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Complete solution for the rainbow numbers of matchings[☆]

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ABSTRACT

For a given graph H and a positive n, the *rainbow number of* H, denoted by $rb(n, H)$, is the minimum integer k so that in any edge-coloring of K_n with k colors there is a copy of H whose edges have distinct colors. In 2004, Schiermeyer determined $rb(n, kK_2)$ for all $n \geq 3k+3$. The case for smaller values of n (namely, $n \in [2k,\ 3k+2]$) remained generally open. In this paper we extend Schiermeyer's result to all plausible n and hence determine the rainbow number of matchings.

1. Introduction

In this paper we consider undirected, finite and simple graphs only, and use standard notations in graph theory (see [3,8]). Let K_n be an edge-colored complete graph on n vertices. If a subgraph H of K_n contains no two edges of the same color, then H is called a *totally multicolored (TMC)* or *rainbow subgraph* of K_n and we say that K_n contains a TMC or rainbow H. Let $f(n, H)$ denote the maximum number of colors in an edge-coloring of K_n with no TMC H. We now define $rb(n, H)$ as the minimum number of colors such that any edge-coloring of K_n with at least $rb(n, H) = f(n, H) + 1$ colors contains a TMC or rainbow subgraph isomorphic to H. The number $rb(n, H)$ is called the *rainbow number of* H.

$f(n, H)$ is called the anti-Ramsey number of H, which was introduced by Erdős, Simonovits and Sós in the 1970s. They showed that it is closely related to the Turán number. The anti-Ramsey number has been studied in [1,2,5,9,11,6,7] and elsewhere. There are very few graphs whose anti-Ramsey numbers have been determined exactly. To the best of our knowledge, $f(n, H)$ is known exactly for large n only when H is a complete graph, a path, a star, a cycle or a broom whose maximum degree exceeds its diameter (a broom is obtained by identifying an end of a path with a vertex of a star) (see [10, 9,11,6,7]).

For a given graph H, let $ext(n, H)$ denote the maximum number of edges that a graph G of order n can have with no subgraph isomorphic to H. For $H = kK_2$, the value $ext(n, kK_2)$ has been determined by Erdős and Gallai [4], where $H = kK_2$ is a matching M of size k.

Theorem 1.1 (*Erdős and Gallai [4]*). *$ext(n, kK_2) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$ for all $n \geq 2k$ and $k \geq 1$, that is, for any given graph G of order n, if $|E(G)| > \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$, then G contains a kK_2, or a matching of size k.*

In 2004, Schiermeyer [10] used a counting technique and determined the rainbow numbers $rb(K_n, kK_2)$ for all $k \geq 2$ and $n \geq 3k+3$.

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Theorem 1.2 (*Schiermeyer [10]*). *$rb(n, kK_2) = ext(n, (k-1)K_2) + 2$ for all $k \geq 2$ and $n \geq 3k + 3$.*

It is easy to see that n must be at least $2k$. So, for $2k \leq n < 3k + 3$, the rainbow numbers remain not determined. In this paper, we will use a technique different from Schiermeyer [10] to determine the exact values of $rb(n, kK_2)$ for all $k \geq 2$ and $n \geq 2k$. Our technique is to use the Gallai–Edmonds structure theorem for matchings.

Theorem 1.3.

$$rb(n, kK_2) = \begin{cases} 4, & n = 4 \text{ and } k = 2; \\ ext(n, (k-1)K_2) + 3, & n = 2k \text{ and } k \geq 7; \\ ext(n, (k-1)K_2) + 2, & \text{otherwise.} \end{cases}$$

2. Preliminaries

Let M be a matching in a given graph G. Then the subgraph of G induced by M, denoted by $\langle M \rangle_G$ or $\langle M \rangle$, is the subgraph of G whose edge set is M and whose vertex set consists of the vertices incident with some edges in M. A vertex of G is said to be *saturated* by M if it is incident with an edge of M; otherwise, it is said to be *unsaturated*. If every vertex of a vertex subset U of G is saturated, then we say that U is saturated by M. A matching with maximum cardinality is called a *maximum matching*.

In a given graph G, $N_G(U)$ denotes the set of vertices of G adjacent to a vertex of U. If $R, T \in V(G)$, we denote $E_G(R, T)$ or $E(R, T)$ as the set of all edges having a vertex from both R and T. Let $G(m, n)$ denote a bipartite graph with bipartition $A \cup B$, and $|A| = m$ and $|B| = n$. Without loss of generality, in the following we always assume that $m \geq n$.

Let $ext(m, n, H)$ denote the maximum number of edges that a bipartite graph $G(m, n)$ can have with no subgraph isomorphic to H. The following lemma is due to Ore and can be found in [8].

Lemma 2.1. *Let $G(m, n)$ be a bipartite graph with bipartition $A \cup B$, and M a maximum matching in G. Then the size of M is $m - d$, where*

$$d = \max\{|S| - |N_G(S)| : S \subseteq A\}.$$

We now determine the value $ext(m, n, H)$ for $H = kK_2$.

Theorem 2.2.

$$ext(m, n, kK_2) = m(k-1) \quad \text{for all } n \geq k \geq 1,$$

that is, for any given bipartite graph $G(m, n)$, if $|E(G(m, n))| > m(k-1)$, then $kK_2 \subset G(m, n)$.

Proof. Suppose that G contains no kK_2. Let M be a maximum matching of G and the size of M be $k - i$, where $i \geq 1$. By Lemma 2.1, there exists a subset $S \subset A$ such that $|S| - |N_G(S)| = m - k + i$. Thus

$$|E(G)| \leq |S||N_G(S)| + n(m - |S|) = (|N_G(S)| + m - k + i)|N_G(S)| + n(k - i - |N_G(S)|).$$

Since $0 \leq |N_G(S)| \leq k - i \leq k - 1$, we obtain

$$|E(G)| \leq \max\{m(k-1), n(k-1)\} = m(k-1).$$

So, $ext(m, n, kK_2) = m(k-1)$. ■

Lemma 2.3.

$$ext(2k, (k-1)K_2) = \begin{cases} \binom{k-2}{2} + (k-2)(k+2), & 2 \leq k \leq 7; \\ \binom{2k-3}{2}, & k = 2 \text{ or } k \geq 7. \end{cases}$$

Proof. From Theorem 1.1, we have that $ext(2k, (k-1)K_2) = \max\left\{\binom{2k-3}{2}, \binom{k-2}{2} + (k-2)(k+2)\right\}$. Since $\binom{2k-3}{2} - \left(\binom{k-2}{2} + (k-2)(k+2)\right) = \frac{1}{2}(k-2)(k-7)$, we have that if $2 \leq k \leq 7$, $ext(2k, (k-1)K_2) = \binom{k-2}{2} + (k-2)(k+2)$, and if $k = 2$ or $k \geq 7$, $ext(2k, (k-1)K_2) = \binom{2k-3}{2}$. ■

Let G be a graph. Denote by $D(G)$ the set of all vertices in G which are not covered by at least one maximum matching of G. Let $A(G)$ be the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally let $C(G) = V(G) - A(G) - D(G)$. We denote the $D(G)$, $A(G)$ and $C(G)$ as the *canonical decomposition* of G.

A *near-perfect* matching in a graph G is a matching of G covering all but exactly one vertex of G. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for every $v \in V(G)$.

Theorem 2.4 (*The Gallai–Edmonds Structure Theorem [8]*). *For a graph G, let $D(G)$, $A(G)$ and $C(G)$ be defined as above. Then:*

(a) *The components of the subgraph induced by $D(G)$ are factor-critical.*
(b) *The subgraph induced by $C(G)$ has a perfect matching.*
(c) *The bipartite graph obtained from G by deleting the vertices of $C(G)$ and the edges spanned by $A(G)$ and by contracting each component of $D(G)$ to a single vertex has positive surplus (as viewed from $A(G)$).*
(d) *Any maximum matching M of G contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$.*
(e) *The size of a maximum matching M is $\frac{1}{2}(|V(G)| - c(D(G)) + |A(G)|)$, where $c(D(G))$ denotes the number of components of the graph spanned by $D(G)$.* ■

3. Main results

For $k = 1$, it is clear that $rb(n, K_2) = 1$. Now we determine the value of $rb(n, 2K_2)$ (for $k = 2$).

Theorem 3.1.

$$rb(4, 2K_2) = 4,$$

and

$$rb(n, 2K_2) = 2 = ext(n, K_2) + 2 \quad \text{for all } n \geq 5.$$

Proof. It is obvious that $rb(4, 2K_2) \leq 4$. Let $V(K_4) = \{a_1, a_2, a_3, a_4\}$. If K_4 is edge-colored with 3 colors such that $c(a_1a_2) = c(a_3a_4) = 1$, $c(a_1a_3) = c(a_2a_4) = 2$ and $c(a_1a_4) = c(a_2a_3) = 3$, then K_4 contains no TMC $2K_2$. So, $rb(4, 2K_2) = 4$.

For $n \geq 5$, let the edges of $G = K_n$ be colored with at least 2 colors. Suppose that K_n contains no TMC $2K_2$. Let $e_1 = a_1b_1$ be an edge with $c(e_1) = 1$, $T = \{a_1, b_1\}$ and $R = V(K_n) - T$. Then $c(e) = 1$ for all edges $e \in E(G[R])$. Moreover, $c(e) = 1$ for all edges $e \in E(T, R)$, since $|R| \geq 3$. But then K_n is monochromatic, a contradiction. So, $rb(n, 2K_2) = 2$ for all $n \geq 5$. ■

The next proposition provides a lower and an upper bound for $rb(n, kK_2)$.

Proposition 3.2. $ext(n, (k-1)K_2) + 2 \leq rb(n, kK_2) \leq ext(n, kK_2) + 1$.

Proof. The upper bound is obvious. For the lower bound, an extremal coloring of K_n can be obtained from an extremal graph S_n for $ext(n, (k-1)K_2)$ by coloring the edges of S_n differently and the edges of $\overline{S_n}$ by one extra color. It is obvious that the coloring does not contain a TMC kK_2. ■

We will show that the lower bound can be achieved for all $n \geq 2k + 1$ and $k \geq 3$, and thus obtain the exact value of $rb(n, kK_2)$ for all $n \geq 2k + 1$ and $k \geq 3$.

For $n = 2k$, we suppose that $H = K_{2k-3}$ is a subgraph of K_n and $V(K_n) - V(H) = \{a_1, a_2, a_3\}$. If K_n is edge-colored such that $c(a_1a_2) = 1$, $c(a_1a_3) = c(a_2a_3) = 2$, $c(e) = 1$ for all edges $e \in E(a_3, V(H))$, $c(e) = 2$ for all edges $e \in E(a_1, V(H)) \cup E(a_2, V(H))$ and the edges of $H = K_{2k-3}$ is colored differently by $\binom{2k-3}{2}$ extra colors. It is easy to check that the coloring does not contain a TMC kK_2 in K_n. So, $rb(2k, kK_2) \geq \binom{2k-3}{2} + 3$ for all $k \geq 3$. Hence, if $k \geq 7$, then $ext(2k, (k-1)K_2) = \binom{2k-3}{2}$ and $rb(2k, kK_2) \geq ext(2k, (k-1)K_2) + 3$. We will show that the lower bound can be achieved for all $n \geq 2k$ and $k \geq 7$.

Theorem 3.3. *For all $n \geq 2k$ and $k \geq 3$, we have*

$$rb(n, kK_2) = \begin{cases} ext(n, (k-1)K_2) + 3, & n = 2k \text{ and } k \geq 7; \\ ext(n, (k-1)K_2) + 2, & \text{otherwise.} \end{cases}$$

Proof. We shall prove the theorem by contradiction. If $n = 2k$ and $k \geq 7$, let the edges of K_n be colored with $ext(n, (k-1)K_2) + 3$ colors; otherwise, let the edges of K_n be colored with $ext(n, (k-1)K_2) + 2$ colors. Suppose that K_n contains no TMC kK_2. Now let $G \subset K_n$ be a TMC spanning subgraph which contains all colors in K_n, i.e., if $n = 2k$ and $k \geq 7$, $|E(G)| = ext(n, (k-1)K_2) + 3$; otherwise $|E(G)| = ext(n, (k-1)K_2) + 2$. Since $|E(G)| \geq ext(n, (k-1)K_2) + 2$, there is a TMC $(k-1)K_2$ in G.

We first need to prove the following two lemmas.

Lemma 3.4. *If two components of G consist of a K_{2k-3} and a K_3, respectively, and the other components are isolated vertices (see Fig. 1), then K_n contains a TMC kK_2.*

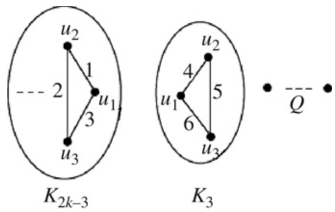

Fig. 1. The special graph SG_1.

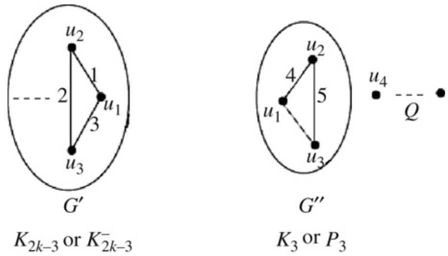

Fig. 2. The special graph SG_2. G' and G'' are a K_{2k-3} and a P_3, respectively, or G' and G'' are a K^-_{2k-3} and a K_3, respectively.

Proof. Denote SG_1 as the special graph G and Q as the set of isolated vertices of G. Without loss of generality, we suppose that $c(u_1u_2) = 1, c(u_2u_3) = 2, c(u_1u_3) = 3, c(v_1v_2) = 4, c(v_2v_3) = 5, c(v_1v_3) = 6$ (see Fig. 1).

The proof of the lemma is given by distinguishing the following two cases:

Case I. $k \geq 4$.

We suppose that G contains no TMC kK_2. We will show $c(u_1v_1) = 5$. If $c(u_1v_1) \neq 5$, then in $G_1 = K_{2k-3} - u_1$ the number of edges whose colors are not $c(u_1v_1)$ is at least $\binom{2k-4}{2} - 1$. Since $k \geq 4$, we have $\binom{2k-4}{2} - 1 > ext(2k-4, (k-2)K_2) = \binom{2k-5}{2}$. Thus we can obtain a TMC $H = (k-2)K_2$ which contains no color $c(u_1v_1)$ in G_1, and hence there is a TMC $kK_2 = H \cup \{u_1v_1, v_2v_3\}$ in K_n. So, $c(u_1v_1)$ must be 5. By the same token, $c(u_2v_2)$ and $c(u_3v_3)$ must be 6 and 4, respectively. Now we can obtain a TMC $H' = (k-3)K_2$ in $G_2 = K_{2k-3} - u_1 - u_2 - u_3$, and hence there is a TMC $kK_2 = H' \cup \{u_1v_1, u_2v_2, u_3v_3\}$ in K_n.

Case II. $k = 3$.

We suppose that K_n contains no TMC $3K_2$. Then $c(u_1v_1) \in \{2, 5\}, c(u_2v_2) \in \{3, 6\}, c(u_3v_3) \in \{1, 4\}$. Now we can obtain a TMC $3K_2 = u_1v_1 \cup u_2v_2 \cup u_3v_3$ in K_n. ■

Lemma 3.5. *If $n \geq 2k+1$ and two components of G are G' and G'', where G' and G'' are a K_{2k-3} and a P_3, respectively, or G' and G'' are a K^-_{2k-3} and a K_3, respectively, and the other components are isolated vertices (see Fig. 2), then K_n contains a TMC kK_2, where P_3 is a path with three vertices and K^-_{2k-3} is obtained from K_{2k-3} by deleting an edge.*

Proof. Denote SG_2 as the special graph G and Q as the set of isolated vertices of G. Without loss of generality, we suppose that $c(u_1u_2) = 1, c(u_2u_3) = 2, c(u_1u_3) = 3, c(v_1v_2) = 4, c(v_2v_3) = 5$ (see Fig. 2). The proof of the lemma is given by distinguishing the following two cases:

Case I. $k \geq 4$.

Since $n \geq 2k+1$, we suppose that $v_4 \in Q$. If $c(u_1v_4) = j$, without loss of generality, we suppose that $j \neq 4$. The number of edges of $G' - u_1$ whose color is not j is at least $\binom{2k-4}{2} - 2$ and $\binom{2k-4}{2} - 2 > ext(2k-4, (k-2)K_2) = \binom{2k-5}{2}$. Then there is a TMC $H = (k-2)K_2$ in $G' - u_1$ which contains no color j. We can obtain a TMC $kK_2 = H \cup u_1v_4 \cup v_1v_2$ in K_n.

Case II. $k = 3$.

Without loss of generality, we suppose that G' and G'' are a K_3 and a P_3, respectively. We suppose that K_n contains no TMC $3K_2$. Then, $c(u_1v_4) \in \{2, 5\} \cap \{2, 4\}$, i.e., $c(u_1v_4) = 2$, $c(u_3v_3) \in \{2, 4\} \cap \{1, 4\}$, i.e., $c(u_1v_4) = 4$, $c(u_2v_1) \in \{2, 5\} \cap \{3, 5\}$, i.e., $c(u_1v_4) = 5$. Now we obtain a TMC $3K_2 = u_1v_4 \cup u_3v_3 \cup u_2v_1$. See Fig. 3. ■

Now we turn back to the proof of Theorem 3.3. Let $D(G), A(G), C(G)$ be the canonical decomposition of G and $c(D(G)) = q$, $|A(G)| = s$, $|V(G)| = n$. Since the size of the maximum matchings of G is $k-1$, by Theorem 2.4 (e), $k - 1 = \frac{1}{2}(n - q + s)$, i.e., $q = n - 2k + 2 + s$. Let the components of $D(G)$ be $D_1, D_2, \ldots, D_q$. By Theorem 2.4 (a), the components of the subgraph induced by $D(G)$ are factor-critical, hence we suppose that $|V(D_i)| = 2l_i + 1$ for $1 \leq i \leq q$, without loss of generality, $l_1 \geq l_2 \geq \cdots \geq l_q \geq 0$. Let the components of $C(G)$ be $C_1, C_2, \ldots, C_{q'}$ with $|V(C_i)| = 2t_i$ for $1 \leq i \leq q'$.

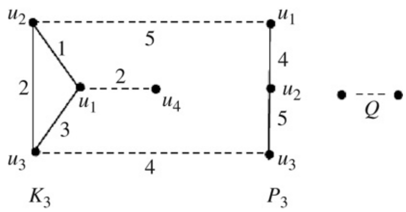

Fig. 3. We can obtain a TMC $3K_2 = u_1v_4 \cup u_3v_3 \cup u_2v_1$ in K_n.

Since $s+q = s+n-2k+2+s \le n$, then $0 \le s \le k-1$. Moreover,

$$\begin{aligned} n = s + \sum_{i=1}^{q}(2l_i+1) + |C(G)| &\ge s + (2l_1+1) + \sum_{i=2}^{q}(2l_i+1) \\ &\ge s + (2l_1+1) + (q-1) \\ &\ge s + (2l_1+1) + (n-2k+2+s-1), \end{aligned}$$

hence $2l_1+1 \le 2k-2s-1$. We distinguish four cases to finish the proof of Theorem 3.3.

Case 1. $s = k-1$.

In this case, since $s+q = (k-1)+n-2k+2+(k-1) = n$, then $C(G) = \emptyset$ and $l_1 = l_2 = \cdots = l_q = 0$. The components of the subgraph induced by $D(G)$ are isolated vertices. We distinguish two subcases to finish the proof of the case.

Subcase 1.1. There is at most one vertex u in $D(G)$ such that $d_G(u) < k-1$.

We suppose $v \in D(G)$ and $u \ne v$. Let $G(n-k-1, k-1)$ be the bipartite graph obtained from G by deleting the vertices u, v and the edges spanned by $A(G)$. It is obvious that $uv \in E(K_n)$ and $uv \notin E(G)$, without loss of generality, we suppose $c(uv) = 1$. Then the number of edges in $G(n-k-1, k-1)$ whose color is not 1 is at least $(n-k-1)(k-1)-1$. Since $n-k-1 \ge 2$, then $(n-k-1)(k-1)-1 > ext(n-k-1, k-1, (k-1)K_2) = (n-k-1)(k-2)$. By Theorem 2.2, there exists a TMC $H = (k-1)K_2$ in $G(n-k-1, k-1)$ which contains no color 1, thus we obtain a TMC $kK_2 = H \cup uv$ in K_n.

Subcase 1.2. There exist at least two vertices u, v in $D(G)$ such that $d_G(u) < k-1$ and $d_G(v) < k-1$.

We suppose that $c(uv) = 1$. Let $G'(n-k-1, k-1)$ be the bipartite graph obtained from G by deleting the vertices u, v and the edges spanned by $A(G)$ and the edge whose color is 1. Thus there is no TMC $(k-1)K_2$ in $G'(n-k-1, k-1)$. Hence, by Theorem 2.2,

$$\begin{aligned} |E(G)| &\le 1 + ext(n-k-1, k-1, (k-1)K_2) + 2(k-2) + \binom{k-1}{2} \\ &\le 1 + (k-2)(n-k-1) + 2(k-2) + \binom{k-1}{2} \\ &= \binom{k-2}{2} + (k-2)(n-k+2) + 1 \\ &< ext(n, (k-1)K_2) + 2, \end{aligned}$$

which contradicts $|E(G)| \ge ext(n, (k-1)K_2) + 2$.

Case 2. $0 \le s \le k-2$ and $2l_1+1 \le 2k-2s-3$.

In this case, if $2k-2s-3 = 1$, then $l_1 = l_2 = \cdots = l_q = 0$, $s = k-2$ and $|C(G)| = 2$, hence

$$\begin{aligned} |E(G)| &\le \binom{s}{2} + s(n-s) + \binom{2}{2} \\ &= \binom{k-2}{2} + (k-2)(n-k+2) + 1 \\ &< ext(n, (k-1)K_2) + 2, \end{aligned}$$

which contradicts $|E(G)| \ge ext(n, (k-1)K_2) + 2$.

If $2k-2s-3 \ge 3$, then $0 \le s \le k-3$ and

$$\begin{aligned} \sum_{i=2}^{q}(2l_i+1) + \sum_{i=1}^{q'}(2t_i) &= n - s - (2l_1+1) \\ &\ge n - s - (2k-2s-3) = (q-1) + 2. \end{aligned}$$

Thus, if $|C(G)| \geq 2$, then

$$\begin{aligned}
|E(G)| &\leq \binom{s}{2} + s(n-s) + \sum_{i=1}^{q}\binom{2l_i+1}{2} + \sum_{i=1}^{q'}\binom{2t_i}{2} \\
&\leq \binom{s}{2} + s(n-s) + \binom{2l_1+1+\sum_{i=2}^{q}2l_i}{2} + \sum_{i=1}^{q'}\binom{2t_i}{2} \\
&\leq \binom{s}{2} + s(n-s) + \binom{2l_1+1+\sum_{i=2}^{q}2l_i + \left(\sum_{i=1}^{q'}2t_i - 2\right)}{2} + \binom{2}{2} \\
&= \binom{s}{2} + s(n-s) + \binom{n-s-(q-1)-2}{2} + \binom{2}{2} \\
&= \binom{s}{2} + s(n-s) + \binom{2k-2s-3}{2} + \binom{2}{2} := f_1(s).
\end{aligned}$$

Hence,

$$f_1(0) = \binom{2k-3}{2} + 1 < ext(n, (k-1)K_2) + 2,$$

$$\begin{aligned}
f_1(k-3) &= \binom{k-2}{2} + (k-2)(n-k+2) - (n-k) + 2 \\
&< \binom{k-2}{2} + (k-2)(n-k+2) < ext(n, (k-1)K_2) + 2.
\end{aligned}$$

Since $0 \leq s \leq k-3$, $|E(G)| \leq \max\{f_1(0), f_1(k-3)\} < ext(n, (k-1)K_2)+2$, which contradicts $|E(G)| \geq ext(n, (k-1)K_2)+2$.

If $|C(G)| = 0$, then $2l_2 + 1 \geq 3$ and

$$\begin{aligned}
|E(G)| &\leq \binom{s}{2} + s(n-s) + \sum_{i=1}^{q}\binom{2l_i+1}{2} + \sum_{i=1}^{q'}\binom{2t_i}{2} \\
&\leq \binom{s}{2} + s(n-s) + \binom{2l_1+1+\sum_{i=3}^{q}2l_i + \sum_{i=1}^{q'}2t_i}{2} + \binom{2l_2+1}{2} \\
&\leq \binom{s}{2} + s(n-s) + \binom{2l_1+1+\sum_{i=3}^{q}2l_i + \sum_{i=1}^{q'}2t_i + (2l_2-2)}{2} + \binom{3}{2} \\
&= \binom{s}{2} + s(n-s) + \binom{n-s-(q-1)-2}{2} + \binom{3}{2} \\
&= \binom{s}{2} + s(n-s) + \binom{2k-2s-3}{2} + \binom{3}{2} := f_2(s).
\end{aligned}$$

Thus,

$$f_2(0) = \binom{2k-3}{2} + 3,$$

$$f_2(1) = \binom{2k-3}{2} + n - 4k + 11,$$

$$\begin{aligned}
f_2(k-3) &= \binom{k-2}{2} + (k-2)(n-k+2) - (n-k) + 4 \\
&\leq \binom{k-2}{2} + (k-2)(n-k+2) + 1 < ext(n, (k-1)K_2) + 2.
\end{aligned}$$

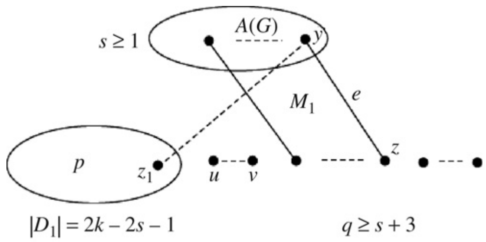

Fig. 4. If $yz_1 \in E_G(y, D_1)$, we can obtain a TMC $kK_2 = M_1' \cup M_2' \cup uv$ in K_n.

If $s = 0$ and $|E(G)| = \binom{2k-3}{2} + 3$, then $G \cong SG_1$. By Lemma 3.4, we can obtain a TMC kK_2 in K_n. If $s = 0$, $n \geq 2k + 1$ and $|E(G)| = \binom{2k-3}{2} + 2$, then $G \cong SG_2$. By Lemma 3.5, we can obtain a TMC kK_2 in K_n. So, if $n \geq 2k + 1$, then $|E(G)| \leq \binom{2k-3}{2} + 1 < ext(n, (k-1)K_2) + 2$, which contradicts $|E(G)| = ext(n, (k-1)K_2) + 2$. If $n = 2k$ and $k \geq 7$, then $|E(G)| \leq \binom{2k-3}{2} + 2 = ext(n, (k-1)K_2) + 2$, which contradicts $|E(G)| = ext(n, (k-1)K_2) + 3$. If $n = 2k$ and $3 \leq k \leq 6$, then $|E(G)| \leq \binom{2k-3}{2} + 2 \leq \binom{k-2}{2} + (k-2)(k+2) = ext(n, (k-1)K_2)$, which contradicts $|E(G)| = ext(n, (k-1)K_2) + 2$.

If $1 \leq s \leq k - 3$, then $k \geq 4$ and $|E(G)| \leq \max\{f_2(1), f_2(k-3)\}$. So, if $f_2(k-3) \geq f_2(1)$, then $|E(G)| \leq f_2(k-3) < ext(n, (k-1)K_2) + 2$, a contradiction. If $f_2(1) > f_2(k-3)$, then

$$\binom{2k-3}{2} + n - 4k + 11 > \binom{k-2}{2} + (k-2)(n-k+2) - (n-k) + 4.$$

Hence $2k \leq n < \frac{1}{2}(5k-7)$, $k > 7$ and

$$\begin{aligned}|E(G)| &\leq f_2(1) = \binom{2k-3}{2} + n - 4k + 11 \\ &< \binom{2k-3}{2} + \frac{1}{2}(15 - 3k) \\ &< ext(n, (k-1)K_2) + 2,\end{aligned}$$

a contradiction.

Case 3. $0 \leq s \leq k - 2$, $2l_1 + 1 = 2k - 2s - 1$ and $n \geq 2k + 1$.

In this case, $s + (2l_1 + 1) + (q - 1) = n$, hence $C(G) = \emptyset$, $l_2 = l_3 = \cdots = l_q = 0$ and each D_i for $2 \leq i \leq q$ is an isolated vertex.

Let $G(q, s)$ be the bipartite graph obtained from G by deleting the edges spanned by $A(G)$ and by contracting the component D_1 to a single vertex p. Thus by Theorem 2.4(c) and (d), we can obtain a maximum matching M of size $k-1$ such that M contains a maximum matching M_1 of $G(q, s)$ which does not match vertex p and a near-perfect matching M_2 of D_1. Since $q = n - 2k + 2 + s \geq s + 3$, there exist two vertices $u, v \in D(G) - D_1$ and $u, v \notin \langle M \rangle$. It is obvious that $uv \in E(K_n)$ and $uv \notin E(G)$. We suppose that $c(uv) = 1$, hence there exists an edge $e = yz \in M$ with $c(e) = 1$. Now we distinguish two subcases to complete the proof of the case.

Subcase 3.1. $e \in M_1$.

In this subcase, $s \geq 1$ and $yz \in E_G(A(G), D(G))$, without loss of generality, we suppose that $y \in A(G)$. If there exists an edge $yz_1 \in E_G(y, D_1)$ with $z_1 \in D_1$, then we can obtain another maximum matching M_1' of $G(q, s)$ with $M_1' = M_1 \cup yz_1 - yz$ and a near-perfect matching M_2' of D_1 which does not match z_1. Thus we obtain a TMC $kK_2 = M_1' \cup M_2' \cup uv$ in K_n. See Fig. 4.

Thus we suppose that $E_G(y, D_1) = \emptyset$. There is no matching of size s in $G'(q-3, s) = G(q, s) - p - u - v - e$. By Theorem 2.2, $|E_G(G')| \leq (s-1)(q-3) = (s-1)(n-2k+s-1)$. Now

$$\begin{aligned}|E(G)| &\leq \binom{s}{2} + \binom{2k-2s-1}{2} + 1 + |E_G(G')| + |E_G(D_1, A(G))| + |E_G(\{u, v\}, A(G))| \\ &\leq \binom{s}{2} + \binom{2k-2s-1}{2} + 1 + (s-1)(n-2k+s-1) + (2k-2s-1)(s-1) + 2s := f_3(s).\end{aligned}$$

Hence,

$$\begin{aligned}f_3(1) &= \binom{2k-3}{2} + 3, \\ f_3(2) &= \binom{2k-3}{2} + n - 4k + 11,\end{aligned}$$

$$\begin{aligned} f_3(k-2) &= \binom{k-2}{2} + (k-2)(n-k+2) - (n-k) + 4 \\ &\le \binom{k-2}{2} + (k-2)(n-k+2) < ext(n, (k-1)K_2) + 2. \end{aligned}$$

If $s = 1$, then $|E(G)| \le \binom{2k-3}{2} + 3$. If $|E(G)| = \binom{2k-3}{2} + 3$, then $(G - e + uv) \cong SG_1$. By the proof of Lemma 3.4, we can obtain a TMC kK_2 in K_n. If $|E(G)| = \binom{2k-3}{2} + 2$, then $(G - e + uv) \cong SG_2$. By the proof of Lemma 3.5, we can obtain a TMC kK_2 in K_n. If $|E(G)| \le \binom{2k-3}{2} + 1 \le ext(n, (k-1)K_2) + 1$, this contradicts $|E(G)| = ext(n, (k-1)K_2) + 2$.

If $2 \le s \le k-2$, then $k \ge 4$ and $|E(G)| \le \max\{f_3(2), f_3(k-2)\}$. So, if $f_3(k-2) \ge f_3(2)$, then $|E(G)| \le f_3(k-2) < ext(n, (k-1)K_2) + 2$, a contradiction. If $f_3(1) > f_3(k-3)$, then

$$\binom{2k-3}{2} + n - 4k + 11 > \binom{k-2}{2} + (k-2)(n-k+2) - (n-k) + 4.$$

Hence, $2k \le n < \frac{1}{2}(5k-7)$, $k > 7$ and

$$\begin{aligned} |E(G)| &\le f_3(2) = \binom{2k-3}{2} + n - 4k + 11 \\ &< \binom{2k-3}{2} + \frac{1}{2}(15 - 3k) \\ &< ext(n, (k-1)K_2) + 2, \end{aligned}$$

a contradiction.

Subcase 3.2. $e \in M_2$.

In this subcase, $y \in D_1$ and $z \in D_1$. By Theorem 2.4 (a), D_1 is factor-critical, there exists a near-perfect matching M_2' which does not match y, So M_2' does not contain $e = yz$. Now we obtain a TMC $kK_2 = M_2' \cup M_1 \cup uv$ in K_n.

Case 4. $0 \le s \le k-2$, $2l_1 + 1 = 2k - 2s - 1$ and $n = 2k$.

In this case, $q = s+2$ and $s + (2l_1 + 1) + (q-1) = 2k$, hence $C(G) = \emptyset$, $l_2 = l_3 = \cdots = l_q = 0$ and each D_i for $2 \le i \le q$ is an isolated vertex. Now we distinguish two subcases to complete the proof of the case.

Subcase 4.1. $1 \le s \le k-2$.

If $E_G(D_1, A(G)) = \emptyset$, then

$$|E(G)| \le \binom{2k-2s-1}{2} + \binom{s}{2} + s(s+1) := f_4(s).$$

Thus,

$$f_4(1) = \binom{2k-3}{2} + 2,$$

$$f_4(k-2) = \binom{k-2}{2} + (k-2)(k+2) + 3 - 3(k-2).$$

Since $k \ge 3$, then $f_4(1) \ge f_4(k-2)$ and $|E(G)| \le \max\{f_4(1), f_4(k-2)\} = f_4(1) = \binom{2k-3}{2} + 2$. If $k \ge 7$, this contradicts $|E(G)| = ext(2k, (k-1)K_2) + 3 = \binom{2k-3}{2} + 3$. If $3 \le k \le 6$, then

$$\begin{aligned} |E(G)| &\le \binom{2k-3}{2} + 2 \\ &\le \binom{k-2}{2} + (k-2)(k+2) = ext(2k, (k-1)K_2), \end{aligned}$$

which contradicts $|E(G)| = ext(2k, (k-1)K_2) + 2$.

So we suppose that $E_G(D_1, A(G)) \neq \emptyset$. Let $G(s+2, s)$ be the bipartite graph obtained from G by deleting the edges spanned by $A(G)$ and by contracting the component D_1 to a single vertex p. Thus by Theorem 2.4 (d), we can obtain a maximum matching M of size $k-1$ such that M contains a near-perfect matching M_1 of D_1 which does not match w with $w \in D_1$ and a matching M_2 of size s which matches all vertices of $A(G)$ with vertices in $\{w\} \cup (D(G) - D_1)$. Since $E_G(D_1, A(G)) \neq \emptyset$, we can suppose that $w \in \langle M_2 \rangle$. There exist exactly two vertices $u, v \in D(G) - D_1$ and $u, v \notin \langle M \rangle$. It is obvious that $uv \in E(K_n)$ and $uv \notin E(G)$. We suppose that $c(uv) = 1$, hence there exists an edge $e = yz \in M$ with $c(e) = 1$. Now we distinguish two subcases to complete the proof of Subcase 4.1.

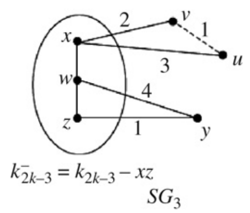

Fig. 5. The special graph SG_3 and $|E(SG_3)| = \binom{2k-3}{2} + 3$.

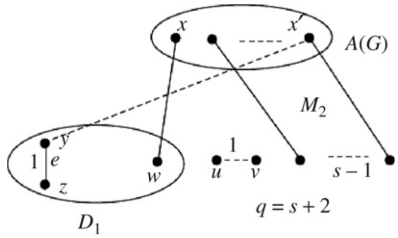

Fig. 6. There is no $(k-s-1)K_2$ in $D_1' = D_1 - w - yz$. If $x'y \in E(G)$, there is no $(s-1)K_2$ in the bipartite graph $G'(s-1, s-1) = G - \{D_1 \cup u \cup v \cup x'\}$.

Subcase 4.1.1. $e = yz \in M_1$.

If $s = 1$, then $|D_1| = 2k-3$ and we suppose $A(G) = \{x\}$. Thus the size of M_1 is $k-2$ and there is no $H = (k-2)K_2$ in $D_1' = D_1 - w - yz$, for otherwise, we can obtain a TMC $kK_2 = H \cup xw \cup uv$ in K_{2k}. If $E_G(x, \{y, z\}) \neq \emptyset$, say $xy \in E(G)$, then we can obtain a perfect matching M_1' of $D_1 - y$ and a TMC $kK_2 = M_1' \cup uv \cup xy$ in K_{2k}. So, $E_G(x, \{y, z\}) = \emptyset$ and

$$\begin{aligned}|E(G)| &= 1 + |E_G(D_1')| + |E_G(w, D_1')| + |E_G(x, D_1)| + |E_G(x, \{u, v\})| \\ &\leq 1 + ext(2k-4, (k-2)K_2) + (2k-4) + (2k-5) + 2 \\ &= \binom{2k-5}{2} + 4k - 6 \\ &= \binom{2k-3}{2} + 3.\end{aligned}$$

Denote SG_3 to be the special graph G shown in Fig. 5, whence $E(SG_3) = E(K_{2k-3}^{-}) \cup xu \cup xv \cup yw \cup yz$. Without loss of generality, we suppose that $c(wy) = 4$. If $|E(G)| = \binom{2k-3}{2} + 3$, it is easy to check that $G \cong SG_3$.

If $k \geq 7$, then by the starting hypothesis $|E(G)| = ext(2k, (k-1)K_2) + 3 = \binom{2k-3}{2} + 3$, whence $G \cong SG_3$. Now $\binom{2k-4}{2} - 1 > ext(2k-4, (k-2)K_2)$, we can obtain a TMC $H = (k-2)K_2$ in $K_{2k-3}^{-} - w$, whence a TMC $kK_2 = H \cup yw \cup uv$ in K_{2k}.

If $3 \leq k \leq 6$, then

$$\binom{2k-3}{2} + 3 \leq \binom{k-2}{2} + (k-2)(k+2) + 1 = ext(2k, (k-1)K_2) + 1,$$

which contradicts $|E(G)| = ext(2k, (k-1)K_2) + 2$. If $2 \leq s \leq k-2$, then $k \geq 4$. We suppose that $x \in A(G)$ and $xw \in M_2$. By the same token, $E_G(x, \{y, z\}) = \emptyset$ and there is no $(k-s-1)K_2$ in $D_1' = D_1 - w - yz$.

If $E_G(A(G) - x, \{y, z\}) \neq \emptyset$, say $x'y \in E(G)$, then there is no $H = (s-1)K_2$ in the bipartite graph $G'(s-1, s-1) = G - \{D_1 \cup u \cup v \cup x'\}$, for otherwise, we can obtain a perfect matching M_1' in $D_1 - y$ and a TMC $kK_2 = M_1' \cup H \cup uv \cup x'y$. See Fig. 6.

Thus,

$$\begin{aligned}|E_G(A(G), D(G))| &= |E_G(A(G), D_1 - y - z)| + |E(A(G), \{y, z\})| + |E_G(A(G), \{u, v\})| + |E_G(G'(s-1, s-1))| \\ &\quad + |E_G(x', D(G) - D_1 - u - v)| \\ &\leq (2k-2s-3)s + 2(s-1) + 2s + ext(s-1, s-1, (s-1)K_2) + (s-1) \\ &= (2k-2s-3)s + 2s + (s-1)(s+1).\end{aligned}$$

If $E_G(A(G) - x, \{y, z\}) = \emptyset$, then

$$\begin{aligned}|E_G(A(G), D(G))| &= |E_G(A(G), D_1 - y - z)| + |E_G(A(G), D(G) - D_1)| \\ &\leq (2k-2s-3)s + s(s+1).\end{aligned}$$

So,

$$\begin{aligned}|E_G(A(G), D(G))| &\le \max\{(2k-2s-3)s+2s+(s-1)(s+1), (2k-2s-3)s+s(s+1)\}\\ &= (2k-2s-3)s+2s+(s-1)(s+1).\end{aligned}$$

Now, we have

$$\begin{aligned}|E(G)| &= \binom{s}{2}+1+|E_G(D_1')|+|E_G(w, D_1')|+|E_G(A(G), D(G))|\\ &\le \binom{s}{2}+1+\binom{2k-2s-3}{2}+(2k-2s-2)+(2k-2s-3)s+2s+(s-1)(s+1) := f_5(s).\end{aligned}$$

Thus,

$$\begin{aligned}f_5(2) &= \binom{2k-3}{2}-2k+11,\\ f_5(k-2) &= \binom{k-2}{2}+(k-2)(k+2)-k+4\\ &< ext(2k, (k-1)K_2)+2.\end{aligned}$$

If $4 \le k \le 6$, then $f_5(k-2) \ge f_5(2)$ and $|E(G)| \le \max\{f_5(2), f_5(k-2)\} = f_5(k-2) < ext(2k, (k-1)K_2)+2$, which contradicts $|E(G)| = ext(2k, (k-1)K_2)+2$.

If $k \ge 7$, then $f_5(2) \ge f_5(k-2)$ and $|E(G)| \le \max\{f_5(2), f_5(k-2)\} = f_5(2) = \binom{2k-3}{2} - 2k + 11 < \binom{2k-3}{2} = ext(2k, (k-1)K_2)$, which contradicts $|E(G)| = ext(2k, (k-1)K_2)+3$.

Subcase 4.1.2. $e = yz \in M_2$.

Without loss of generality, we suppose that $y \in A(G)$.

If $s = 1$, then $A(G) = \{y\}$, $yz = yw$ and $c(yw) = c(uv) = 1$. Then $E_G(y, D_1 - w) = \emptyset$, for otherwise, say $yw' \in E_G(y, D_1 - w)$ with $w' \in (D_1 - w)$, we can obtain a TMC $H = (k-2)K_2$ in $D_1 - w'$ and a TMC $kK_2 = H \cup yw' \cup uv$ in K_{2k}. So,

$$|E(G)| = |E_G(D_1)| + |E_G(y, \{w, u, v\})| \le \binom{2k-3}{2}+3.$$

If $3 \le k \le 6$, then

$$\binom{2k-3}{2}+3 \le \binom{k-2}{2}+(k-2)(k+2)+1 = ext(2k, (k-1)K_2)+1,$$

which contradicts $|E(G)| = ext(2k, (k-1)K_2)+2$.

If $k \ge 7$, since $|E(G)| = \binom{2k-3}{2}+3$, it is easy to check that $(G - e + uv) \cong SG_1$. By the proof of Lemma 3.4, we can obtain a TMC kK_2 in K_{2k}.

If $2 \le s \le k-2$, first we look at the bipartite graph $G(s+2, s)$. We suppose that M_2' is any maximum matching of size s in $G(s+2, s)$ with $p \in \langle M_2' \rangle$ and $u_1, v_1 \notin \langle M_2' \rangle$. By Subcase 4.1.1, we can suppose that there exists an edge $e_1 \in M_2'$ such that $c(e_1) = c(u_1v_1)$. If $d_{G(s+2,s)}(p) = s$ and there is at most one vertex u_2 in $D(G) - D_1$ such that $d_{G(s+2,s)}(u) \le s-1$, we suppose $v_2 \in D(G) - D_1$ and $u_2 \ne v_2$. Let $G(s, s)$ be the bipartite graph obtained from $G(s+2, s)$ by deleting the vertices u_2, v_2. It is obvious that $u_2v_2 \in E(K_n)$ and $u_2v_2 \notin E(G)$. Then the number of edges in $G(s, s)$ whose color is not $c(u_2v_2)$ is at least $s^2 - 1$. Since $s \ge 2$, then $s^2 - 1 \ge ext(s, s, sK_2) = s(s-1)+1$. By Theorem 2.2, there exists a TMC $H = sK_2$ in $G(s, s)$ which contains no color $c(u_2v_2)$, thus we obtain a TMC $(s+1)K_2 = H \cup u_2v_2$. By Theorem 2.4, we can obtain a TMC kK_2 in K_{2k}.

So, if $d_{G(s+2,s)}(p) = s$, then we suppose there exist at least two vertices u_3, v_3 in $D(G) - D_1$ such that $d_{G(s+2,s)}(u_3) \le s-1$ and $d_{G(s+2,s)}(v_3) \le s-1$. Let $G'(s, s)$ be the bipartite graph obtained from $G(s+2, s)$ by deleting the vertices u_3, v_3 and the edge whose color is $c(u_3v_3)$. Thus there is no TMC sK_2 in $G'(s, s)$. By Theorem 2.2, $E(G(s+2, s)) \le 1 + 2(s-1) + s(s-1)$ and

$$|E_G(A(G), D(G))| \le 1+2(s-1)+s((2k-2s-1)+(s-2)) = 1+2(s-1)+s(2k-s-3).$$

Now we suppose that $d_{G(s+2,s)}(p) \le s-1$. Since $E(A(G), D_1) \ne \emptyset$, if there exists an edge $w''x' \in E(A(G), D_1)$ with $x' \in A(G)$, $w'' \in D_1$ and $w''x' \ne wx$. Thus there is no TMC $H = (s-1)K_2$ in $G(s+2, s) - \{p \cup u \cup v \cup x'\} - yz$, for otherwise, we can obtain a TMC $(s+1)K_2 = H \cup uv \cup w''x'$, a TMC $(k-s-1)K_2$ in $D_1 - w''$ and a TMC kK_2 in K_{2k}. We have

$$\begin{aligned}|E_G(A(G), D(G))| &\le |E_G(A(G), D_1)| + (s-1)(s-2)+1+|E_G(x', D(G)-D_1-u-v)|+|E_G(A(G), \{u, v\})|\\ &\le (2k-2s-1)(s-1)+(s-1)(s-2)+1+(s-1)+2s\\ &= (2k-2s-1)(s-1)+s^2+2.\end{aligned}$$

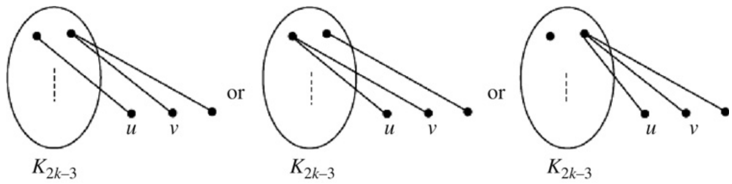

Fig. 7. G is isomorphic to one of the above three graphs.

If $E(A(G), D_1) = \{xw\}$, then

$$|E_G(A(G), D(G))| \leq 1 + s(s+1).$$

Thus,

$$\begin{aligned}|E_G(A(G), D(G))| &\leq \max\{1 + 2(s-1) + s(2k-s-3), (2k-2s-1)(s-1) + s^2 + 2, 1 + s(s+1)\}\\ &= 1 + 2(s-1) + s(2k-s-3).\end{aligned}$$

So,

$$|E(G)| \leq \binom{s}{2} + \binom{2k-2s-1}{2} + 1 + 2(s-1) + s(2k-s-3) := f_6(s).$$

We have

$$\begin{aligned}f_6(2) &= \binom{2k-3}{2} + 3,\\ f_6(3) &= \binom{2k-3}{2} - 2k + 12,\\ f_6(k-2) &= \binom{k-2}{2} + (k-2)(k+2) - k + 4\\ &< ext(2k, (k-1)K_2) + 2.\end{aligned}$$

If $s = 2$ and $|E(G)| = f_6(2) = \binom{2k-3}{2} + 3$, then it is easy to check that G has a structure shown in Fig. 7. By the proof Lemma 3.4, we can obtain a TMC kK_2 in K_{2k}.

If $3 \leq s \leq k-2$, then $k \geq 5$. If $5 \leq k \leq 6$, then $f_6(k-2) = f_6(3)$ and $|E(G)| \leq f_6(k-2) < ext(2k, (k-1)K_2) + 2$, which contradicts $|E(G)| = ext(2k, (k-1)K_2) + 2$. If $k \geq 7$, then $f_6(3) > f_6(k-2)$ and $|E(G)| \leq f_6(3) = \binom{2k-3}{2} - 2k + 12 < \binom{2k-3}{2} = ext(2k, (k-1)K_2)$, which contradicts $|E(G)| = ext(2k, (k-1)K_2) + 3$.

Subcase 4.2. $s = 0$.

In this subcase, $|V(D_1)| = 2k - 1$ and $q = 2$. We suppose that $z_1 \in D_1$ and $D_2 = \{z_2\}$. Let M be a perfect matching of $D_1 - z_1$. Then there exists an edge $e \in M$ such that $c(e) = c(z_1z_2)$. So, there is no TMC $(k-1)K_2$ in $D_1 - z_1 - e$. Let D'_1 be $D_1 - z_1 - e$, and $D(D'_1)$, $A(D'_1)$ and $C(D'_1)$ be the canonical decomposition of D'_1. We look at the graph $G_1 = G - e + z_1z_2$. Let $A'(G_1) = A(D'_1) \cup z_1$ and $D'(G_1) = D(D'_1) \cup z_2$ and $C'(G_1) = C(D'_1)$. Let $|A'(G_1)| = s'$, $q' = c(D'(G_1)) = c(D(D'_1)) + 1 = (2k-2) - 2(k-2) + s - 1 + 1 = s + 2$. Obviously, $1 \leq s' \leq k - 1$. Employing a similar technique as in the proofs of Cases 1, 2 and Subcase 4.1, we can obtain contradictions. The details are omitted. Now, the proof is complete. ■

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