$.

If $c_3 \to b_3$, then we obtain $b_3 \to u_2$ and $b_3 \Rightarrow C_5$, and we arrive at the complementary cycles C_5^* and $c_1u_1v_1a_2v_3c_3b_3u_2b_2a_3c_1$.

Case 7. Assume that $c = 5$ and $r = 2$. This case was solved with the help of an algorithm programmed in GAP [16] (cf. the Appendix).

Case 8. Assume that $c = 4$ and $r = 6$. This implies that D is 9-regular and $\alpha(H) = 5$. Since $i_l(H) \leq 4$, Theorem 2.9 yields $\kappa(H) \geq 3$. If H has a cycle factor, then Theorem 2.10 shows that H is Hamiltonian, and we are done.

Assume next that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 3$ and $\alpha(H) = 5$, we see that $3 \leq |Z| < |Y| \leq 5$. Let, without loss of generality, $|R_1| \leq |R_2|$. Since D is 9-regular, we see that $d_H^+(x), d_H^-(x) \geq 5$ for every $x \in V(H)$ and $d_H^+(x), d_H^-(x) \geq 6$ for $x \in V_1'$. Let $V_1 = \{u_1, u_2, \ldots, u_6\}$, $V_2 = \{x_1, x_2, \ldots, x_6\}$, $V_3 = \{y_1, y_2, \ldots, y_6\}$, $V_4 = \{w_1, w_2, \ldots, w_6\}$ and, without loss of generality, $V(C_5) = \{u_5, u_6, x_6, y_6, w_6\}$.

Case 8.1. Assume that $|Z| = 3$ and $|Y| = 5$. In this case, Theorem 2.15 with $k = 1$ and $t = 0$ leads to the contradiction $i_g(H) \geq 5$.

Case 8.2. Assume that $|Z| = 3$ and $|Y| = 4$. If $R_1 = \emptyset$, then $Y \Rightarrow R_2$ and $|Z| = 3$ yields the contradiction $d_H^-(y) \leq 3$ for every $y \in Y$. There remain the cases $1 \leq |R_1| \leq 6$. If there exists a vertex $u \in R_1$ such that $d_{D[R_1]}^-(u) \leq 1$, then $|Z| = 3$ implies the contradiction $d_H^-(u) \leq 4$. Hence we assume in the following that $d_{D[R_1]}^-(x) \geq 2$ for every $x \in R_1$. This immediately leads to $|R_1| = 6$. If $D[R_1]$ is bipartite, then we arrive at the contradiction $12 \leq |E(D[R_1])| \leq 9$. If $D[R_1]$ is exactly 3-partite, then it follows that $d_{D[R_1]}^-(x) = 2$ for every vertex $x \in R_1$. Since there are two vertices $u \in R_1$ and $v \in Z$ that belong to the same partite set, we obtain the contradiction $d_H^-(u) \leq 4$. If $D[R_1]$ is exactly 4-partite, then we arrive at the contradiction

$$31 \leq \sum_{x \in R_1} d_H^-(x) = \sum_{x \in R_1} d_{D[R_1]}^-(x) + d^+(Z, R_1) \leq 13 + 18 - 3 = 28.$$

Case 8.3. Assume that $|Z| = 4$ and $|Y| = 5$. Assume, without loss of generality, that $Y = V_4' = \{w_1, w_2, w_3, w_4, w_5\}$. If $R_1 = \emptyset$, then $Y \Rightarrow R_2$ and $|Z| = 4$ yields the contradiction $d_H^+(y) \leq 4$ for every $y \in Y$. There remain the cases $1 \leq |R_1| \leq 5$. If there exists a vertex $u \in R_1$ such that $d_{D[R_1]}^-(u) = 0$, then $|Z| = 4$ implies the contradiction $d_H^-(u) \leq 4$. Hence we assume in the following that $d_{D[R_1]}^-(x) \geq 1$ for every $x \in R_1$. This immediately leads to $|R_1| \geq 3$.

Case 8.3.1. Assume that $|R_1| = 3$. We deduce that $D[R_1]$ is a 3-cycle. Since $|Z| = 4$, we arrive at the contradiction

$$16 \leq \sum_{x \in R_1} d_H^-(x) = \sum_{x \in R_1} d_{D[R_1]}^-(x) + d^+(Z, R_1) \leq 3 + 12 - 4 = 11.$$

Case 8.3.2. Assume that $|R_1| = 4$. If $D[R_1]$ is exactly 3-partite, then we obtain the contradiction

$$21 \leq \sum_{x \in R_1} d_H^-(x) = \sum_{x \in R_1} d_{D[R_1]}^-(x) + d^+(Z, R_1) \leq 5 + 16 - 4 = 17.$$

In the case that $D[R_1]$ is bipartite, we arrive at a contradiction, or $R_1 \subset (V_2' \cup V_3')$, $Z = V_1' = \{u_1, u_2, u_3, u_4\}$, $D[R_1]$ is a 4-cycle, $Z \to R_1$, and $C_5 \Rightarrow R_1$. If, without loss of generality, $R_1 = \{x_1, x_2, y_1, y_2\}$, then $R_2 = \{x_3, x_4, x_5, y_3, y_4, y_5\}$. Because of $d_H^+(x) \geq 5$ for every vertex $x \in V(H)$, we deduce that $d_{D[R_2]}^+(x) \geq 1$ for every vertex $x \in R_2$. Now let, without loss of generality, $d_{D[R_2]}^+(x_3) = d_{D[R_2]}^+(y_3) = d_{D[R_2]}^+(y_4) = 1$. This implies that $\{x_3, y_3, y_4\} \to Z$ and $\{x_3, y_3, y_4\} \Rightarrow C_5$.

Assume that $C_5 = x_6u_5y_6u_6w_6x_6$. Since $Y \to R_2$, the 9-regularity of D shows that every vertex of Y has an in-neighbor in Z as well as in $V(C_5)$. Assume, without loss of generality, that $y_6 \to w_5$ and $u_1 \to w_1$. Since at least one of the vertices in $\{x_4, x_5, y_5\}$ has at least three out-neighbors in Z and the remaining two vertices at least two out-neighbors in Z, we have, without loss of generality, the two possibilities $y_5 \to u_2$, $x_5 \to u_3$, and $x_4 \to u_4$ or $y_5 \to u_1$, $x_5 \to u_3$, and $x_4 \to u_4$. Now there are the two complementary cycles $C_5' = y_6w_5y_4x_6u_5y_6$ and

$$C_{19} = u_6w_6y_1w_4y_3u_1w_1y_5u_2x_1w_2x_5u_3x_2w_3x_4u_4y_2x_3u_6.$$

or $C_5' = y_6w_5y_4x_6u_5y_6$ and

$$C_{19} = u_6w_6y_1w_4y_5u_1w_1y_3u_2x_1w_2x_5u_3x_2w_3x_4u_4y_2x_3u_6.$$

Since we can change the vertices x_i and y_i in R_1 for $i = 1, 2$ as well as x_3 with y_3 and y_4 arbitrary when we search arcs between these vertices and vertices from Y or Z, we see that all other cases are analogous.

Case 8.3.3. Assume that $|R_1| = 5$. If $D[R_1]$ is exactly 3-partite, then we obtain the contradiction

$$26 \leq \sum_{x \in R_1} d_H^-(x) = \sum_{x \in R_1} d_{D[R_1]}^-(x) + d^+(Z, R_1) \leq 8 + 20 - 4 = 24.$$

In the case that $D[R_1]$ is bipartite, we arrive at a contradiction, or $R_1 \subset (V_2' \cup V_3')$ and $Z = V_1' = \{u_1, u_2, u_3, u_4\}$. Let, without loss of generality, $R_1 = \{x_1, x_2, x_3, y_1, y_2\}$ and $R_2 = \{x_4, x_5, y_3, y_4, y_5\}$. Because of $d_H^+(x), d_H^-(x) \geq 5$ for every vertex $x \in V(H)$, we deduce that there are exactly four vertices $x \in R_1$ with $d_{D[R_1]}^-(x) = 1$ and four vertices $y \in R_2$ with $d_{D[R_2]}^+(y) = 1$. Assume that $C_5 = x_6u_5y_6u_6w_6x_6$.

We only discuss the case $d_{D[R_1]}^-(x_1) = d_{D[R_1]}^-(x_2) = d_{D[R_1]}^-(y_1) = d_{D[R_1]}^-(y_2) = 1$ and $d_{D[R_2]}^+(x_4) = d_{D[R_2]}^+(x_5) = d_{D[R_2]}^+(y_3) = d_{D[R_2]}^+(y_4) = 1$ completely, because the other cases are similar.

If we assume, without loss of generality, that $y_1 \to x_1$, then we obtain $x_1 \to y_2$. This implies $y_2 \to \{x_2, x_3\}$ and thus $x_2 \to y_1 \to x_3$. The 9-regularity of D leads to $Z \to \{x_1, x_2, y_1, y_2\}$ and $C_5 \Rightarrow \{x_1, x_2, y_1, y_2\}$.

In addition, if we assume, without loss of generality, that $y_3 \to x_4$, then we obtain $x_5 \to y_3$. This implies $\{y_4, y_5\} \to x_5$ and thus $y_5 \to x_4 \to y_4$. The 9-regularity of D leads to $\{x_4, x_5, y_3, y_4\} \to Z$ and $\{x_4, x_5, y_3, y_4\} \Rightarrow C_5$.

In the case that $Z \to Y$, it follows that $Y \Rightarrow C_5$. This leads to $d_D^-(u_6) \geq 10$, a contradiction to the 9-regularity of D.

Otherwise there exists an arc from Y to Z, say $w_1 \to u_1$. This implies that there is an arc from C_5 to w_1, say $x_6 \to w_1$. The 9-regularity of D shows that x_3 has at least three in-neighbors in Z and that y_5 has at least three out-neighbors in Z. We assume, without loss of generality, that $u_4 \to x_3$ and $y_5 \to u_2$. Now we obtain the two complementary cycles $C_5' = x_6w_1u_1x_2y_4x_6$ and

$$C_{19} = u_5y_6u_6w_6x_1w_2y_3u_4x_3w_3x_5u_3y_1w_4y_5u_2y_2w_5x_4u_5.$$

Case 9. Assume that $c = 4$ and $r = 4$. This implies that D is 6-regular and $\alpha(H) = 3$. Since $i_l(H) \le 4$, Theorem 2.9 yields $\kappa(H) \ge 1$. Since D is 6-regular, we see that $d_H^+(x), d_H^-(x) \ge 2$ for every $x \in V(H)$ and $d_H^+(x), d_H^-(x) \ge 3$ for $x \in V_1'$.

Subcase 9.1. Assume that H has a cycle-factor. If H is Hamiltonian, then we are done. If not, then let $C_1', C_2', \dots, C_t'$ be a minimal cycle-factor with the properties described in Theorem 2.13. Because of $|V^*| \le 3$, it follows from Theorem 2.13 that there are at most two incident arcs from $H - V(C_1')$ to C_1'. Since $\kappa(H) \ge 1$, there exists at least one arc from $H - V(C_1')$ to C_1'.

If C_1' is a 3-cycle, then we arrive at the contradiction $d_H^-(x) \le 1$ for at least two vertices $x \in V(C_1')$. If C_1' is a 4-cycle, then we arrive at the contradiction $d_H^-(x) \le 1$ for at least one vertex $x \in V(C_1')$ or $d_H^-(y) \le 2$ for a vertex $y \in V_1'$. Let now C_1' be a 5-cycle $c_1c_2c_3c_4c_5c_1$.

Subcase 9.1.1. Assume that $H[V(C_1')]$ is 3-partite. The 6-regularity of D easily yields $\sum_{x \in V(C_1')} d_H^+(x) = 30$ and there are exactly two incident arcs from $H - V(C_1')$ to C_1'.

If these two arcs are incident with c_1, then $c_5 \in V^*$, $V(c_1) = V(c_3)$ and $V(c_2) = V(c_4)$. In addition, it follows that $c_1 \to c_4$ and $c_5 \to \{c_2, c_3\}$, and we arrive at the contradiction $d_H^-(c_5) = 1$.

Assume that the two arcs from $H - V(C_1')$ to C_1' are incident with c_1 and c_4. It follows that $V(c_5) = V(c_3) = V^*$, and the 6-regularity of D leads to $V(c_1) = V(c_4)$, $c_1 \to c_3$, $c_2 \to c_5$ and $c_4 \to c_2$. We deduce that $V_1' \cap V(C_1') = \emptyset$.

Subcase 9.1.1.1. Assume that $t = 3$. Let $C_2' = y_1y_2y_3y_1$ such that $y_1 \to \{c_1, c_4\}$, and let $C_3' = x_1x_2x_3x_1$. This leads to $y_2 \in V^*$.

Assume first that $y_1 \in V_1'$. It follows that $y_3 \in V(c_2)$. Furthermore assume, without loss of generality, that $x_1 \in V_1'$, $x_2 \in V(c_1)$ and $x_3 \in V(c_2)$. The 6-regularity of D implies that $x_1 \to \{y_2, y_3\}$. Now C_5 and $x_2x_3x_1y_2y_3y_1c_4c_5c_1c_2c_3x_2$ are complementary cycles of D.

Assume second that $y_1 \in V(c_2)$. It follows that $y_3 \in V_1'$. Furthermore assume, without loss of generality, that $x_1 \in V_1'$, $x_2 \in V(c_1)$ and $x_3 \in V(c_2)$. The 6-regularity of D implies that $x_1 \to \{y_1, y_2\}$. Again C_5 and $x_2x_3x_1y_2y_3y_1c_4c_5c_1c_2c_3x_2$ are complementary cycles of D.

Subcase 9.1.1.2. Assume that $t = 2$. Let $C_2' = y_1y_2y_3y_4y_5y_6y_1$ and $y_1 \to c_1$ as well as $y_1 \to c_4$. This implies that $y_2 \in V^*$, and it is straightforward to verify that $y_1 \in V_1'$.

Assume that $V(c_2) = V(y_3) = V(y_6)$, $V(y_4) = V(y_1)$ and thus $V(y_5) = V(c_1) = V(c_4)$. We conclude that $y_4 \to \{y_2, y_6\}$. If $y_5 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5y_5y_2y_3y_4y_6y_1$ are complementary cycles. If $y_2 \to y_5$, then C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_2y_5y_6y_1$ are complementary cycles.

Assume that $V(c_2) = V(y_3) = V(y_6)$ and $V(y_5) = V(y_1)$. This implies that $V(y_4) = V(c_1) = V(c_4)$. We conclude that $y_5 \to \{y_2, y_3\}$ and thus $y_3 \to y_1$. If $y_2 \to y_6$, then C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_5y_2y_6y_1$ are the desired complementary cycles. If $y_6 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5y_4y_5y_6y_2y_3y_1$ are complementary cycles.

Assume that $V(c_2) = V(y_3) = V(y_5)$, $V(y_4) = V(y_1)$ and thus $V(y_6) = V(c_1) = V(c_4)$. We conclude that $y_4 \to \{y_2, y_6\}$. If $y_5 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5y_5y_2y_3y_4y_6y_1$ are complementary cycles. If $y_2 \to y_5$, then C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_2y_5y_6y_1$ are complementary cycles.

Assume that $V(c_2) = V(y_4) = V(y_6)$, $V(y_3) = V(y_1)$ and thus $V(y_5) = V(c_1) = V(c_4)$. We conclude that $y_3 \to \{y_5, y_6\}$ and so $y_6 \to y_2$. In the case that $y_5 \to y_2$, there are the complementary cycles C_5 and $y_1c_1c_2c_3c_4c_5y_4y_5y_2y_3y_6y_1$. If $y_5 \to y_1$, then C_5 and $y_1c_1c_2c_3c_4c_5y_6y_2y_3y_4y_5y_1$ are complementary cycles. Otherwise we obtain the contradiction $d^-(y_5) \ge 7$.

Assume that $V(c_2) = V(y_4) = V(y_6)$, $V(y_5) = V(y_1)$ and thus $V(y_3) = V(c_1) = V(c_4)$. We conclude that $y_5 \to \{y_2, y_3\}$. If $y_2 \to y_6$, then C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_5y_2y_6y_1$ are complementary cycles. If $y_6 \to y_2$, then it follows that $y_2 \to y_4$ and thus $y_4 \to y_1$. But now C_5 and $y_1c_1c_2c_3c_4c_5y_5y_6y_2y_3y_4y_1$ are complementary cycles.

Subcase 9.1.2. Assume that $H[V(C_1')]$ is 4-partite. The 6-regularity of D easily yields $\sum_{x \in V(C_1')} d_H^+(x) = 30$ and there are exactly two incident arcs from $H - V(C_1')$ to C_1'.

Subcase 9.1.2.1. Assume that these two arcs are incident with c_1. This implies that $c_5 \in V^*$, and it is a simple matter to verify that $V(c_2) = V(c_4)$. In addition, it follows that $c_1 \to c_4$ and $c_5 \to c_2$ and thus $c_3 \to c_5$ and so $c_1 \to c_3$. This leads to $c_1 \in V_1'$.

In the case $t = 2$ assume that $C_2' = y_1y_2y_3y_4y_5y_6y_1$. If $y_1 \to c_1$ and so $y_2 \in V^*$, then C_5 and $y_1c_1c_3c_4c_5c_2y_2y_3y_4y_5y_6y_1$ are complementary cycles.

In the other case $t = 3$, let $C_2' = x_1x_2x_3x_1$ and $C_3' = y_1y_2y_3y_1$ such that $\{x_1, y_1\} \to c_1$. This implies that $x_2, y_2 \in V^*$. We assume, without loss of generality, that $x_3 \in V_1'$. If $x_3 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5x_1x_2x_3y_2y_3y_1$ are complementary cycles. If otherwise $y_2 \to x_3$, then the 6-regularity of D yields $x_3 \to \{y_1, y_3\}$. Since $x_2 \to y_3$ leads to the 6-cycle $y_1y_2x_3x_1x_2y_3y_1$ and thus to $t = 2$, it remains the case that $y_3 \to x_2$. But now C_5 and $x_1c_1c_2c_3c_4c_5y_1y_2y_3x_2x_3x_1$ are complementary cycles.

Subcase 9.1.2.2. Assume that the two arcs from $H - V(C_1')$ to C_1' are incident with c_1 and c_4. It follows that $V(c_5) = V(c_3) = V^*$, and the 6-regularity of D leads to $c_1 \to c_3$ and $c_2 \to c_5$ and thus $c_4 \to c_2$ and so $c_1 \in V_1'$ or $c_4 \in V_1'$. Now it is easy to show that $t = 2$. Let $C_2' = y_1y_2y_3y_4y_5y_6y_1$ and $y_1 \to c_1$ as well as $y_1 \to c_4$. This implies that $y_2 \in V^*$.

Subcase 9.1.2.2.1. Assume that $y_3 \in V_1'$. It follows that $V(y_4) = V(y_6)$ and $V(y_1) = V(y_5)$. If $y_1 \to y_3$, then we deduce that $y_3 \to \{y_5, y_6\}$ and thus $\{y_5, y_6\} \to y_2$. This leads to $y_2 \to y_4$ and so $y_4 \to y_1$. Now C_5 and $y_1c_1c_2c_3c_4c_5y_5y_6y_2y_3y_4y_1$ are complementary cycles. Otherwise we have $y_3 \to y_1$. If $y_6 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5y_4y_5y_6y_2y_3y_1$ are complementary cycles. In the remaining case that $y_2 \to y_6$, we deduce that $y_3 \to y_5$ and thus $y_5 \to y_2$. Now C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_5y_2y_6y_1$ are complementary cycles.

Subcase 9.1.2.2.2. Assume that $y_4 \in V_1'$. It follows that $V(y_3) = V(y_6)$ and $V(y_1) = V(y_5)$. If $y_1 \to y_4$, then we deduce that $y_4 \to \{y_2, y_6\}$ and thus $y_6 \to y_2$. This leads to $y_2 \to y_5$ and so $y_5 \to y_3$. This implies that $y_3 \to y_1$ and hence

C_5 and $y_1c_1c_2c_3c_4c_5y_4y_5y_6y_2y_3y_1$ are complementary cycles. Otherwise we have $y_4 \to y_1$. If $y_6 \to y_2$, then there are the complementary cycles C_5 and $y_1c_1c_2c_3c_4c_5y_5y_6y_2y_3y_4y_1$. In the remaining case that $y_2 \to y_6$, we deduce that $y_6 \to y_4$ and thus $y_4 \to y_2$. If $y_5 \to y_3$, then C_5 and $y_1c_1c_2c_3c_4c_5y_5y_3y_4y_2y_6y_1$ are complementary cycles. If $y_3 \to y_5$, then $y_5 \to y_2$, and there are the complementary cycles C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_5y_2y_6y_1$.

Subcase 9.1.2.2.3. Assume that $y_5 \in V_1'$. It follows that $V(y_3) = V(y_6)$ and $V(y_1) = V(y_4)$ or $V(y_1) = V(y_3)$ and $V(y_4) = V(y_6)$. We only discuss the first case, the second one is similar. If $y_5 \to y_2$ and $y_4 \to y_6$, then C_5 and $y_1c_1c_2c_3c_4c_5y_5y_2y_3y_4y_6y_1$ are complementary cycles. If $y_5 \to y_2$ and $y_6 \to y_4$, then $y_4 \to y_2$ and thus $y_2 \to y_6$. Now C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_5y_2y_6y_1$ are complementary cycles. If $y_2 \to y_5$, then $y_5 \to \{y_1, y_3\}$ and thus $y_3 \to y_1$. If $y_6 \to y_2$, then C_5 and $y_1c_1c_2c_3c_4c_5y_4y_5y_6y_2y_3y_1$ are complementary cycles. If $y_2 \to y_6$, then $y_6 \to y_4$ and thus $y_4 \to y_2$. Now C_5 and $y_1c_1c_2c_3c_4c_5y_3y_4y_2y_5y_6y_1$ are complementary cycles.

Subcase 9.1.2.2.4. Assume that $y_6 \in V_1'$. This case is similar to Subcase 9.1.2.2.1 and is therefore omitted.

Using the converse D^{-1} of D, we obtain the desired results by the cases discussed above when C_1' is a 6-, 7-, or 8-cycle.

Subcase 9.2. Assume next that H has no cycle-factor. Then, with respect to Lemma 2.12, the vertex set $V(H)$ can be partitioned into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$, and Y is an independent set. Since $\kappa(H) \geq 1$ and $\alpha(H) = 3$, we see that $1 \leq |Z| < |Y| \leq 3$. Let, without loss of generality, $|R_1| \leq |R_2|$. Let $V_1 = \{a_1, a_2, a_3, a_4\}$, $V_2 = \{b_1, b_2, b_3, b_4\}$, $V_3 = \{u_1, u_2, u_3, u_4\}$, $V_4 = \{v_1, v_2, v_3, v_4\}$ and $C_5 = x_1x_2x_3x_4x_5x_1$ such that $V(C_5) = \{a_3, a_4, b_4, u_4, v_4\}$ and, without loss of generality, $x_1 = b_4$.

Subcase 9.2.1. Assume that $|Z| = 1$ and $|Y| = 3$. If $R_1 = \emptyset$, then we arrive at the contradiction $d^+(y) \geq 7$ for $y \in Y$. If $1 \leq |R_1| \leq 3$, then we obtain the contradiction $d_H^-(x) \leq 1$ for at least one vertex $x \in R_1$.

Subcase 9.2.2. Assume that $|Z| = 1$ and $|Y| = 2$. If $R_1 = \emptyset$, then we arrive at the contradiction $d^+(y) \geq 7$ for $y \in Y$. If $1 \leq |R_1| \leq 2$, then we obtain the contradiction $d_H^-(x) \leq 1$ for at least one vertex $x \in R_1$.

Subcase 9.2.2.1. Assume that $|R_1| = 3$. In this case we arrive at a contradiction, unless $\{a_1\} = Z \subset V_1'$ and $H[R_1]$ is a 3-cycle C_3, say $C_3 = b_1u_1v_1b_1$. Let, without loss of generality, $Y = \{v_2, v_3\}$. The 6-regularity of D shows that $C_5 \Rightarrow R_1$, that there are at least three arcs from R_2 to a_1, and that $D[R_2]$ contains a cycle.

Assume first that $D[R_2]$ contains a 3-cycle, say $a_2b_2u_2a_2$. We assume, without loss of generality, that $b_3 \to a_1$. If $u_3 \to b_3$, then there exists the 5-cycle $u_3b_3a_1u_1v_2u_3$. Since there is at least one of the three arcs $a_2 \to x_2$, $b_2 \to x_2$, or $u_2 \to x_2$, say $u_2 \to x_2$, we obtain the complementary cycle $x_2x_3x_4x_5x_1v_1b_1v_3a_2b_2u_2x_2$. If $b_3 \to u_3$ and $u_3 \to a_1$, then we have the same situation as before. In the remaining case that $b_3 \to u_3$ and $a_1 \to u_3$, it follows that $u_3 \to a_2$, and this yields the contradiction $d^-(a_2) \geq 7$.

Assume next that $D[R_2]$ contains a 4-cycle C_4 but no 3-cycle. This is only possible when, without loss of generality, $C_4 = b_2u_2b_3u_3b_2$ and $a_2 \to \{b_2, b_3, u_2, u_3\}$. This implies that $C_4 \Rightarrow C_5$. If we assume, without loss of generality, that $b_2 \to a_1$, then $a_2b_2a_1u_1v_2a_2$ is a 5-cycle. If $x_2 \neq u_4$, then $x_2x_3x_4x_5x_1v_1b_1v_3u_2b_3u_3x_2$ is a complementary cycle. If $x_2 = u_4$, then $x_3x_4x_5x_1x_2v_1b_1v_3u_2b_3u_3x_3$ is a complementary cycle.

Subcase 9.2.2.2. Assume that $|R_1| = |R_2| = 4$. Let, without loss of generality, $Y = \{v_2, v_3\}$. It is straightforward to verify that $Z \subset V_1$ or $Z \subset V_4$.

Assume that $Z = \{a_1\}$. In this case we arrive at a contradiction without the case that, without loss of generality, $R_1 = \{a_2, b_1, u_1, v_1\}$ such that $\{b_1, u_1, v_1\} \to a_2$ and $b_1 \to v_1 \to u_1 \to b_1$ and $D[R_2]$ consists of the 4-cycle $C_4 = b_2u_3b_3u_2b_2$. It follows that $R_2 \Rightarrow C_5 \Rightarrow R_1$ and $R_2 \to a_1 \Rightarrow R_1$. Now we obtain the complementary cycles $b_2u_3a_1u_1v_2b_2$ and $x_1x_2x_3x_4x_5b_1v_1a_2v_3b_3u_2x_1$.

Assume that $Z = \{v_1\}$. In this case we arrive at a contradiction, unless $D[R_1]$ consists, without loss of generality, of the cycle $a_1b_1u_1b_2a_1$ such that $u_1 \to a_1$ and $D[R_2]$ consists of the cycle $a_2u_2b_3u_3a_2$ such that $a_2 \to b_3$. It follows that $R_2 \Rightarrow C_5 \Rightarrow R_1$ and $R_2 \to a_1 \to R_1$. Now we obtain the complementary cycles $u_2v_1b_1u_1v_2u_2$ and $x_1x_2x_3x_4x_5b_2a_1v_3b_3u_3a_2x_1$.

Subcase 9.2.3. Assume that $|Z| = 2$ and $|Y| = 3$. Let, without loss of generality, $Y = \{v_1, v_2, v_3\}$.

Subcase 9.2.3.1. Assume that $R_1 = \emptyset$. This implies that $Z \to Y \to R_2$ and $C_5 \Rightarrow Y$. This case was solved with the help of an algorithm programmed in GAP [16] (cf. the Appendix).

Subcase 9.2.3.2. Assume that $|R_1| = 1$. This implies that $Z \to R_1$ and, without loss of generality, $R_1 = \{b_1\}$. This case was solved with the help of an algorithm programmed in GAP [16] (cf. the Appendix).

Subcase 9.2.3.3. Assume that $|R_1| = 2$. We have to discuss the following three cases: $R_1 = \{b_1, b_2\}$ and $Z \to R_1$, $R_1 = \{a_1, b_1\}$ such that $b_1 \to a_1$ and $Z \to R_1$, $R_1 = \{b_1, u_1\}$ such that, without loss of generality, $b_1 \to u_1$ and $Z \to b_1$.

Subcase 9.2.3.3.1. Assume that $R_1 = \{b_1, b_2\}$ and $Z = \{a_1, a_2\}$. It follows that $R_2 = \{u_1, u_2, u_3, b_3\}$ and $C_5 \Rightarrow R_1$.

If $\{u_1, u_2, u_3\} \to b_3$, then $b_3 \to Z$ and there is an arc from u_1 to Z, say $u_1 \to a_2$. Assume that there is an arc from a_2 to Y, say $a_2 \to v_3$. Since u_2 has at least three out-neighbors in C_5, and b_2 four in-neighbors in C_5, there exist two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \to b_2$ and $u_2 \to x_{i+1}$. Hence there are the complementary cycles $u_3b_3a_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_ib_2v_2u_1a_2v_3u_2x_{i+1}$. Otherwise we have $Y \to a_2$. This implies that v_3 has at least three in-neighbors in C_5. Since u_2 has at least three out-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \to v_3$ and $u_2 \to x_{i+1}$. Hence there are the complementary cycles $u_3b_3a_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3u_1a_2b_2v_2u_2x_{i+1}$.

If $\{u_1, u_2\} \to b_3$ and $b_3 \to u_3$, then $u_3 \to Z$ and there is an arc from b_3 to Z, say $b_3 \to a_2$. Assume that there is an arc from a_1 to Y, say $a_1 \to v_3$. Since u_2 has at least three out-neighbors in C_5, and b_2 four in-neighbors in C_5, there exist two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow b_2$ and $u_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_1b_3a_2b_1v_1u_1$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_ib_2v_2u_3a_1v_3u_2x_{i+1}$. Otherwise we have $Y \rightarrow a_1$. This implies that v_3 has at least three in-neighbors in C_5. Since u_2 has at least three out-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_1b_3a_2b_1v_1u_1$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3u_3a_1b_2v_2u_2x_{i+1}$.

The cases $b_3 \rightarrow \{u_2, u_3\}$ and $u_1 \rightarrow b_3$ as well as $b_3 \rightarrow \{u_1, u_2, u_3\}$ are similar and are therefore omitted.

Subcase 9.2.3.3.2. Assume that $R_1 = \{b_1, b_2\}$ and $Z = \{a_1, u_1\}$. It follows that $R_2 = \{a_2, b_3, u_2, u_3\}$ and $C_5 \Rightarrow R_1$.

Assume that $D[R_2]$ contains a cycle. This implies that $D[R_2]$ has a 3-cycle, say $a_2u_2b_3a_2$. It follows that $a_2 \rightarrow \{u_1, u_3\}$ and thus $u_3 \rightarrow b_3$ and so $\{u_2, u_3\} \rightarrow a_1$ and $\{u_2, u_3\} \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_3a_2u_1b_1v_1b_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3u_3a_1b_2v_2u_2x_{i+1}$.

If $D[R_2]$ has no cycle, then there remain the two possibilities $a_2 \rightarrow \{u_2, u_3, b_3\}$ and $\{u_2, u_3\} \rightarrow b_3$ or $\{a_2, u_2, u_3\} \rightarrow b_3$ and $u_2 \rightarrow a_2 \rightarrow u_3$.

In the first case it follows that $u_2 \rightarrow a_1$, $b_3 \rightarrow u_1$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $a_2u_2a_1b_1v_1a_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3u_1b_2v_2u_3x_{i+1}$.

In the second case it follows that $a_2 \rightarrow u_1$, $b_3 \rightarrow a_1$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_2a_2u_1b_1v_1u_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3a_1b_2v_2u_3x_{i+1}$.

Subcase 9.2.3.3.3. Assume that $R_1 = \{b_1, b_2\}$ and $Z = \{u_1, u_2\}$. It follows that $R_2 = \{a_1, a_2, b_3, u_3\}$ and $C_5 \Rightarrow R_1$.

Assume that $D[R_2]$ contains a cycle. This implies that $D[R_2]$ has a 3-cycle, say $a_1u_3b_3a_1$. It follows that $u_3 \rightarrow a_2$ and thus $a_2 \rightarrow b_3$ and so $\{a_1, a_2\} \rightarrow Z$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_3a_1u_1b_1v_1b_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2b_2v_2u_3x_{i+1}$.

If $D[R_2]$ has no cycle, then there remain the two possibilities $u_3 \rightarrow \{a_1, a_2, b_3\}$ and $\{a_1, a_2\} \rightarrow b_3$ or $\{a_1, a_2, u_3\} \rightarrow b_3$ and $a_1 \rightarrow u_3 \rightarrow a_2$.

In the first case it follows that $a_1 \rightarrow u_1$, $a_2 \rightarrow u_2$ and $b_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3a_1u_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2b_2v_2b_3x_{i+1}$.

In the second case it follows that $b_3 \rightarrow u_1$, $a_2 \rightarrow u_2$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $a_1b_3u_1b_1v_1a_1$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2b_2v_2u_3x_{i+1}$.

Subcase 9.2.3.3.4. Assume that $R_1 = \{a_1, b_1\}$. This implies that $b_1 \rightarrow a_1$ and $Z = \{u_1, u_2\} \rightarrow R_1$.

Assume that $D[R_2]$ contains a cycle. This implies that $D[R_2]$ has a 3-cycle, say $a_2u_3b_2a_2$. It follows that $u_3 \rightarrow b_3$ and $u_3 \Rightarrow C_5$. Assume that $a_2 \rightarrow b_3$ and, without loss of generality, that $a_2 \rightarrow u_2$. It follows that $b_3 \rightarrow u_1$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2a_2u_2b_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3u_1a_1v_2u_3x_{i+1}$. If otherwise $b_3 \rightarrow a_2$ and, without loss of generality, $b_3 \rightarrow u_1$, then it follows that $a_2 \rightarrow u_2$ and we arrive at the same complementary cycles.

If $D[R_2]$ has no cycle, then there remain the following four possibilities:

$u_3 \rightarrow \{a_2, b_2, b_3\}$ and $a_2 \rightarrow \{b_2, b_3\}$ or

$u_3 \rightarrow \{a_2, b_2, b_3\}$ and $b_3 \rightarrow a_2 \rightarrow b_2$ or

$u_3 \rightarrow \{b_2, b_3\}$ and $a_2 \rightarrow \{b_2, b_3, u_3\}$ or

$u_3 \rightarrow \{a_2, b_3\}$, $b_2 \rightarrow \{a_2, u_3\}$ and $a_2 \rightarrow b_3$.

In the first case it follows that $b_2 \rightarrow \{u_1, u_2\}$ and $b_3 \Rightarrow C_5$. Assume, without loss of generality, that $a_2 \rightarrow u_2$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3a_2u_2b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_2u_1a_1v_2b_3x_{i+1}$.

In the second case it follows that $b_2 \rightarrow \{u_1, u_2\}$ and $b_2 \Rightarrow C_5$. Assume, without loss of generality, that $a_2 \rightarrow u_2$. Next we distinguish two further cases. Assume that $u_1 \rightarrow b_3$. This implies that $b_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3b_2u_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2a_1v_2b_3x_{i+1}$. Now assume that $b_3 \rightarrow u_1$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3b_3u_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2a_1v_2b_2x_{i+1}$.

In the third case it follows that $\{b_1, b_2\} \rightarrow \{u_1, u_2\}$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $a_2b_2u_2b_1v_1a_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3u_1a_1v_2u_3x_{i+1}$.

In the fourth case it follows that $\{a_2, b_3\} \rightarrow \{u_1, u_2\}$ and $u_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2a_2u_1b_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3u_2a_1v_2u_3x_{i+1}$.

Subcase 9.2.3.3.5. Assume that $R_1 = \{b_1, u_1\}$ and $Z = \{a_1, a_2\}$. Assume, without loss of generality, that $b_1 \rightarrow u_1$. It follows that $\{a_1, a_2\} \rightarrow b_1$ and, without loss of generality, that $a_2 \rightarrow u_1$. This implies that $C_5 \Rightarrow b_1$.

Assume that $D[R_2]$ is a 4-cycle, say $b_2u_2b_3u_3b_2$. If $a_1 \rightarrow R_2$, then we deduce that $R_2 \rightarrow a_2$, $R_2 \Rightarrow C_5$, and there are at least two vertices in Y, say v_2, v_3, such that $\{v_2, v_3\} \rightarrow a_1$. Since v_3 has at least three in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3b_2a_2u_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_1b_1v_2u_2b_3x_{i+1}$. Otherwise we have, without loss of generality, $b_3 \rightarrow a_1$. If there is an arc from Y to a_2, say $v_3 \rightarrow a_2$, then v_3 has at least three in-neighbors in C_5 and b_2 has at least three out-neighbors in C_5. Hence there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_2 \rightarrow x_{i+1}$. This leads to the complementary cycles $u_2b_3a_1b_1v_1u_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_1v_2u_3b_2x_{i+1}$. Otherwise we have $a_2 \rightarrow Y$. This yields $b_2 \rightarrow a_2$ or $u_3 \rightarrow a_2$, say $b_2 \rightarrow a_2$. Since u_1 has at least three in-neighbors in C_5 and u_3 has at least three out-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow u_1$ and $u_3 \rightarrow x_{i+1}$. This leads to the complementary cycles $u_2b_3a_1b_1v_1u_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iu_1v_3b_2a_2v_2u_3x_{i+1}$.

If $D[R_2]$ has no cycle, then there remain, without loss of generality, the two possibilities $\{u_2, u_3\} \rightarrow \{b_2, b_3\}$ or $b_2 \rightarrow \{u_2, u_3\}$ and $u_2 \rightarrow b_3 \rightarrow u_3$.

In the first case it follows that $\{b_2, b_3\} \rightarrow \{a_1, a_2\}$ and $b_2 \Rightarrow C_5$. Assume that $u_2 \rightarrow a_2$ or $u_2 \rightarrow a_1$, say $u_2 \rightarrow a_2$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3b_3a_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3u_2a_2u_1v_2b_2x_{i+1}$. In the case that $Z \rightarrow u_2$, we observe that $u_2 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $u_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $u_3b_2a_1b_1v_1u_3$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3a_2u_1v_2u_2x_{i+1}$.

In the second case it follows that $u_3 \rightarrow \{a_1, a_2\}$ and, without loss of generality, that $u_2 \rightarrow a_2$. Assume that there exists an arc from a_1 to Y, say $a_1 \rightarrow v_2$. Since b_3 has at least three out-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow b_1$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2u_2a_2u_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_ib_1v_3u_3a_1v_2b_3x_{i+1}$. In the case that $Y \rightarrow a_1$, we observe that v_3 has at least three in-neighbors in C_5. Since b_3 has at least three out-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2u_2a_2u_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3u_3a_1b_1v_2b_3x_{i+1}$.

Subcase 9.2.3.3.6. Assume that $R_1 = \{b_1, u_1\}$ and, without loss of generality, that $Z = \{a_1, u_2\}$. It follows that $\{a_1, b_1\} \rightarrow u_1$ and $\{a_1, u_2\} \rightarrow b_1$.

Assume that $D[R_2]$ contains a cycle. This implies that $D[R_2]$ has a 3-cycle, say $a_2b_2u_3a_2$. It follows that $a_2 \rightarrow u_2$, $a_2 \rightarrow b_3$, and $a_2 \Rightarrow C_5$.

If $b_3 \rightarrow u_3$, then it follows that $u_3 \rightarrow a_1$. Assume that $b_3 \rightarrow u_2$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $a_2 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2u_3a_1u_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3b_3u_2b_1v_2a_2x_{i+1}$. Otherwise we have $u_2 \rightarrow b_3$ and thus $b_3 \Rightarrow C_5$. Since v_3 has at least two in-neighbors in C_5, there are two consecutive vertices x_i and x_{i+1} on C_5 such that $x_i \rightarrow v_3$ and $b_3 \rightarrow x_{i+1}$. Hence there are the complementary cycles $b_2u_3a_1u_1v_1b_2$ and $x_{i+1}x_{i+2}x_{i+3}x_{i+4}x_iv_3a_2u_2b_1v_2b_3x_{i+1}$. If $u_3 \rightarrow b_3$, then we obtain similarly the two desired complementary cycles.

If $D[R_2]$ has no cycle, then there are the four possibilities $b_2 \rightarrow a_2 \rightarrow u_3 \rightarrow b_3$ and $b_2 \rightarrow u_3$ and $a_2 \rightarrow b_3$ or $a_2 \rightarrow b_2 \rightarrow u_3 \rightarrow b_3$ and $a_2 \rightarrow b_3$ and $a_2 \rightarrow u_3$ or $a_2 \rightarrow \{u_3, b_2, b_3\}$ and $u_3 \rightarrow b_2$ and $u_3 \rightarrow b_3$ or $u_3 \rightarrow \{a_2, b_2, b_3\}$ and $a_2 \rightarrow b_2$ and $a_2 \rightarrow b_3$. All these cases are analogue to the cases above and therefore are omitted.

Subcase 9.2.3.4. Assume that $|R_1| = |R_2| = 3$. This case was solved with the help of an algorithm programmed in GAP [16] (cf. the Appendix). □

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Appendix

With the following algorithm programmed in GAP [16] we tested Case 5 of Theorem 3.1. Case 7, Case 9.2.3.1, Case 9.2.3.2 and Case 9.2.3.4 were tested the same way using minor modifications in Algorithm 2 (the initialization of the adjacency matrix A), Algorithm 3 (the values concerning the number of vertices and the regularity) and Algorithm 4 (the values concerning the chosen subsets). A similar program has also confirmed that the digraph $D^*_{4,2}$ is the only regular 4-partite tournament with two vertices in each partite set that does not contain two complementary cycles of length 4.

Algorithm 1 (*Tests Via Backtracking Whether the Vertices of the List* `subset` *Induce a Hamiltonian Subdigraph of the Digraph with Adjacency Matrix* A).

```
Hamiltontest:=function(subset,A)

local l,a,recursion;

recursion:=function(ll) #'ll' is a local list
 local ii,rest;
```

```
 rest:=Difference(subset,ll);
 if rest=[]

  then
   #test whether the last vertex of ‘ll’ dominates the first vertex of ‘ll’
   return A[ll[Length(ll)]][ll[1]]=1;

  else
   #test all possibilities to extend the list ‘ll’
   for ii in rest do
    if A[ll[Length(ll)]][ii]=1
     then if recursion(Concatenation(ll,[ii]))=true
           then return true;
          fi;
    fi;
   od;
 fi;

 return false;
end;

l:= [subset[1]];
a:= A{subset}{subset};

#shortcut: test whether there exists a vertex x in ‘subset’ such that
```

#$d^+(x) = 0$ or $d^-(x) = 0$

```
if ForAny(a,x->not (1 in x and -1 in x))
 then return false;
fi;

return recursion(l);
end;
```

Algorithm 2 (*Initialization of the Following Global Variables: Adjacency Matrix A, and Degree Vectors* $dplus$ *and* $dminus$).

```
Matrix_A_Init:=function();

A:=List([1..12],x->List([1..12],y->2));
#all entries of A are initialized with 2

for i in [1..12] do
 for j in [1..12] do
  if (i-j) mod 6 =0 #if i and j are in the same partition
   then A[i][j]:=0; #then there is no arc between them
  fi;
 od;
od;

#without loss of generality, there exists a cycle through
#the vertices 1,2,...,6
for i in [1..6] do
 A[i][i mod 6+1]:=1;
od;

dplus:=List([1..12],x->Number([1..12],y->A[x][y]=1));
 #the vector of all outdegrees

dminus:=List([1..12],x->Number([1..12],y->A[x][y]=-1));
 #the vector of all indegrees

end;
```

Algorithm 3 (*Changes* A, dplus, dminus; *Recursive Computation of the Adjacency Matrices of at Least all Non-Isomorphic 5-regular 6-partite Tournaments*).

```
AllMat:=function(n) #recursive computation, n is a list of vertices

local new,i,j;

new:=ShallowCopy(n[Length(n)]);

repeat
 if new[2]=12
  then
   new[1]:=new[1]+1;
   new[2]:=new[1]+1;
  else
   new[2]:=new[2]+1;
 fi;

 #if the recursive construction is complete, then test if there are
 #complementary cycles
 if new[1]>11
  then
   TestCC(A);
   return;
 fi;
until A[neu[1]][neu[2]]=2; #2 indicates that an arc has to be chosen

#update ‘dplus’ and ‘dminus’
if dplus[neu[1]]<5 and dminus[neu[2]]<5
 then
  dplus[neu[1]]:=dplus[neu[1]]+1;
  dminus[neu[2]]:=dminus[neu[2]]+1;
  A[neu[1]][neu[2]]:=1;A[neu[2]][neu[1]]:=-1;
  AllMat(Concatenation(n,[neu]));
  A[neu[1]][neu[2]]:=2;A[neu[2]][neu[1]]:=2;
  dplus[neu[1]]:=dplus[neu[1]]-1;
  dminus[neu[2]]:=dminus[neu[2]]-1;
fi;

if dplus[neu[2]]<5 and dminus[neu[1]]<5
 then
  dplus[neu[2]]:=dplus[neu[2]]+1;
  dminus[neu[1]]:=dminus[neu[1]]+1;
  A[neu[1]][neu[2]]:=-1;A[neu[2]][neu[1]]:=1;
  AllMat(Concatenation(n,[neu]));
  A[neu[1]][neu[2]]:=2;A[neu[2]][neu[1]]:=2;
  dplus[neu[2]]:=dplus[neu[2]]-1;
  dminus[neu[1]]:=dminus[neu[1]]-1;
fi;

end;
```

Algorithm 4 (*Tests for Complementary Cycles*).

```
TestCC:=function(mat)

local i,j;

#SubsetN(m,n) computes all subsets of {1,2,...,n} of size m
if ForAny(SubsetsN(5,12),x->Hamiltontest(x)
                             and Hamiltontest(Difference([1..12],x)))
 then return true;
```

```
 else return false;
fi;

end;
```

Algorithm 5 (*Concatenation of Algorithms* 1–4).

```
Matrix_A_Init();
AllMat([[1,1]]);
```

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