Discrete Mathematics 309 (2009) 2990–2997

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cube intersection concepts in median graphs[☆]

Boštjan Brešar [a,*], Tadeja Kraner Šumenjak [b]

[a] FEECS, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia

[b] FA, University of Maribor, Vrbanska 30, 2000 Maribor, Slovenia

ARTICLE INFO

Article history:
Received 24 September 2007
Received in revised form 27 July 2008
Accepted 29 July 2008
Available online 30 August 2008

Keywords:
Cartesian product
Median graph
Cube graph
Intersection graph
Convexity

ABSTRACT

In this paper, we study different classes of intersection graphs of maximal hypercubes of median graphs. For a median graph G and $k \geq 0$, the intersection graph $\mathcal{Q}_k(G)$ is defined as the graph whose vertices are maximal hypercubes (by inclusion) in G, and two vertices H_x and H_y in $\mathcal{Q}_k(G)$ are adjacent whenever the intersection $H_x \cap H_y$ contains a subgraph isomorphic to Q_k. Characterizations of clique-graphs in terms of these intersection concepts when $k > 0$, are presented. Furthermore, we introduce the so-called maximal 2-intersection graph of maximal hypercubes of a median graph G, denoted $\mathcal{Q}_{m2}(G)$, whose vertices are maximal hypercubes of G, and two vertices are adjacent if the intersection of the corresponding hypercubes is not a proper subcube of some intersection of two maximal hypercubes. We show that a graph H is diamond-free if and only if there exists a median graph G such that H is isomorphic to $\mathcal{Q}_{m2}(G)$. We also study convergence of median graphs to the one-vertex graph with respect to all these operations.

1. Introduction

Intersection concepts play an important role in graph theory, and were briefly surveyed in McKee and McMorris [30]. Probably the greatest attention has been given to the concept of the *clique graph* of a graph, where vertices of the graph $K(G)$ are cliques (maximal complete subgraphs) of G, and adjacency is defined by non-empty intersections of two cliques. It is known which graphs are clique graphs of graphs, see Roberts and Spencer [37]. Many authors studied the behavior of the clique operator K in various classes of graphs that are often related to chordal graphs. For instance, clique graphs of interval graphs were considered in Hedman [22], while chordal and dually-chordal graphs were connected through the operator K in [8,11], see also Brandstädt, Le and Spinrad [12] for many related results. Several papers consider the question of which graph classes are fixed under K [4,20], and which graphs are *clique-convergent* to the one-vertex graph [4,9]. The latter problem is derived from the iterated clique graph concept, defined recursively by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ which was studied for instance in [10,21,22,28,34–36,38].

Intersection graphs of maximal hypercubes of graphs (also called *cube graphs*) have been studied in the context of median graphs and related classes of graphs that are rich in hypercubes [2,6,13,14]. There is a surprising similarity between classes of chordal and dually chordal graphs on the one side and median graphs and graphs of acyclic cubical complexes on the other side, which pertains to the geometric (or topological) interpretation where graphs arise from (simplicial or cubical) complexes, see Chepoi [18]. Furthermore, dually chordal graphs are precisely the clique graphs of chordal graphs [11] and precisely the cube graphs of acyclic cubical complexes [2,13].

Median graphs are defined as connected graphs in which for every three vertices u, v, and w there exists a unique vertex simultaneously lying on a shortest path between any pair of the three. They are among the most extensively studied classes

☆ The work supported in part by the Ministry of Education, Science and Sport of Slovenia. The authors are also with the Institute of Mathematics, Physics and Mechanics, Jadranska 19, 1000 Ljubljana.

* Corresponding author.
E-mail addresses: bostjan.bresar@uni-mb.si (B. Brešar), tadeja.kraner@uni-mb.si (T.K. Šumenjak).

0012-365X/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2008.07.032

of graphs that appear under different guises in many other discrete structures. In addition, there are now an abundance of applications, like in computer science, consensus theory, chemistry, genetics, location theory, conflict models, etc.; we refer to a comprehensive survey on median graphs by Klavžar and Mulder [26]. Due to a connection with the class of triangle-free graphs, the recognition complexity of median graphs is roughly the same as that of triangle-free graphs, see Imrich, Klavžar and Mulder [24].

One of the earliest characterizations of median graphs states, expressed in terms of ternary algebras, they are precisely the graphs that can be obtained by a sequence of convex amalgamations from hypercubes [25]. This in particular indicates the importance of the role that hypercubes are playing in median graphs. Bandelt and van de Vel studied median graphs with respect to the ordinary cube graph operation [6] that we denote in this paper by $\mathcal{Q}_0$. They proved that for any median graph G the cube graph $\mathcal{Q}_0(G)$ is a Helly graph, and they posed a question whether every Helly graph is the cube graph of some median graph. This problem is still open which gives a motivation to consider different kinds of intersection graphs of maximal hypercubes in median graphs. For many of these natural concepts we are able to prove "if and only if" results that yield new connections of median graphs to some well-known classes of graphs.

In the next section we fix the notation and recall several properties of median graphs that will be needed in the sequel. In Section 3 the concepts $\mathcal{Q}_k$ are introduced, and the edge-intersection graphs $\mathcal{Q}_1(G)$ of median graphs G are studied in detail. We prove that a graph is clique-graph (that is, can be realized as $K(G)$ for some graph G) if and only if it is the edge-intersection graph $\mathcal{Q}_1(H)$ for some median graph H. Then we consider the question of which median graphs converge to the one-vertex graph, and when restricted to the so-called simple median graphs we prove that these are precisely certain grid-like graphs. Section 4 is devoted to other operators $\mathcal{Q}_k$, where $k \geq 2$, and we show similar results for them as in the preceding section. Finally, in the last section the concept $\mathcal{Q}_{m2}$ is studied, and diamond-free graphs are characterized as $\mathcal{Q}_{m2}(G)$ graphs where G runs through all median graphs. The convergence to the one-vertex graphs is also briefly considered for $\mathcal{Q}_{m2}$. Several open questions are posed along the way.

2. Preliminaries

Throughout the paper we consider finite, simple, undirected graphs. *The distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. A subgraph H of a graph G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. *The interval* $I_G(u, v)$ is the set of vertices on shortest paths between u and v in G. A graph G is a *median graph* if for every triple of vertices $u, v, w \in V(G)$ the intersection $I(u, v) \cap I(u, w) \cap I(v, w)$ consists of precisely one vertex. A connected subgraph H of G is called *convex* if for every two vertices from H all shortest paths are contained in H. It is easy to see that the intersection of two convex subgraphs is also convex. A *convex closure* of a subgraph H of G is defined as the smallest convex subgraph of G which contains H. A subgraph U of G is 2-*convex* if for any two vertices u and v of U with $d_G(u, v) = 2$, every common neighbor of u and v belongs to U. In median graphs convexity is equivalent to 2-convexity [15, 23]. It is also clear that a convex subgraph of a median graph is a median graph as well.

The cycle on 3 (respectively 4) vertices will be called the *triangle* (respectively the *square*). The *hypercube (of dimension* k) or k-*cube* is the graph Q_k with the vertex set $\{0, 1\}^k$ where two vertices are adjacent whenever they differ in exactly one position. For instance, Q_1 is K_2, Q_2 is the square C_4, and Q_3 is the cube. Denote by Q_3^- the graph obtained from Q_3 by removing a vertex. Note that Q_3^- is not a median graph. The *bipartite wheel* BW_k is formed by the cycle C_{2k} and another vertex (*center of* BW_k) that is adjacent to every second vertex of the cycle. Bipartite wheels BW_k with $k \geq 4$ are median graphs.

Theorem 1 ([1]). *A connected graph G is a median graph if and only if the convex closure of any isometric cycle is a hypercube in G.*

A graph G satisfies the *quadrangle property* if for any $u, x, y, z \in V(G)$ such that $d(u, x) = d(u, y) = d(u, z) - 1$ and $d(x, y) = 2$ with z a common neighbor of x and y, there exists a common neighbor v of x and y such that $d(u, v) = d(u, x) - 1$.

Theorem 2 ([19]). *A connected bipartite graph G is a median graph if and only if it satisfies the quadrangle property.*

Let G be a connected graph and G_1 a convex subgraph. Then the *peripheral expansion* of G is the graph G' constructed as follows. Let G_1' be an isomorphic copy of G_1, and $G \cup G_1'$ the disjoint union of G and G_1'. For each vertex u of G_1 we denote the corresponding vertex in G_1' by u'. Now, G' is the graph obtained from $G \cup G_1'$ by adding edges between u and u' for all vertices $u \in V(G_1)$. We also say that we *expand* G_1 in G to obtain G'. This operation is a special case of expansion as introduced by Mulder [31,32], by which one of the first characterizations of median graphs was obtained. He later proved that median graphs are precisely the graphs that can be obtained by a sequence of peripheral expansions from K_1 [33].

A graph is called *cube-free* if it does not contain the 3-cube as an induced subgraph. The following proposition easily follows from Mulder's peripheral expansion theorem, cf. [27].

Proposition 3. *A graph is a cube-free median graph if and only if it can be obtained from the one-vertex graph by a sequence of peripheral expansions such that in every expansion step a convex tree is expanded.*

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with the vertex set $V(G) \times V(H)$ and $(a, x)(b, y) \in E(G \Box H)$ whenever either $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Cartesian products of median graphs are median graphs [23]. The *grid graph* $P_n \Box P_m$ is the Cartesian product of two paths (grid graphs are clearly median graphs).

We will also use some hypergraph notions from [7]. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. The dual hypergraph of $\mathcal{H}$ is denoted by $\mathcal{H}^*$. A hypergraph $\mathcal{H}$ is *conformal* if any subset $S \subseteq V$ of elements is contained in an edge of $\mathcal{H}$ provided any pair of elements of S does. A hypergraph is a *Helly hypergraph* if any family of pairwise intersecting edges contain a common vertex. Conformal hypergraphs and Helly hypergraphs are mutually dual. The 2-*section* $[\mathcal{H}]_2$ of a hypergraph $\mathcal{H}$ is the graph whose vertices are the vertices of $\mathcal{H}$ and two vertices are adjacent in $[\mathcal{H}]_2$ if and only if they belong to a common edge of $\mathcal{H}$. If $\mathcal{H}$ is conformal, then its edges are exactly the maximal cliques of its 2-section.

Given a median graph G the graph G^{Δ} was introduced in [6], whose vertices are the vertices of G and two vertices are adjacent whenever they belong to a common hypercube in G. It was shown that every G^{Δ} is a Helly graph (i.e. balls enjoy the Helly property), and also $K(G^{\Delta})$ is a Helly graph. Hence G^{Δ} is also a clique-Helly graph (i.e. cliques enjoy the Helly property). Since cliques of G^{Δ} are induced by vertices of maximal hypercubes of G, this implies that maximal hypercubes of G enjoy the Helly property:

Lemma 4. *In a median graph any family of pairwise non-disjoint hypercubes has a non-empty common intersection.*

Finally, since $K(G^{\Delta})$ is a Helly graph this implies that $\mathcal{Q}_0(G)$ is a Helly graph for every median graph G [6].

3. Edge-intersection of maximal hypercubes

Let G be a median graph. For $k \geq 0$ the intersection graph $\mathcal{Q}_k(G)$ is the graph with the vertex set consisting of all maximal hypercubes (by inclusion) in G and two vertices H_x, H_y in $\mathcal{Q}_k(G)$ are adjacent whenever the intersection $H_x \cap H_y$ contains a hypercube Q_k. Note that $\mathcal{Q}_0$ coincides with the usual intersection concept, while other operators $\mathcal{Q}_k$ are covered by certain tolerance intersection concepts [30] (notably two vertices in $\mathcal{Q}_k(G)$ are adjacent whenever the intersection of the corresponding hypercubes in G has at least 2^k vertices).

Among the series of concepts $\mathcal{Q}_k$ we are particularly interested in the operation $\mathcal{Q}_1$, in which two vertices are adjacent whenever the corresponding hypercubes have an edge in common. For $k > 1$, the $\mathcal{Q}_k$ operation is similar to this case in several ways as will be seen in the next section.

We start with the following problem: which are the graphs that can be obtained as $\mathcal{Q}_1(G)$ for some median graph G. In fact, we will first solve the opposite direction of this problem, by showing that every *clique-graph* (that is, a graph that can be obtained as $K(H)$ for some graph H) can be obtained by this operation from a median graph.

By *simplex* of a graph we mean a set of vertices that induce a complete subgraph in G (of any possible size, including the empty set). Given a graph G denote by $\kappa(G)$ its *simplex graph*, that is the graph whose vertices are simplices of G, and two simplices are adjacent if and only if they differ in one vertex and are comparable. Clearly the simplex graph of the complete graph K_n is the hypercube Q_n (using the fact that hypercubes are characterized as subset graphs via positions of 1's in the binary representation of Q_n). It is also not surprising that the simplex graph of any graph is always a median graph, as shown in [5]. There it was also proved that a hypercube in $\kappa(G)$ of maximum dimension d corresponds to some largest clique of order d in G. In fact any maximal hypercube in $\kappa(G)$ corresponds precisely to some maximal complete subgraph (clique) of G. Furthermore, we find that two maximal hypercubes share an edge in $\kappa(G)$ whenever the corresponding cliques in G share a vertex (this fact was discovered by Chepoi [17]; for a related study we refer to a commendable survey [3]). Hence the clique-graph $K(G)$ of a graph G coincides with the edge-intersection graph of the simplex graph of G, that is

$$\mathcal{Q}_1(\kappa(G)) = K(G).$$

We derive that every clique-graph (i.e. a graph which can be represented as the clique graph of some graph), can also be represented as the edge-intersection graph of maximal hypercubes of a median graph, notably of the median graph $\kappa(H)$. We mention that clique-graphs have been characterized by Roberts and Spencer [37], and we can now present another characterization of clique-graphs.

Theorem 5. *A graph H is clique-graph if and only if there exists a median graph G such that $\mathcal{Q}_1(G) = H$.*

The theorem says two things: every clique graph $K(H)$ can be represented as the edge-intersection graph of maximal cubes of a median graph, and conversely, for every median graph G the graph $\mathcal{Q}_1(G)$ is a clique-graph. The latter we still need to prove. We will use the following lemma which is a sort of conformality property of 1-cubes with respect to hypercubes in median complexes.

Lemma 6. *If all pairs of edges from $S = \{e_1, \dots, e_k\}$ in a median graph G belong to a common hypercube then there exists a hypercube that contains all edges of S.*

Proof. It is known that in a median graph any set of vertices that are pairwise in the same hypercube, all belong to a common hypercube, cf. [6] (in terms of median complexes as cube hypergraphs this is precisely the conformality property). Consider the set of edges $S = \{e_1, \dots, e_k\}$ that pairwise belong to a common hypercube. Then for any two edges the endvertices of both pairwise belong to a hypercube, hence this holds for all endvertices of the edges from S. By the conformality we infer that all the endvertices belong to a common hypercube, and so all the edges belong to the very same hypercube. ■

Let G be a median graph. Denote by G^e the graph with $V(G^e) = E(G)$ and two vertices e_1, e_2 are adjacent in G^e if and only if the edges e_1 and e_2 belong to a common hypercube in G. We call G^e the *line-cube graph* of a graph G. The connection with

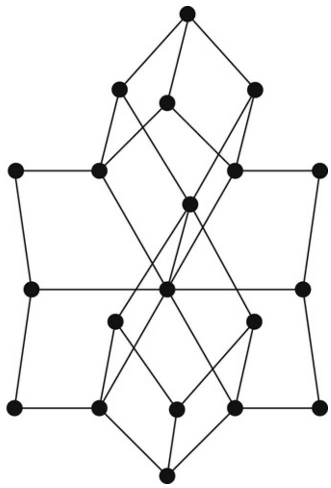

Fig. 1. Graph that converges to K_1.

line graph is only in their vertex sets which are the same, while G^e is in general quite different from $L(G)$. For instance, G^e of a tree is a totally disconnected graph, and G^e is a complete graph if G is a hypercube.

From Lemma 6 we infer that every complete subgraph in G^e is induced by edges that belong to a common hypercube of G. Hence maximal hypercubes of G correspond bijectively to cliques in G^e where vertices of a clique are just the edges of the corresponding maximal hypercube. Clearly two maximal hypercubes share an edge if and only if the cliques in G^e share a vertex. We derive that

$$K(G^e) = \mathcal{Q}_1(G),$$

and so every edge-intersection graph of maximal hypercubes of a median graph is a clique-graph, by which Theorem 5 is finally proved. We state this direction explicitly as follows.

Corollary 7. *For any median graph G the graph $\mathcal{Q}_1(G)$ is a clique-graph. More precisely $\mathcal{Q}_1(G) = K(G^e)$.*

If G is a median graph, $\mathcal{Q}_1(G)$ can be any clique-graph, hence in general need not be a median graph (for instance if G is a bipartite wheel BW_n then $\mathcal{Q}_1(G)$ is a cycle which is not median as soon as $n > 4$). We are now interested in median graphs which have the additional property that they converge to the one-vertex graph in the following sense: there exists a natural number n such that the iterated graph $\mathcal{Q}_1^n(G) = \mathcal{Q}_1(\mathcal{Q}_1^{n-1}(G))$ consists of only one vertex, and every graph in the sequence of $\mathcal{Q}_1$-iterations is a median graph.

Clearly if G is isomorphic to a hypercube then $\mathcal{Q}_1(G) = K_1$. By using induction on the number n we can easily show that $\mathcal{Q}_1^n(P_{n+1}\Box P_n) = K_1$ and $\mathcal{Q}_1^n(P_{n+1}\Box P_{n+1}) = K_1$. On the other hand, if n and m differ by more than 1 then $P_n\Box P_m$ converges to a path of length more than 1, and in turn to a totally disconnected graph of order more than 1.

Of course there are many examples of median graphs that converge to K_1. For instance such is the graph obtained from the bipartite wheel BW_6, by the simultaneous expansion of two middle hypercubes, see Fig. 1. Using that median graphs are clique-graphs (as are all bipartite graphs), we infer from Theorem 5 the following construction. If G is a median graph that converges to K_1, then there is a median graph G' with $\mathcal{Q}_1(G') = G$, and so G' also converges to K_1. Now argue the same for G', and so on. In addition, more complex constructions can be used to obtain median graphs that converge to K_1. However, we can show that under certain additional restrictions that do not allow such constructions, the grid graphs, mentioned above, are the only examples of median graphs that converge to K_1.

First we will need a lemma that shows in which way one can obtain the square in a median graph $\mathcal{Q}_1(G)$.

Lemma 8. *Let G and $\mathcal{Q}_1(G)$ be median graphs, and let $S = x_1x_2x_3x_4x_1$ be an induced square in $\mathcal{Q}_1(G)$. Denote by $H_{x_1}, H_{x_2}, H_{x_3}, H_{x_4}$ the maximal hypercubes of G that correspond to x_1, x_2, x_3, x_4. Then there exists a convex bipartite wheel BW_4 with squares $S_1, \ldots, S_4$, such that $S_i \subseteq H_{x_i}$, for all $i \in \{1, \ldots, 4\}$.*

Proof. Let us first prove that the common intersection of the four hypercubes consists of precisely one vertex, that is $\cap_{i=1}^4 V(H_{x_i}) = \{v\}$ for some $v \in V(G)$. Note that $|\cap_{i=1}^4 V(H_{x_i})| \geq 2$ implies that all the hypercubes are adjacent in $\mathcal{Q}_1(G)$ and so K_4 appears in $\mathcal{Q}_1(G)$, a contradiction. Hence $|\cap_{i=1}^4 V(H_{x_i})| \leq 1$.

Consider first the case when three of the hypercubes share a vertex (they cannot share two vertices because a K_3 would appear in $\mathcal{Q}_1(G)$). Without loss of generality we may assume that $\cap_{i=1}^3 V(H_{x_i}) = \{v\}$, and $v \notin V(H_{x_4})$. Since $|H_{x_1} \cap H_{x_4}| \geq 2$, $|H_{x_3} \cap H_{x_4}| \geq 2$ and $H_{x_1} \cap H_{x_3} = \{v\}$, it follows from Lemma 4 (Helly property for hypercubes) that $H_{x_1} \cap H_{x_3} \cap H_{x_4} = \{v\}$ (again they cannot share two vertices because a K_3 would appear in $\mathcal{Q}_1(G)$). We infer that $\cap_{i=1}^4 V(H_{x_i}) = \{v\}$.

Hence suppose that $\cap_{i=1}^{4} H_{x_i} = \emptyset$, and no three hypercubes share a vertex. Let $H_{x_1} \cap H_{x_2} = H_{1,2}$, $H_{x_2} \cap H_{x_3} = H_{2,3}$, $H_{x_3} \cap H_{x_4} = H_{3,4}$ and $H_{x_4} \cap H_{x_1} = H_{4,1}$. Note that $H_{1,2} \cap H_{2,3} \cap H_{3,4} \cap H_{4,1} = \emptyset$, moreover they are pairwise disjoint. Let C be a shortest cycle that consecutively passes $H_{1,2}, H_{x_2}, H_{2,3}, H_{x_3}, H_{3,4}, H_{x_4}, H_{4,1}, H_{x_1}, H_{1,2}$, and denote the set of these eight hypercubes by $\mathcal{H}$. Note that some of the vertices in C may be used once in two of the consecutive hypercubes from $\mathcal{H}$, but not in three. If C is isometric then by Theorem 1 its convex closure is a hypercube H which is clearly not any of the four hypercubes. In addition H contains all edges of this cycle and so it adjacent to all of the four hypercubes, which implies that there are triangles in $\mathcal{Q}_1(G)$, which is a contradiction with $\mathcal{Q}_1(G)$ being a median graph. Suppose C is not isometric. By the way C is constructed (noting that hypercubes are convex in G, and so there can be no shortcut between two vertices of any hypercube from $\mathcal{H}$), we derive that a shortcut can only be between two (almost) opposite hypercubes from $\mathcal{H}$. In any such case an isometric cycle appears whose convex closure is a hypercube H that is adjacent in $\mathcal{Q}_1$ with at least two of the hypercubes $H_{x_1}, H_{x_2}, H_{x_3}, H_{x_4}$ which is again a contradiction with $\mathcal{Q}_1(G)$ being a median graph. We conclude there is a vertex v in $\cap_{i=1}^{4} V(H_{x_i}) = \{v\}$.

Hence, v is the center of a bipartite wheel W (isomorphic to BW_4) whose squares belong to the four hypercubes, respectively. We claim that W is convex. Since G is a median graph, there are not many possibilities that W would not be convex. Since convex and 2-convex sets coincide in median graphs, one must only check the possibility of additional common neighbors. If two vertices whose degree in W is 3 have a common neighbor then a $K_{2,3}$ appears which is not possible in median graphs. Hence, the only remaining way to prevent convexity of W is that two of the vertices whose degree in W is 2 have a neighbor in common. But then the convex closure of W contains the graph isomorphic to $P_3 \Box C_4$. Hence two new hypercubes appear (since each of the squares of W belongs to exactly one of the four hypercubes $H_{x_1}, H_{x_2}, H_{x_3}, H_{x_4}$) and each of these two hypercubes contain two squares of W which have an edge in common. This is again impossible, since these edges would belong to three hypercubes, and (two) triangles would appear in $\mathcal{Q}_1(G)$. We conclude that W is a convex bipartite 4-wheel, as claimed. ■

In the construction of $\mathcal{Q}_1(G)$ which yields a median graph H via the simplex graph $\kappa(H) = G$, there is a vertex v that is common to all hypercubes of G, hence v is the center of all convex bipartite wheels in G. We are trying to avoid such constructions, and concentrate on median graphs of different nature. According to Lemma 8 we say that a wheel W from G corresponds to the square S from $\mathcal{Q}_1(G)$. Then we say that a median graph G is *simple* if for any set of squares $S_1, S_2 \dots, S_k$ in $\mathcal{Q}_1(G)$ there exist bipartite wheels $W_1, W_2, \dots, W_k$ in G that have pairwise different centers and W_i corresponds to S_i for $i = 1, 2, \dots, k$. The following result shows that the structure of $\mathcal{Q}_1(G)$ is quite restricted when G is a simple median graph.

Proposition 9. *Let G be a simple median graph and let $\mathcal{Q}_1(G)$ be a median graph. Then $\mathcal{Q}_1(G)$ contains no Q_3 as an induced subgraph.*

Proof. Suppose that a subgraph isomorphic to Q_3 lies in $\mathcal{Q}_1(G)$. Let

$$\mathcal{H} = \left\{H_{x_1}, H_{x_2}, H_{x_3}, H_{x_4}, H_{x_5}, H_{x_6}, H_{x_7}, H_{x_8}\right\}$$

be the family of the maximal hypercubes of G corresponding to this Q_3. Consider 4-cycles $C_1 = x_1x_2x_4x_3x_1$, $C_2 = x_2x_8x_6x_4x_2$, $C_3 = x_6x_8x_7x_5x_6$, $C_4 = x_5x_7x_1x_3x_5$ in Q_3. By using Lemma 8 and the fact that G is a simple median graph there exist convex bipartite wheels with different central vertices. Denote by v_1, v_2, v_3, v_4 the central vertices that correspond to C_1, C_2, C_3, C_4, respectively. Let P_1 be a shortest path in $H_{x_2} \cap H_{x_4}$ from v_1 to v_2, P_2 a shortest path in $H_{x_6} \cap H_{x_8}$ from v_2 to v_3, P_3 a shortest path in $H_{x_5} \cap H_{x_7}$ from v_3 to v_4 and P_4 a shortest path in $H_{x_1} \cap H_{x_3}$ from v_4 to v_1. Let C be the cycle formed by $P_1 \cup P_2 \cup P_3 \cup P_4$, and note that C is an induced cycle. If C is isometric then by Theorem 1 its convex hull is a hypercube H, which is not any of the eight hypercubes. Hypercube H contains all edges of this cycle, so the vertex x which corresponds to H is adjacent to every vertex of the Q_3, a contradiction. Now suppose that cycle C is not isometric. Then there is a shortcut between two vertices of C, such that together with the path on C it forms an isometric cycle (note that this cycle contains at least four of the hypercubes from $\mathcal{H}$). Its convex closure is the hypercube which is adjacent (has an edge in common) to two adjacent hypercubes from $\mathcal{H}$, yielding a triangle in $\mathcal{Q}_1(G)$ which again is a contradiction with $\mathcal{Q}_1(G)$ being a median graph. ■

Theorem 10. *Let G be a simple median graph that converges to K_1. Then $\mathcal{Q}_1(G) = P_n \Box P_m$, where $|m - n| \le 1$.*

Proof. By Proposition 9, $\mathcal{Q}_1(G)$ is cube-free. Hence by Proposition 3, $\mathcal{Q}_1(G)$ can be obtained from the one-vertex graph by the expansion procedure in which every expansion step is taken with respect to a convex tree. First we observe that every expansion step is taken with respect to a path, otherwise three hypercubes share an edge in $\mathcal{Q}_1(G)$ and so K_3 appears in $\mathcal{Q}_1(\mathcal{Q}_1(G))$. Moreover, by convexity of the expanded graph, every edge of the path in an expansion step is contained in exactly one hypercube. Let e_1, e_2 be incident edges of the path and denote by H_1 and H_2 corresponding hypercubes. Then $|H_1 \cap H_2| \ge 2$, otherwise an isometric cycle C_n, where $n > 4$ appears in $\mathcal{Q}_1(\mathcal{Q}_1(G))$. We infer by induction that $\mathcal{Q}_1(G)$ is a subgraph of grid graph $P_n \Box P_m$. It is easy to see that the only such graphs that converge to K_1 are the graphs $P_n \Box P_m$ where $|m - n| \le 1$. ■

The question remains which simple median graphs G are such that $\mathcal{Q}_1(G) = P_n \Box P_m$ where $|m - n| \le 1$. Beside grids there are also other examples, for instance any graphs obtained from grids by expanding some of their hypercubes (see Fig. 2 for such an example).

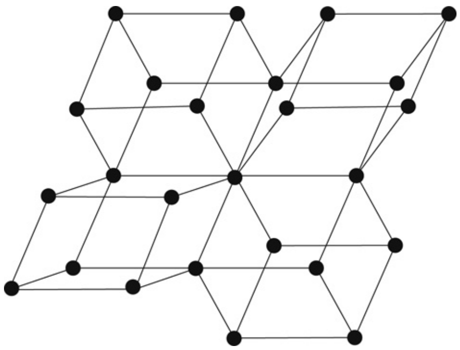

Fig. 2. Edge-intersection graph of this graph is the square.

4. $\mathcal{Q}_k$-intersections of median graphs for $k > 1$

The following lemma is an easy consequence of the fact that there is natural bijective correspondence between maximal hypercubes of G and of $G\Box Q_\ell$ by which we can identify the pairs of hypercubes in both graphs. In addition, the intersections of maximal hypercubes in $G\Box Q_\ell$ are hypercubes of dimension larger by ℓ as the corresponding intersections in G.

Lemma 11. *Let G be a median graph. For any nonnegative integers k and ℓ we have*

$$\mathcal{Q}_k(G) = \mathcal{Q}_{k+\ell}(G\Box Q_\ell).$$

Hence, any graph that can be realized as $\mathcal{Q}_k(G)$, can also be realized as $\mathcal{Q}_{k+\ell}$ of some graph, but the converse is not always true. For instance, $\mathcal{Q}_1(P_3\Box P_3) = C_4$ while C_4 cannot be realized as $\mathcal{Q}_0$ of some graph, since it is not a Helly graph.

From Lemma 11 and Theorem 5 we infer the following result.

Theorem 12. *For every $k > 1$ and every clique-graph H there exists a median graph G such that $\mathcal{Q}_k(G) = H$.*

Indeed, for any graph G we have $\mathcal{Q}_k(\kappa(G)\Box Q_{k-1}) = K(G)$.

The converse of Theorem 12 is also true and the construction is similar as in the case of edge-intersection. Namely, given a median graph G, we construct the graph G^{Q_k} whose vertex set is the union two sets: the set of hypercubes of G isomorphic to Q_k and the set of maximal hypercubes in G of dimension less than k. Two vertices are adjacent in G^{Q_k} if and only if the hypercubes belong to a common (larger) hypercube. Note that G^e is precisely G^{Q_1}. One can prove a similar (conformality) result analogous to Lemma 6 (following its proof) by which we infer that maximal complete subgraphs in G^{Q_k} correspond bijectively to maximal hypercubes of G, and moreover

$$K(G^{Q_k}) = \mathcal{Q}_k(G).$$

Theorem 13. *For any median graph G the graph $\mathcal{Q}_k(G)$ is a clique-graph. More precisely $\mathcal{Q}_k(G) = K(G^{Q_k})$.*

Let P be a generalized grid graph isomorphic to $P_{n+1}\Box P_n\Box\cdots\Box P_n$ or $P_{n+1}\Box P_{n+1}\Box\cdots\Box P_{n+1}$ in which there are k factors. Then $\mathcal{Q}_k^n(P) = K_1$. Are there any other nice examples of median graphs that converge to K_1? Can a similar results be proved for simple median graphs as in the case of $\mathcal{Q}_1$?

5. Maximal 2-intersection graph of maximal hypercubes

By $\mathcal{Q}_{\mathrm{m2}}(G)$ we denote *maximal 2-intersection graph of maximal hypercubes* of a graph G. Vertices of $\mathcal{Q}_{\mathrm{m2}}(G)$ correspond to maximal hypercubes (by inclusion) of G, and two vertices are adjacent if the corresponding hypercubes have a maximal 2-intersection in common (intersection of two maximal hypercubes is called *maximal 2-intersection* if it is not a proper subcube of some (other) intersection of two maximal hypercubes). This concept appeared implicitly in [13] where it was invented for the purposes of proving the main theorem (it was shown that every such maximal 2-intersection forms a cut set in the graphs of acyclic cubical complexes).

Lemma 14. *Let G be a median graph, let vertices x, y, z induce a triangle in $\mathcal{Q}_{\mathrm{m2}}(G)$, and let H_x, H_y, H_z be the corresponding maximal hypercubes in G. Then $H_x \cap H_y \cap H_z = H_x \cap H_y = H_y \cap H_z = H_x \cap H_z$.*

Proof. Since vertices x, y, z induce a triangle in $\mathcal{Q}_{m2}(G)$, $I_1 = H_x \cap H_y$, $I_2 = H_y \cap H_z$, $I_3 = H_x \cap H_z$ are maximal 2-intersections. Hence they are all non-empty, and so $H_x \cap H_y \cap H_z \neq \emptyset$ by Lemma 4. Moreover, since I_i's are maximal, either they all coincide with $H_x \cap H_y \cap H_z$ or each I_i has a vertex (and also a Θ-class) that does not belong to the other two. We assume that the latter case appears which will lead us to a contradiction.

Let v be a vertex of $H_x \cap H_y \cap H_z$. Then v is incident with all Θ-classes that appear in H_x, H_y and H_z. Denote by $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ the set of Θ-classes that appear in $H_x \cap H_y$, $H_y \cap H_z$, $H_x \cap H_z$, respectively, and not in $H_x \cap H_y \cap H_z$. Now, consider the edges $e_1 = vu_1$, $e_2 = vu_2$, $e_3 = vu_3$ that belong to Θ-classes from $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$, respectively. Denote by v_i the unique common neighbor of u_i and u_{i+1} (modulo 3) for $i = 1, 2, 3$, different from v. Then the vertices u_i,v_i for $i = 1, 2, 3$ and v induce a Q_3^- in G. Since G is a median graph, its convex closure in G is a 3-cube, in which we denote by u the vertex of the convex closure. Let H_a be a maximal hypercube that contains this 3-cube, and note that it is distinct from H_x, H_y and H_z. Suppose there exists another Θ-class in I_i (if not, we already derive a contradiction with I_i being maximal 2-intersections).

Without loss of generality let w be any other vertex in $I_1 = H_x \cap H_y$. Then in the square $S = \{v, u_3, v_2, u_2\} \subseteq H_z$ the closest vertex to w is v and the farthest vertex to w is v_2. Let w_3 be the neighbor of w whose closest vertex in S is u_3, and let w_2 be the neighbor of w whose closest vertex in S is u_2. (Then $w_3 \in H_x \setminus H_y$ and $w_2 \in H_y \setminus H_x$.) Note that $d(v_2, w_3) = d(v_2, w_2) = d(v_2, w) - 1$. Hence by the quadrangle property there exists a common neighbor of w_3 and w_2, whose closest vertex in S is v_2. By applying this argument for all vertices of $H_x \cap H_y$ we derive that the convex closure of these vertices is the hypercube $S \Box (H_x \cap H_y)$ that lies in a maximal hypercube H_a, distinct from H_x, H_y and H_z. Clearly I_1 is a proper subgraph of $H_a \cap H_x$ (as well as of $H_a \cap H_x$), and so I_1 is not a maximal 2-intersection which is the final contradiction. We derive $I_i = H_x \cap H_y \cap H_z$ for all i. ■

The *diamond* is the graph $K_4 - e$ (obtained from K_4 by deletion of an edge). A graph is called *diamond-free* if it does not contain any diamond as an induced subgraph. It is clear that diamond-free graphs are precisely the graphs in which every edge lies in a unique clique [29].

Theorem 15. *For any median graph G the graph $\mathcal{Q}_{m2}(G)$ is diamond-free.*

Proof. Let G be a median graph and $\mathcal{Q}_{m2}(G)$ its maximal 2-intersection graph. Let T be a triangle with vertices x, y and z in $\mathcal{Q}_{m2}(G)$ and denote by H_x, H_y, H_z the corresponding hypercubes of G. Then by Lemma 14 we have $I = H_x \cap H_y \cap H_z = H_x \cap H_y = H_y \cap H_z = H_x \cap H_z$. Hence if a vertex $u \in V(\mathcal{Q}_{m2}(G))$ is adjacent to x and y, then by Lemma 14 again $H_x \cap H_y \cap H_u = H_x \cap H_y = I$, which implies that u is adjacent also to z. We derive that induced diamond is not possible in $\mathcal{Q}_{m2}(G)$. ■

Proposition 16. *If G is a diamond-free graph then $\mathcal{Q}_1(\kappa(G)) = \mathcal{Q}_{m2}(\kappa(G))$.*

Proof. Let G be a diamond-free graph and $\kappa(G)$ its simplex graph which is a median graph. Note that maximal hypercubes in $\kappa(G)$ correspond to cliques of G and two such hypercubes share an edge whenever the corresponding cliques share a vertex. Since every two cliques in G share at most one vertex, for any non-disjoint maximal hypercubes H_x, H_y from $\kappa(G)$ we have $\left|H_x \cap H_y\right| = 2$ (that is, maximal 2-intersections of maximal hypercubes of $\kappa(G)$ are only edges). We infer $\mathcal{Q}_1(\kappa(G)) = \mathcal{Q}_{m2}(\kappa(G))$. ■

Theorem 17. *For every diamond-free graph H there exists a median graph G such that $\mathcal{Q}_{m2}(G) = H$.*

Proof. Let H be a diamond-free graph. By a result from [16] every diamond-free graph is the clique graph of a diamond-free graph G. Hence there is a diamond-free graph G such that $K(G) = H$. By Theorem 5 we have $\mathcal{Q}_1(\kappa(G)) = K(G) = H$, which combined with Proposition 16 yields the desired formula $\mathcal{Q}_{m2}(G) = H$. ■

We conclude with the formula concerning $\mathcal{Q}_{m2}$ of Cartesian products of median graphs.

Proposition 18. *Let G and H be median graphs. Then $\mathcal{Q}_{m2}(G \Box H) = \mathcal{Q}_{m2}(G) \Box \mathcal{Q}_{m2}(H)$.*

Proof. Let A and B be maximal hypercubes in median graphs G and H, respectively. Then $V(A) \times V(B)$ clearly induces a maximal hypercube in $G \Box H$. On the other hand, every maximal hypercube in $G \Box H$ is a subproduct of $G \Box H$, hence $V(\mathcal{Q}_{m2}(G \Box H)) = V(\mathcal{Q}_{m2}(G) \Box \mathcal{Q}_{m2}(H))$.

Let $(a, b)(x, y)$ be an arbitrary edge from $\mathcal{Q}_{m2}(G \Box H)$. Then the corresponding endvertices (hypercubes) $A \times B$ and $X \times Y$ from $G \Box H$ have a maximal 2-intersection in common. We claim that $A = X$ or $B = Y$. Indeed, otherwise the intersection of $A \times B$ and $X \times Y$ is a proper subcube of $A \times B \cap A \times Y$. Hence we may assume without loss of generality that $A = X$. Then the intersection of maximal hypercubes B and Y in H must clearly be a maximal 2-intersection. Hence the vertices b and y (corresponding to B and Y) are adjacent in $\mathcal{Q}_{m2}(H)$. Thus $(a, b)(x, y) \in E(\mathcal{Q}_{m2}(G) \Box \mathcal{Q}_{m2}(H))$. The converse can be easily proved by following similar arguments. ■

Proposition 18 is not true for operator $\mathcal{Q}_k$. For example if G is isomorphic to $P_3 \Box P_3$ and H is isomorphic to K_2, then $\mathcal{Q}_2(G \Box H)$ is isomorphic to K_4 and $\mathcal{Q}_2(G) \Box \mathcal{Q}_2(H)$ is isomorphic to Q_2.

It follows from Proposition 18 that the convergence (with respect to $\mathcal{Q}_{m2}$) of two graphs to the one-vertex graph implies the convergence of their Cartesian product to K_1. Hence, for instance, all generalized grid graphs $P_{n_1} \Box P_{n_2} \Box \cdots \Box P_{n_k}$ converge to K_1. It would be interesting to find more examples of median graphs that converge to the one-vertex graph with respect to operator $\mathcal{Q}_{m2}$. We believe a characterization of such median graphs is not an easy problem.

References

[1] H.-J. Bandelt, Characterizing median graphs, manuscript, 1982.
[2] H.-J. Bandelt, V. Chepoi, Graphs of acyclic cubical complexes, European J. Combin. 17 (1996) 113–120.
[3] H.-J. Bandelt, V. Chepoi, Metric graph theory and geometry — a survey, Contemp Math. (in press).
[4] H.-J. Bandelt, E. Prisner, Clique graphs and Helly graphs, J. Combin. Theory Ser. B 51 (1991) 34–45.
[5] H.-J. Bandelt, M. van de Vel, Embedding topological median algebras in products of dendrons, Proc. London Math. Soc. (3) 58 (1989) 439–453.
[6] H.-J. Bandelt, M. van de Vel, Superextensions and the depth of median graphs, J. Combin. Theory Ser. A 57 (1991) 187–202.
[7] C. Berge, Hypergraphs, North-Holland, Amsterdam, 1989.
[8] C.F. Bornstein, J.L. Szwarcfiter, Clique graphs of chordal graphs and path graphs, SIAM J. Discrete Math. 7 (1994) 331–336.
[9] C.F. Bornstein, J.L. Szwarcfiter, On clique convergent graphs, Graphs Combin. 11 (1995) 213–220.
[10] C.F. Bornstein, J.L. Szwarcfiter, Iterated clique graphs with increasing diameters, J. Graph Theory 28 (1998) 147–154.
[11] A. Brandstädt, F. Dragan, V. Chepoi, V. Voloshin, Dually chordal graphs, SIAM J. Discrete Math. 11 (1998) 437–455.
[12] A. Brandstädt, V.B. Le, J.P. Spinrad, Graphs Classes: A Survey, SIAM, Philadelphia, 1999.
[13] B. Brešar, Intersection graphs of maximal hypercubes, European J. Combin. 24 (2003) 195–209.
[14] B. Brešar, W. Imrich, S. Klavžar, Tree-like isometric subgraphs of hypercubes, Discuss. Math. Graph Theory 23 (2003) 227–240.
[15] B. Brešar, S. Klavžar, R. Škrekovski, Quasi-median graphs, their generalizations, and tree-like equalities, European J. Combin. 24 (2003) 557–572.
[16] L. Chang-Keang, P. Yee-Hock, On graphs without multicliqual edges, J. Graph Theory 5 (1981) 443–451.
[17] V. Chepoi, personal communication.
[18] V. Chepoi, Graphs of some CAT(0) complexes, Adv. Appl. Math. 24 (2000) 125–179.
[19] F.R.K. Chung, R.L. Graham, M.E. Saks, Dynamic search in graphs, in: D.S. Johnson, T. Nishizeki, N. Takao, H.S. Wilf (Eds.), Discrete Algorithms and Complexity, Academic Press, San Diego, 1987, pp. 351–388.
[20] M. Gutierrez, Intersection graphs and the clique operator, Graphs Combin. 17 (2001) 237–244.
[21] S.T. Hedetniemi, P.J. Slater, Line Graphs of Triangleless Graphs and Iterated Clique Graphs, in: Lecture Notes in Mathematics, vol. 303, Springer, Berlin, 1972, pp. 139–147.
[22] B. Hedman, Clique graphs of time graphs, J. Combin. Theory 3 (1984) 270–278.
[23] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.
[24] W. Imrich, S. Klavžar, H.M. Mulder, Median graphs and triangle-free graphs, SIAM J. Discrete Math. 12 (1999) 111–118.
[25] J.R. Isbell, Median algebra, Trans. Amer. Math Soc. 260 (1980) 319–362.
[26] S. Klavžar, H.M. Mulder, Median graphs: Characterizations, location theory and related structures, J. Combin. Math. Combin. Comput. 30 (1999) 103–127.
[27] S. Klavžar, H.M. Mulder, R. Škrekovski, An Euler-type formula for median graphs, Discrete Math. 187 (1998) 255–258.
[28] F. Larrion, V. Neumann-Lara, A family of clique divergent graphs with linear growth, Graphs Combin. 13 (1997) 263–266.
[29] N.V.R. Mahadev, T.M. Wang, On uniquely intersectable graphs, Discrete Math. 207 (1999) 149–159.
[30] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, SIAM, Philadelphia, 1999.
[31] H.M. Mulder, The structure of median graphs, Discrete Math. 24 (1978) 197–204.
[32] H.M. Mulder, The Interval Function of a Graph, Mathematical Centre Tracts, vol. 132, Matematisch Centrum, Amsterdam, 1980.
[33] H.M. Mulder, The expansion procedure for graphs, in: R. Bodendiek (Ed.), Contemporary Methods in Graph Theory, B.I.-Wissenschaftsverlag, Manhaim, Wien, Zürich, 1990, pp. 459–477.
[34] C. Peyrat, D.F. Rall, P.J. Slater, On iterated clique graphs with increasing diameters, J. Graph Theory 10 (1986) 167–171.
[35] M.A. Pizaña, Distances and diameters on iterated clique graphs, Disrete Appl. Math. 141 (2004) 255–261.
[36] E. Prisner, Graph Dynamics, Longman, Harlow, 1995.
[37] F.S. Roberts, J.H. Spencer, A characterization of clique graphs, J. Combin. Theory 10 (1971) 102–108.
[38] J.L. Szwarcfiter, A survey on clique graphs, in: B.A. Reed, C. Linhares-Sales (Eds.), Recent Advances in Algorithms and Combinatorics, Springer, New York, 2003, pp. 109–136.