# On Jacobi's Remarkable Curve Theorem 

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FOR DIRK STRUIK ON HIS 100TH BIRTHDAY

One of the prettiest results in the global theory of curves is a theorem of Jacobi (1842): The spherical image of the normal directions along a closed differentiable curve in space divides the unit sphere into regions of equal area. The statement of this theorem is an afterthought to a paper in which Jacobi responds to the published correction by Thomas Clausen (1842) of an earlier paper, Jacobi (1836). In this note the context for this theorem and its proof are presented as well as a discussion of the 'error' corrected by Clausen. © 1994 Academic Press, Inc.

Einer der schönsten Sätze der globalen Theorie von Kurven ist der Satz von Jacobi (1842): Das sphärische Bild der normallen Richtungen auf einer geschlosser differenzierbaren Raumcurve teilt die Kugelfläche in zwei gleiche Teile. Dieser Satz erscheint als Nachtrag zu einer Arbeit, in der Jacobi auf die veröffentlichte Berichtigung Thomas Clausens (1842) einer vorherigen Publikation Jacobis (1836) entgegnet. Die vorliegende Arbeit beschäftigt sich mit dem Zusammenhang dieses Satz und seines Beweis, ebenso wie mit dem 'Fehler,' den Clausen berichtigt hat. O 1994 Academic Press, Inc.

Un des plus beaux résultats de la théorie globale des courbes est celui de Jacobi (1842): l'image sphérique des directions normales d'une courbe fermée et différentiable divise la sphère-unité en deux régions égales. Ce théorème semble être une réflexion après coup d'un article dans lequel Jacobi répond à une correction publiée par Thomas Clausen (1842) d'un travail de Jacobi (1836). Ici, je mets ce théorème et sa démonstration en contexte, et je mets en discussion la "faute" corrigée par Clausen. 1994 Academic Press. Inc.
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In this work JACOBI touches on differential geometry in the large. Another example is his theorem that the spherical image of the principal normals to a closed continuously curved space curve divides the surface of a sphere into two equal parts, a corollary of GAUSS' theorem on geodesic triangles.
D. J. Struik, Isis 1934

## I. INTRODUCTION

In his Disquisitiones generales circa superficies curvas [6, Sect. 20] Gauss proved the following classic theorem: a triangle $\triangle X Y Z$ with sides made up of geodesics on a surface $S$ in space $\mathbb{R}^{3}$ satisfies the relation

$$
\angle X+\angle Y+\angle Z-\pi=\iint_{\Delta X Y Z} K d A=\operatorname{area}(A B C)
$$

where $K$ denotes the Gaussian curvature, and $A B C$ is the region on a sphere of radius 1 bounded by the curves $A B, B C$, and $C A$ which are determined by radii parallel to the normals to the surface $S$ along the curves $X Y, Y Z$, and $Z X$, respectively. This theorem, already remarkable, attracted the attention of the generation of geometers following Gauss (see [13]). In 1848 Ossian Bonnet [2] gave a more general formulation for triangles whose sides are arbitrary curves on a surface. This result involved the geodesic curvature, the key to the Gauss-Bonnet theorem.

Between the publication of these two famous results, C. G. J. Jacobi proved a different sort of generalization of Gauss's theorem in Demonstratio et amplificatio nova theorematis Gaussiani de curvatura integra trianguli in data superficie e lineis brevissimis formati [7]. Suppose one dispenses with the surface $S$ and considers a triangle $\triangle X Y Z$ made up of three space curves. The unit normal vectors along each curve, that is, the unit vector in the direction of the radius of the osculating circle at each point on the curve, determine a triangle $A B C$ of curves on the unit sphere if the space curves have normals that agree at the vertices. Under these assumptions Jacobi proved the following result.

Theorem I. (Theorem I of [7] and Theorem II of [8]). The area of the triangle $A B C$ on the sphere of radius 1 satisfies

$$
\operatorname{area}(A B C)=\angle X+\angle Y+\angle Z-\pi
$$

The purpose of this article is to reconsider Jacobi's proofs of this theorem. One reason for doing so concerns the historical context that prompted Jacobi's second paper on this subject. In the opening sentence of that paper [8] he wrote:

> In No. 457 of the Astronomische Nachrichten Mr. Clausen put forward some unfounded doubt about the correctness of a theorem proved by me in the 16 th volume of Crelle's Journal of which a famous theorem of Gauss is a special case.

Thus, this paper was written as part of a public mathematical dispute between Jacobi and Thomas Clausen (1801-1885), and we examine the nature of this conflict of opinion below.

A second reason to look at these papers by Jacobi arises from their intrinsic mathematical interest, as they provide a snapshot of the methods employed by differential geometers during this pivotal period. The first half of the 19th century brought a groundswell of activity in geometry. Jacobi, in particular, lectured often throughout his career on the analytic theory of curves and surfaces in space. What emerges from a close reading of his papers [7] and [8] is not an extension of Gauss's methods, but an attempt by Jacobi to get at the geometric principles that underlie Gauss's result and its generalization. In these papers Jacobi not only proved a beautiful result on space curves but demonstrated the power of methods of spherical trigonometry to obtain new results. This is differential geometry driven by spherical geometry together with infinitesimal arguments. The methods go back to Jacobi's idol, Leonhard Euler, and Euler's study of duality in spherical trigonometry (see, e.g., [5]).

The contrast between the analytic and synthetic approaches to geometry is clear
in the mathematical argument between Jacobi and Clausen as well as from views expressed by Jacobi in [7].

## II. THE MATHEMATICAL DISPUTE

Jacobi opens the paper [7] with an argument for Theorem I:
To consider arbitrary curves, however, it may be granted that they are geodesics on some surface: . . . .

From this statement he concludes that Theorem I follows from Gauss's result. The rest of the paper is occupied with Jacobi's real point-to reduce Theorem I, and hence Gauss's theorem, to another key theorem concerning the normal images of space curves, a result connected with the images of space curves provable by using spherical trigonometry and duality. Jacobi describes this main result, Theorem II of [7], as one "quibus genuina eius indoles melius prespicitur," that is, from which the innate quality is seen better. The details of his proof are given in Sections III and IV. Our presentation of the results follows the reverse order of Jacobi's argument, however. He begins by attempting to establish Theorem I via an argument that utilizes Gauss's Theorem and then goes on to review some results from spherical trigonometry. Through a series of substitutions and analyses of certain angles, he obtains a formula which forms the crux of his Theorem II and which he proves independently of Gauss's Theorem. Moreover, Jacobi closes the paper with an analytic proof of Theorem II, a tour de force in the spirit of Gauss's Disquisitiones that he prefaces with the remarks:

> If Theorem II . . . needs a proof by analytic formulas one falls into rather complicated calculations . . they become so troublesome that one easily shrinks back from them.

Given Jacobi's prowess at computation and his obvious understanding of Gauss's methods, it is fairly certain that the inclusion of this second proof was meant to support his search for appropriate foundations for the proof of Gauss's famous theorem and his own generalization of it.

In fact, Jacobi's appeal to Gauss's result to prove Theorem I was incorrect. Six years after the publication of [7], Thomas Clausen, the self-taught mathematician and astronomer, published Berichtigung eines von Jacobi aufgestellten Theorems [4], in which he gave an argument using the methods of Gauss's Disquisitiones to show that it may not be granted that even a pair of space curves with common normal at their intersection can be thought to lie on a surface as a pair of geodesics.

While hardly a well-known figure today, Clausen produced a number of significant mathematical contributions that earned him the respect of Gauss, Bessel, and Jacobi. His best known results concerned the factorization of the seventh Fermat number, $2^{2^{6}}+1$ (by a method still unpublished), results on the Bernoulli numbers (shared with von Staudt), and his computations of the paths of comets for which he received a prize from the Copenhagen Academy in 1840. As related in Biermann's biographical essay [1], Clausen's professional life was made difficult through bouts with mental illness. He spent the years 1824 to 1828 at the Altona

Astronomical Observatory as an assistant to H. C. Schumacher, editor of Astronomische Nachrichten and a close colleague of Gauss. After a falling out with Schumacher, Clausen worked under J. von Utzschneider at the Optical Institute in Munich, where he was able to develop his mathematical and astronomical ideas freely. In 1840, after several years of difficulty with his physical and mental health, Clausen returned to Altona and Schumacher without any means of support. Schumacher made many appeals to well-placed mathematicians for the support of a position for Clausen. He was finally called to Dorpat in 1842 as observateur and later succeeded J. H. Mädler as director of the observatory.

It was during this particularly vulnerable period, 1840-1842, that Clausen published his correction of Jacobi's paper [7]. By that time, Schumacher had prompted Bessel to ask Jacobi to seek support for Clausen while he was in Altona. Jacobi succeeded in securing 250 Talers from the Berlin Academy for Clausen in order for him to complete certain calculations associated with Jacobi's perturbation theory. Nothing seems to have come from this commission. Jacobi related to his brother (letter of February 28, 1841 [9]) that Clausen suffered severe headaches which stood in the way of his computations.

On the receipt of Clausen's paper [4] Schumacher wrote Gauss to inquire about its appropriateness for publication in the Astronomische Nachrichten. Gauss replied (September 3, 1842) that Clausen's refutation of the "alleged generalization" was completely founded and appropriate (see [1]). In short order (between September and October 1842) Jacobi replied with a new proof of Theorem I [8], this time relying solely on area relations in spherical geometry. The proof develops directly without the searching rhetoric of reduction that is found in [7]. Jacobi sought to establish not only the correctness of his results, but the correctness of his approach.

At the end of [8] Jacobi called attention to a corollary of Theorem I:
Corollary. If an arbitrary continuously curved closed curve is given in space and one takes radii from the center of a sphere parallel to the radii of curvature of the curve, the curve on the sphere so constructed divides the sphere into two equal parts.

The corollary, stated without proof, follows from the theorem by choosing three points on the closed curve and calling them vertices of a triangle of space curves. This is the only result from this pair of interesting papers that has remained in the standard literature on differential geometry, where it is usually presented in the context of the Gauss-Bonnet Theorem, that is, through analytic formulas (see, e.g., [14, 407-409]).

In Sections III and IV below we present Jacobi's first proof ([7]) of Theorem I which, in fact, is correct. By reversing the rhetoric, we expose the "innate qualities" at the foundations of geometry that Jacobi judged to be the "fount" of Gauss's result and his generalization.

## III. POLAR RECIPROCITY

"Quae ea est reciprocitas"-What is reciprocity? wrote Jacobi in his preparatory remarks to the proof of his main theorem in [7]. Given a great circle on a unit sphere, its pole is the point on the sphere determined by the line through the
center of the sphere in the direction normal to the plane that gives the great circle. This can be one of two antipodal points and the choice is made by orienting the great circle and choosing the binormal direction (see [12]). Given a spherical triangle $\triangle A B C$ made up of great circle segments, the poles of each side determine another spherical triangle $\Delta \alpha \beta \gamma$ where $\alpha$ is the pole of side $B C$, etc. This triangle is called the polar triangle associated to $\triangle A B C$ and it has the following properties that are recalled by Jacobi (see [3] for a discussion of polar reciprocity from Euler onward):

1. The polar triangle of $\triangle \alpha \beta \gamma$ is $\triangle A B C$.
2. The lengths of the sides of $\triangle \alpha \beta \gamma$ are given by

$$
\alpha \beta=\pi-\angle B C A ; \quad \alpha \gamma=\pi-\angle A B C ; \quad \beta \gamma=\pi-\angle C A B
$$

More generally, an angle between great circle arcs has polar reciprocal equal to an arc whose length is the supplement of the given angle, and vice versa.

It follows immediately from these properties and the formula for the area of a spherical triangle that

$$
\operatorname{area}(\triangle A B C)+\text { circumference }(\triangle \alpha \beta \gamma)=2 \pi
$$

We can extend this relation to spherical $n$-gons and their polar reciprocals, giving the sum of the area of the $n$-gon and the circumference of its polar reciprocal equal to $2 \pi$. Given an arbitrary curve on the sphere, its polar reciprocal is another curve with each point corresponding to the pole of the great circle determined by a point on the given curve and the tangent to the curve at that point.

Let $\lambda:[0, l] \rightarrow \mathbb{R}^{3}$ denote a unit speed space curve with $X=\lambda(0), Y=\lambda(l)$, and denote by $X Y$ the image of $\lambda$. For convenience we fix a sphere of radius 1 , namely $S^{2}$ with center $O=(0,0,0)$. The tangents at each point on $X Y$ determine a curve $a b$ on $S^{2}$. The basic construction of Jacobi is the normal image of $X Y$ which is given by the curve $A B$ on $S^{2}$ of points with radii parallel to the radii of curvature of $X Y$. The following remark links these curves and lies at the heart of Jacobi's proof of theorem I:

The curve of tangents $a b$ is the polar reciprocal of the normal image $A B$ of the curve $X Y$.

## IV. JACOBI'S MAIN THEOREM

Let $\lambda:[0, l] \rightarrow \mathbb{R}^{3}$ be as above with the trace of $\lambda$ denoted by $X Y$. The analytic description of the normal directions $\mathbf{N}(s)$ along $X Y$ given by the radius of the osculating circle at each point is fixed by $\lambda^{\prime \prime}(s)=\kappa(s) \mathbf{N}(s)$ with $\kappa(s)>0$ and $\mathbf{N}(s)$ of length one.

The osculating plane along $X Y$ is the plane spanned by the tangent $\lambda^{\prime}(s)$ and the normal $\mathbf{N}(s)$. Jacobi defines the plane of the osculating radii to be the plane spanned by the normal $\mathbf{N}(s)$ and its derivative $(d / d s)(\mathbf{N}(s))$. Jacobi's main theorem is the following result:

ThEOREM II (in [7]). Consider the curve $a b$ on $S^{2}$ determined by the tangents
$\lambda^{\prime}(s)$ along $X Y$. The length of $a b$ is equal to the difference of the angles which the plane of the osculating radii makes with the osculating plane at the extremities $X$ and $Y$.

Let $0<s_{1}<l$ and write $p=\lambda^{\prime}\left(s_{1}\right), q=\lambda^{\prime}\left(s_{1}+d s_{1}\right)=\lambda^{\prime}\left(s_{2}\right), r=\lambda^{\prime}\left(s_{2}+\right.$ $\left.d s_{2}\right)=\lambda^{\prime}\left(s_{3}\right)$, etc. Let $P=\mathbf{N}\left(s_{1}\right), Q=\mathbf{N}\left(s_{2}\right), R=\mathbf{N}\left(s_{3}\right)$, etc. We have then

$$
p P=q Q=r R=\pi / 2
$$

For $d s_{i}$ infinitesimal the arc of the great circle $p q$ goes through $P, q r$ through $Q$, etc. With this notation the osculating plane to $X Y$ at $\lambda\left(s_{1}\right)$ is $O p P=O q P$, at $\lambda\left(s_{2}\right)$ it is $O p Q=\operatorname{Or} Q$, etc., where $O$ denotes the center of the sphere. Let $Q Q^{\prime}$ be the direction $(d / d s)(\mathbf{N}(s))$ at $s=s_{1}$; the plane of the osculating radii at $\lambda\left(s_{1}\right)$ is $O P Q=O Q Q^{\prime}$. We can then write the angle between the osculating plane and the plane of the osculating radii as $\angle q P Q=\angle p P Q$. The differential of that angle is given by $\angle q Q R-\angle q P Q$ :


A typical element of arc length along $a b$ is given by $p q$. Jacobi next showed that summing $p q, q r, r s, \ldots$ is the same as summing the angles $\angle q Q R-\angle q P Q$.

By the earlier discussion the curve that is polar reciprocal to the normal image of $X Y$ is the curve of the tangent directions $a b$. By the polar correspondence $\angle P Q R$ has polar reciprocal given by the great circle segment $p q$ of length $\pi-$ $\angle P Q R=\angle R Q Q^{\prime}$. Thus to sum $p q, q r, \ldots$ we sum (integrate) $\angle R Q Q^{\prime}$.

In the limits involved in integration, it suffices to show that if $p q, q r, \ldots$ are of first order, then $\angle q P Q-\angle q Q Q^{\prime}$ is of second order. This implies that $\angle q Q R-\angle R Q Q^{\prime}-\angle q Q Q^{\prime}$ is of second order or that $\angle R Q Q^{\prime}=\angle q Q R-$ $\angle q Q Q^{\prime}=\angle q Q R-\angle q P Q$ up to second order.

Consider the spherical triangle $\triangle q P Q$. By the spherical law of sines we have

$$
\frac{\sin (\angle q P Q)}{\sin q Q}=\frac{\sin (\angle q Q P)}{\sin q P}
$$

Since $\angle q Q Q^{\prime}=\pi-\angle q Q P$, we have that $\sin \left(\angle q Q Q^{\prime}\right)=\sin (\angle q Q P)$. Since $p P=\pi / 2$ we have

$$
q P=p P-p q=\pi / 2-p q .
$$

It follows that

$$
\frac{\sin (\angle q P Q)-\sin \left(\angle q Q Q^{\prime}\right)}{\sin (\angle q P Q)}=1-\frac{\sin (\pi / 2-p q)}{\sin (q Q)}=\frac{1-\cos (p q)}{1}=\frac{2 \sin ^{2}(1 / 2 p q)}{1}
$$

Thus if $p q$ is of first order, then $\sin (\angle q P Q)-\sin \left(\angle q Q Q^{\prime}\right)$ is of second order. This is Jacobi's proof of the theorem.

Theorem I of the Introduction follows from Theorem II. Suppose that $\triangle X Y Z$ is a triangle of space curves satisfying the condition that the radii of curvature of each pair of curves point in the same direction at their intersection. Let $A B C$ denote the normal images of these curves on $S^{2}$ with $A B=\mathbf{N}(X Y), B C=\mathbf{N}(Y Z)$, and $C A=\mathbf{N}(Z X)$.

The polar reciprocal of $A B C$ is a hexagon $a a_{1} b b_{1} c c_{1}$, where $a a_{1}, b b_{1}$, and $c c_{1}$ are arcs of great circles of length $\pi-\angle B A C, \pi-\angle A B C$, and $\pi-B C A$, respectively, and the curves $a_{1} b, b_{1} c$, and $c_{1} a$ correspond to the tangent images of $X Y, Y Z$, and $Z A$, respectively.

By the discussion of polar reciprocity we know that

$$
\operatorname{area}(A B C)+\operatorname{circumference}\left(a a_{1} b b_{1} c c_{1}\right)=2 \pi
$$

Now introduce angles

$$
\begin{aligned}
& x^{\prime}=\text { angle between } \lambda_{X Y}^{\prime}(0) \text { and } \frac{d N_{X Y}}{d s}(0) \\
& x^{\prime \prime}=\text { angle between } \lambda_{X Z}^{\prime}(0) \text { and } \frac{d N_{X Z}}{d s}(0)
\end{aligned}
$$


where $\lambda_{X Y}(s)$ is a unit speed parametrization of the curve $X Y$ and $\lambda_{X Z}(s)$ is a unit speed parametrization of the curve $X Z$. Since $\angle Y X Z+x^{\prime \prime}=\angle B A C+x^{\prime}$, by Theorem II we have that

$$
\angle A-\angle X=\angle B A C-\angle Y X Z=x^{\prime}-x^{\prime \prime}=\text { length }\left(a_{1} b\right)
$$

If we define angles $y^{\prime}, y^{\prime \prime}, z^{\prime}$, and $z^{\prime \prime}$ in a similar manner, then Theorem II implies that

$$
\text { length }\left(b_{1} c\right)=y^{\prime}-y^{\prime \prime}=\angle B-\angle Y, \quad \text { length }\left(c_{1} a\right)=z^{\prime}-z^{\prime \prime}=\angle C-\angle Z
$$

It follows that

$$
\begin{aligned}
& \text { circumference }\left(a a_{1} b b_{1} c c_{1}\right) \\
= & \pi-\angle A+\angle A-\angle X+\pi-\angle B+\angle B-\angle Y+\pi-\angle C+\angle C-\angle Z \\
= & 3 \pi-\angle X-\angle Y-\angle Z .
\end{aligned}
$$

Inserting this identity into our formula from polar reciprocity we obtain Theorem I:

$$
\operatorname{area}(A B C)=\angle X+\angle Y+\angle Z-\pi
$$

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