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Essay Review

Sufficient conditions, fields and the calculus of variations

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Von der Bernoullischen Brachistochrone zum Kalibrator-Konzept. Ein historischer Abriß zur Entstehung der Feldtheorie in der Variationsrechnung (hinreichende Bedingungen in der Variationsrechnung)
By Rüdiger Thiele. De Diversis Artibus: Collection de Travaux de l'Académie Internationale d'Histoire des Sciences, vol. 80. Turnhout, Belgium (Brepols Publishers). 2007. ISBN 978-2-503-52438-2, 828 pp. €66.50

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The problem of establishing sufficient conditions for the existence of a maximum or minimum was the outstanding problem of the calculus of variations in the 19th century. By the 1860s the investigation of the second variation had been developed into a successful theory by Carl Jacobi (1804–1851), Alfred Clebsch (1833–1872) and Adolph Mayer (1837–1907). A new approach to the subject was initiated by Karl Weierstrass (1815–1897) in his Berlin lectures of the late 1870s and early 1880s. Although Weierstrass's theory was purely analytic, certain aspects of his approach were amenable to geometric treatment, a fact that was made clear in Adolf Kneser's (1862–1930) exposition in his 1900 *Lehrbuch der Variationsrechnung* [Kneser, 1900]. Kneser introduced explicitly the concept of a "field," a set of solution curves to the Euler–Lagrange equations that was essential to the application of Weierstrass's method. In later mathematics Weierstrass's method became identified with the field theoretic approach to the problem of sufficiency. In the same year as Kneser's book, David Hilbert (1862–1943) introduced the concept of the invariant integral that greatly simplified some of the proofs in Weierstrass's theory. It turned out that Hilbert's insight, presented in analytic form, possessed similarities to ideas underlying earlier geometrical work by Eugenio Beltrami (1835–1900) from 1868. Beginning in 1904, the Göttingen researcher Constantin Carathéodory (1873–1950) assembled a complex of ideas that extended Kneser's geometric approach, and provided a way of connecting Hamilton–Jacobi theory to the calculus of variations. During the 1930s the German mathematicians Hermann Weyl (1885–1955) and Hermann Boerner (1906–1982) and the Belgian researcher Théophile Lepage (1901–1991) investigated the application of Carathéodory's method to variational problems involving multiple integrals. Boerner became something of an apostle

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for Carathéodory's theory, which he characterized as a "royal road" to the calculus of variations. (In September of 1943 Boerner delivered a lecture at the Oberwolfach Institute titled "Carathéodory's Königsweg ins Herz der Variationsrechnung," a theme that was developed by him in an article published as [Boerner, 1953].) During the 1960s several researchers continued Boerner's line of investigation, including the Leipzig mathematician Rolf Klötzler (b. 1931).

Rüdiger Thiele's book is a detailed and comprehensive history of the field theoretic approach to the calculus of variations. In his narrative the crucial moment in the history occurs with the appendix to Carathéodory's 1904 doctoral dissertation. Carathéodory understood his work to be related to some of Johann Bernoulli's (1667–1748) results, originally obtained by Bernoulli in 1697 and published in 1718. Hence Thiele begins the book with an account of the work of Johann Bernoulli and then follows this with a survey of Carathéodory's theory. The narrative then moves back in time to a detailed study of Weierstrass's methods and their elaboration and publication by investigators at the end of the century. This is followed by a survey of relevant 19th-century research in dynamics

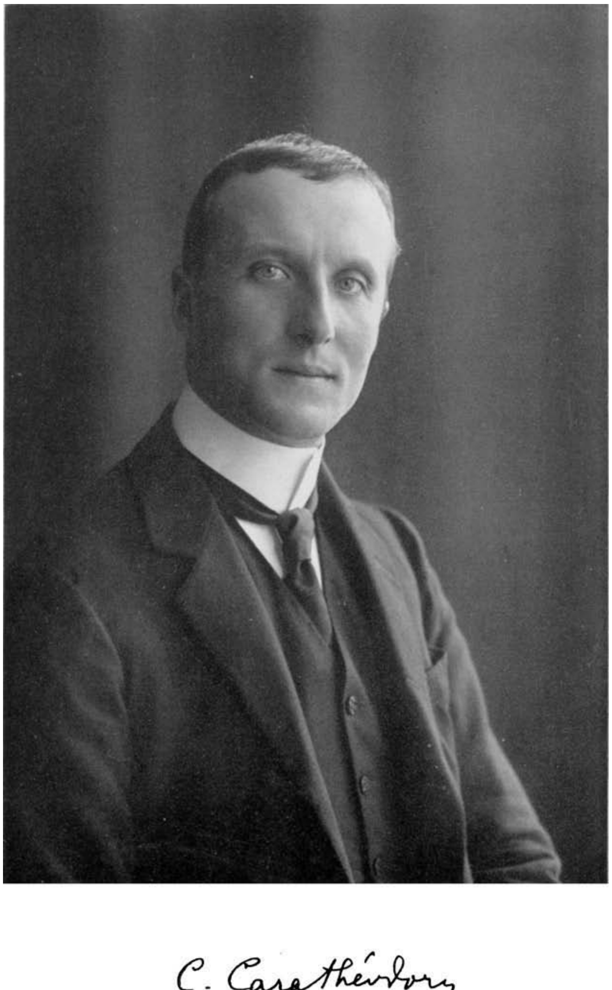

Fig. 1. Constantin Carathéodory (1873–1950).

and differential geometry, a prominent figure here being Kneser. A major chapter is devoted to Hilbert's contributions to the calculus of variations and related subjects, and to Mayer's research on fields. There is also an account of work on the field concept in the first decade of the new century by mathematicians in the United States, Germany (mainly the circle around Hilbert), France and Austria. Carathéodory's theory being seen in some sense as the natural culmination of the Weierstrass–Kneser–Hilbert development, the last part of the book examines how this theory was extended by such researchers as Boerner and Lepage. The term calibrator in the title refers to invariant integrals defined on differential manifolds, objects that occur in parts of recent abstract variational analysis in which some of Carathéodory's original ideas have remained important.

The organization of Thiele's book and its valuation of the theory involve a definite perspective on the history of the calculus of variations. Weierstrass's method as modified by Hilbert constitutes the right approach to the problem of sufficiency, and Carathéodory's theory succeeds in a definitive and natural way in connecting this method to a complementary geometric understanding of the subject. The sense of the subject's historical development is a fairly typical reflection of mathematical modernism, of the implicit belief widely held in the first part of the 20th century that the different parts of mathematics had reached their final and natural form.

Carathéodory (Fig. 1) in his dissertation did not illustrate his method with concrete examples, working instead at a somewhat more general level of analysis. Although his method was analytic, the following simple example may help to illustrate some aspects of his theory. Suppose that space is filled with an optical medium whose index of refraction varies from point to point. The transmission of light in the medium is governed by the variational principle of least time. The family of curves in Fig. 2 originating at the origin are the paths of the rays of light that emanate into the medium from a source located at the origin. In this example the index of refraction decreases with increasing distance below the horizontal y-axis. The set of curved rays constitute a field of extremals for the variational problem. The integral expression for the time taken from the origin to y along a

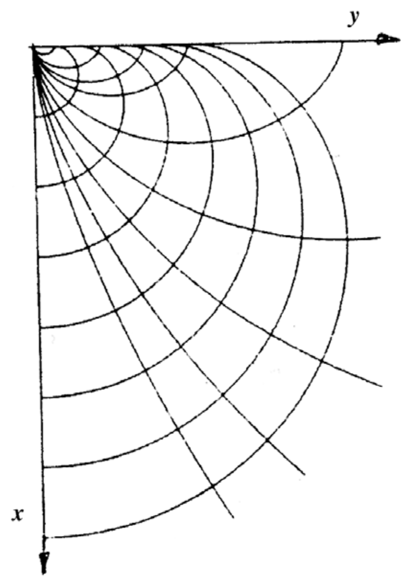

Abb. 1.8. Synchronenschar (orthogonale Trajektorien der Extremalen)

Fig. 2. Figure 1 from p. 49 of Thiele's book.

curved ray is known as a field function. Take some given time and consider the locus of points on the rays reached in this time. The curve formed by this locus is a curve of equal time and was called a synchrone by Johann Bernoulli. Physically, the synchrone is the wave front at the given time of the rays emanating from the origin. It turns out that the synchronal curve cuts each of the field curves orthogonally, a fact that was recognized by Christiaan Huygens (1629–1695) in his optical investigations. The family of synchrones make up what Carathéodory called a geodetically equidistant family of curves with respect to the variational problem defined by the principle of least time. In Carathéodory's analytic development the following are connected:

- The field of extremals consisting of the solution curves to the Euler–Lagrange differential equation;
- The Hamilton–Jacobi partial-differential equation satisfied by the field function;
- The lines defined by setting the field function equal to a constant, these lines being transversal (i.e. normal) to the extremals;
- The expression that appears in the integrand of the Hilbert invariant integral, and the Legendre and Weierstrass necessary conditions.

In order to provide some indication of the analytic character of Carathéodory's theory we show that a curve defined by setting the field function equal to a constant is normal to the extremals. We have the field function $\phi(x,y) = \int_0^x f(x,y,y')dx$, an integral that is understood to be evaluated along the minimizing or maximizing arc from the origin to (x,y). Setting $\phi(x,y) = \lambda$ where λ is a parameter, we differentiate to obtain $\phi_x dx + \phi_y p dx = d\lambda$, where ϕ_x and ϕ_y are the partial derivatives of ϕ with respect to x and y. The integrand $f\,dx$ is then equal to $\frac{f\,d\lambda}{\phi_x + p\phi_y}$, where $p = y' = \frac{dy}{dx}$. For an extremal or curve of "steepest descent" the derivative with respect to p of this last expression is equal to zero:

$$\frac{d\left(\frac{f\,d\lambda}{\phi_x + p\phi_y}\right)}{dp} = 0, \tag{1}$$

or,

$$f_p(\phi_x + p\phi_y) - f\phi_y = 0, \tag{2}$$

where f_p is the partial derivative of $f(x,y,p)$ with respect to p. It is possible to show from (2) that the Euler–Lagrange equation holds for the given arc. Using this fact, one can differentiate the original field function $\phi(x,y) = \int_0^x f(x,y,y')dx$ with respect to x to obtain the relation $f = \phi_x + p\phi_y$. Combining this relation with (2) there follows

$$\begin{aligned} f &= \phi_x + p\phi_y, \\ f_p &= \phi_y. \end{aligned} \tag{3}$$

The condition that the extremal curve be normal to the curve $\phi(x,y) = const.$ at (x,y) is

$$f + (q-p)f_p = 0, \tag{4}$$

where q is the slope of $\phi(x,y) = const.$ at (x,y). Because $q = -\frac{\phi_x}{\phi_y}$, (4) becomes

$$f - \left(\frac{\phi_x}{\phi_y} + p\right)f_p = 0. \tag{5}$$

It is clear that (5) follows from (3). Hence the extremal is normal or transversal to the curve $\phi(x,y) = const.$

An aspect of the history that is not altogether straightforward and merits some discussion concerns the relationship of Johann Bernoulli's research to that of Carathéodory. At the conclusion of a memoir published in 1718, Bernoulli presented two demonstrations that the cycloid is the curve of quickest descent in the brachistochrone problem [Bernoulli, 1718]. He referred to these as his "analytic" and "synthetic" solutions. From a modern perspective, the first solution is equivalent to a derivation of the differential equation of the cycloid as a necessary condition, while the second is a proof that the cycloid results in a smaller time than neighboring comparison arcs. Historian Herman Goldstine (1908–2004) characterized the second solution as the first "sufficiency proof" in the history of the calculus of variations and praised it for its elegance [Goldstine, 1981, 66], an evaluation echoed by Thiele (p. 41) in his account. It is worth noting as a general logical point that all verifications in mathematics can be construed as sufficiency proofs.[1] However, in the calculus of variations such verifications even for very simple examples can give rise to questions of considerable difficulty and subtlety.

In his analytic solution Johann Bernoulli worked with an infinitesimal element of the variational integral and minimized this element using the ordinary methods of the calculus. In this aspect of Bernoulli's method, Carathéodory seemed to find the essence of Johann's theory, and the extent that he regarded his own work to be a continuation of Bernoulli's was apparently because of this belief. Thus Carathéodory's account in his dissertation of Bernoulli's 1718 paper was primarily a description of the latter's analytic solution; it is only at the end of his account and only in a few sentences that Carathéodory observed that the solution may be verified directly. In a paper published one year later [Carathéodory, 1905, 162] Carathéodory stated explicitly that Bernoulli's method consisted of reducing the problem to one of ordinary calculus; this was the approach followed by Bernoulli in his analytic solution and also followed by Carathéodory in his investigation.

In his dissertation Carathéodory obtained the differential equation for the general variational problem by ordinary differentiation of an infinitesimal element of the variational integrand, just as Bernoulli had done for the brachistochrone in his analytic solution of 1718. It should be noted that there were important differences: Bernoulli differentiated with respect to a variable given in terms of the radius of curvature at a point, while Carathéodory differentiated with respect to the general slope variable p. Also Carathéodory understood his method to be direct (one of his early papers [Carathéodory, 1908] was titled "Sur une méthode directe du calcul des variations"), and it would indeed be today grouped with what are called direct methods in the calculus of variations. By contrast, for Bernoulli it was the synthetic solution that was direct; the analytic solution would have been termed indirect because the answer (assuming there was one) was not exhibited at the outset but found by an analytic process.

In a historical article published in 1937 Carathéodory commented on the origins of his own early work in the calculus of variations. In his original dissertation he seemed to be saying that his method was a development of Johann Bernoulli's analytic solution of 1718, a fact that was evident in the procedure actually followed in the dissertation.

[1] There were results before Johann Bernoulli's that could be understood as sufficiency proofs in the calculus of variations. For example, the ray of light that satisfies the law of reflection obeys the principle that light travels the path of shortest distance. Beginning with a broken ray obeying the reflection law (angle of incidence equals angle of reflection), one can show by elementary geometry that the distance along this ray is smaller than the distance along neighboring broken rays. Such an argument is a sufficiency proof and appears in Hero of Alexandria's (ca. 10–75) *Catoptrics*.

However, in 1937 [Carathéodory, 1937, 98] he suggested that it was Johann's synthetic solution that inspired him. A few years later in another historical essay [Carathéodory, 1945, 111] he reported that this solution contained the germ of the modern ideas of the calculus of variations. Thiele (p. 42) accepts this opinion, stating more specifically that the germ of field theory can be found in Johann's synthetic solution. Thiele implies that the solution may even have been of philosophical significance, in advance of its time, since the proof involved the identification of minimality as a property and a demonstration that the cycloid possessed this property.

There are nevertheless some difficulties with Carathéodory's claim. First, it is not clear from a detailed examination of the content of Bernoulli's synthetic solution how the idea of a field is present there. Carathéodory [1937, 106] and Herman Goldstine [1981, 387] claim that Carathéodory's method may be obtained from Johann Bernoulli's synthetic solution if the normals to the brachistochrone are replaced by synchronal curves intersecting this curve. Unfortunately, no details are provided on how this would be done, and it is not at all clear to me that it is possible without moving completely beyond what is in Bernoulli and simply duplicating Carathéodory's analysis. The cycloid is not exhibited as a member of a family of solution curves. The proof depends on a special property of the radius of curvature of the cycloid, a property that is not generalizeable to other curves. Second, it is clear from all of Carathéodory's early writings and from what he actually did during this early period that, if he was indeed following a much older historical precedent, he was following Bernoulli's analytic solution (i.e. derivation of a differential equation as a necessary condition).

In order to elaborate on the claim that the idea of field theory and the inspiration for Carathéodory's theory may be found in Johann Bernoulli's writings, Thiele turns to a paper published by Bernoulli in 1697. It is in this paper that Bernoulli introduced his famous analogy of the descending body in the brachistochrone problem with the emission of light through a medium of variable refractive index [Bernoulli, 1697]. The latter provides an elementary geometric illustration of some of the ideas in Carathéodory's theory, hence our presentation of it as an example above. Unfortunately, Carathéodory in his formative mathematical writings referred only to Johann's paper of 1718, and not once to the article of 1697. The Göttingen researcher did mention Huygens, which would indicate he was familiar with some of the more geometric work of the period. In addition, Johann Bernoulli's optical analogy was well known at the time because of its description in Ernst Mach's widely read 1883 book on the historical development of mechanics. Nevertheless, Carathéodory was very specific that the source of his inspiration was the conclusion of Johann's 1718 memoir. Thiele (p. 52, bottom) suggests that the analytic character of this memoir captured Carathéodory's attention; it is also true that the field concept was less apparent in the memoir than it was in Johann's 1697 investigation.

In his historical writings Carathéodory's somewhat personal perspective is apparent from his overall opinion of Johann Bernoulli's 1718 memoir. The large majority of the memoir was devoted to an exposition of the methods of Johann's late brother Jakob (1654–1705); the resulting account was historically and mathematically a very important contribution to 18th-century analysis and provided the beginning for Leonhard Euler's (1707–1783) seminal researches on what later became known as the calculus of variations. Carathéodory characterizes the memoir as "a rather tedious tract" [Carathéodory, 1937, 98] redeemed only by the appearance in its final pages of the synthetic solution. One might also take exception with Carathéodory's estimate of the elegance of this solution [Carathéodory, 1904, 70]. To me it seems interesting but rather complicated; while it does what it is supposed to do, a reader coming to it without any knowledge of the identity of its author may not find it elegant. Certainly no one before Carathéodory was impressed enough to even mention it. The concluding section of the 1718 paper acted as a stimulus to Carathéodory's early thinking, and the analytic solution presented there may have given him the idea of using ordinary differentiation to maximize or minimize the differential element of the variational integrand. Beyond this, the value he attributed to it may have been more a reflection of his subjective creative process than of what can sensibly be said to exist in the historical record.

Another general point concerning the history being recounted here relates to the assessment of Carathéodory's theory by some of the researchers who followed in his footsteps. Boerner believed that Carathéodory [1935] opened up a royal road to the calculus of variations because of the way in which his theory linked many of the fundamental ideas of the subject. In 1909 Oskar Bolza (1857–1942) had published a major work on the calculus of variations which remains today a valuable guide to the subject; while Carathéodory's theory (then fairly recent) was mentioned [Bolza, 1909, 140-3], the book showed that it is possible to present a coherent and hugely detailed exposition of the subject without invoking this theory as a magic key. Several decades later the American authority Gilbert Bliss (1876–1951) published a rather succinct account [Bliss, 1946, 77–80] of Carathéodory's theory in the course of a comprehensive textbook, referring to it as "a very interesting approach to the calculus of variations" [Bliss, 1946, 77]. The viewpoints of Boerner and Bliss express a certain underlying difference of opinion concerning the centrality of Carathéodory's theory. Boerner's perspective may in part have reflected his excitement and the instinctive tendency of the working mathematician to elevate his own research.

There is no doubt that Carathéodory's theory is a beautiful part of modern mathematics, not altogether widely known or appreciated today. Thiele's book is a masterful account both of its history as well as of the wider origin and development of field theoretic methods in the calculus of variations. Exhaustive in its coverage, the book is filled with interesting observations and informative comments on many different aspects of its subject. The appendix is a diverting essay discussing in an accessible way the many uses of the field concept in variational mathematics. *Von der Bernoullischen Brachistochrone zum Kalibrator-Konzept* is the product of decades of historical investigation and reflection on the part of the author, and will become a standard work of reference for the history of modern analysis.

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