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Automatic All-Hex Topology Operations Using Edge Valence Prediction with Application to Localized Coarsening

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Master of Science

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ABSTRACT

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In this work, we propose using edge valence as a quality predictor when used as a driver for adapting all hexahedral meshes. Edge valence, for hexahedra, is defined as the number of faces attached to an edge. It has shown to be a more reliable quality predictor than node valence for hexahedral meshes. An edge valence of 3, 4, or 5 within the volume of a hexahedral mesh has provided at least a positive scaled Jacobian for all observed meshes, without the presence of over constraining geometry. It is often desirable to adapt an existing mesh through sheet operations such as column collapse, sheet insertion, or sheet extraction. Examples of hexahedral mesh adaptation include refining and coarsening. This work presents a general algorithm for a priori prediction of edge valence when used with column collapse and sheet extraction operations. Using the predicted edge valence we present a method for guiding the mesh adaptation procedure which will result in an overall higher quality mesh than when driven by mesh quality alone. Other quality metrics such as the Jacobian are unfit for predictive algorithms because of their heavy dependence on node positioning instead of hex topology. Results have been derived from application of the algorithm towards the localized coarsening process.

Keywords: hexahedral, adaptation, valence

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1. INTRODUCTION

The Finite Element Method (FEM) is a powerful analytical tool used to solve differential equations for many science and engineering applications. While the FEM has been used since the 1950's [1], the proliferation of personal computers and the rapid increase in affordable computational power has made the method much more common for science and engineering applications. Today the FEM has been applied to a variety of numerical problems including stress analysis, fluid dynamics, and structural vibrations.

As part of the FEM, the domain must be discretized into smaller elements. For two dimensional problems, the most common element shapes are triangles and quadrilaterals. For three dimensional problems the most common element shapes are tetrahedra and hexahedra. Each of these different element types has advantages and disadvantages over the other element type. For example, tetrahedral meshing algorithms are more general purpose than hexahedral algorithms but hexahedral elements provide greater accuracy for the same number nodes [2].

The accuracy of the FEM solution is very dependent on the number of nodes and quality of elements in the domain. In the FEM, each element is mapped to a unit element of the same shape. For example, a hexahedron is mapped to the unit cube. Thus, hexahedra with elemental dihedral angles of approximately 90° are considered ideal. These ideal elements will yield the greatest accuracy but dihedral angles much greater than, or less than, the ideal 90° often occur due to topological constraints. Increasing the number of nodes will increase the number of degrees of freedom which will increase the accuracy of the FEM solution. Increasing the number of degrees of freedom will also increase the computational time which may be significant on large meshes or slow processors.

A compromise in node density may be reached by increasing the number of nodes in regions of the mesh where greater accuracy is needed and reducing the number of nodes in regions where less accuracy is needed. Regions that may need greater accuracy include locations of high stress or strain, complex geometry, or a high error count. When generating the initial mesh, the user may know what areas will require a high node density and what areas are appropriate for a lower node density, but often these areas can only be identified after the FEM has been performed and error estimates are obtained [3]. In response to this initial FEM solution, the user may change the mesh by either re-meshing the entire domain or modifying specific regions of the initial mesh.

For hexahedron meshes, many mesh modification techniques include some type of sheet operation[1] [4-7]. A sheet can be defined as a set of contiguous hexahedron with each hexahedron sharing geometrically opposite faces with other hexahedron of the same sheet, as shown in Figure 1-1(a). The intersection of two sheets forms one or more columns, as shown in Figure 1-1(b).

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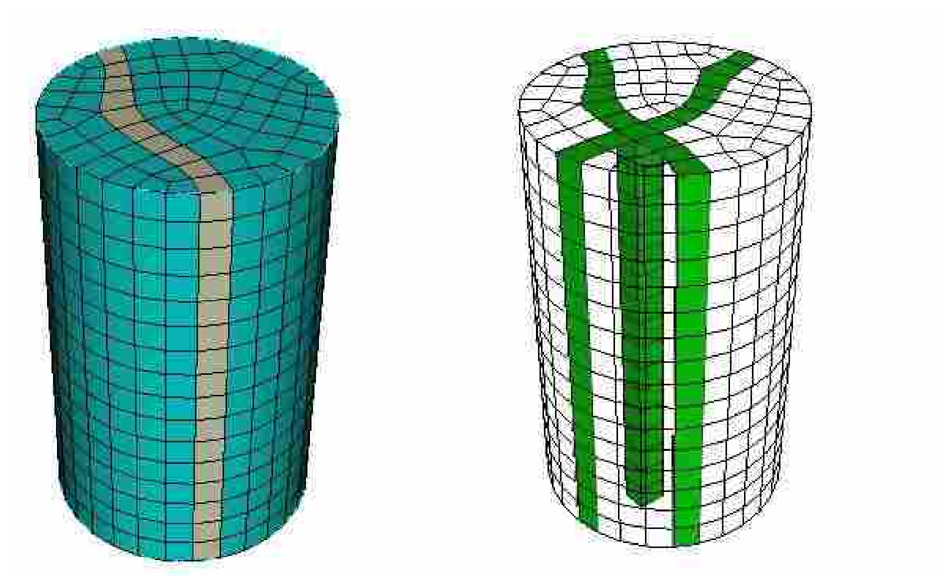
[1] See Appendix for descriptions of hexahedral topology modification operations

The purpose of this work is to develop a method for guiding the localized hexahedral coarsening process based on a reliable quality metric. This method must accurately predict mesh quality for sheet operations without actually performing those operations in order to guide the modification process. This new method must also show an improvement over existing methods.

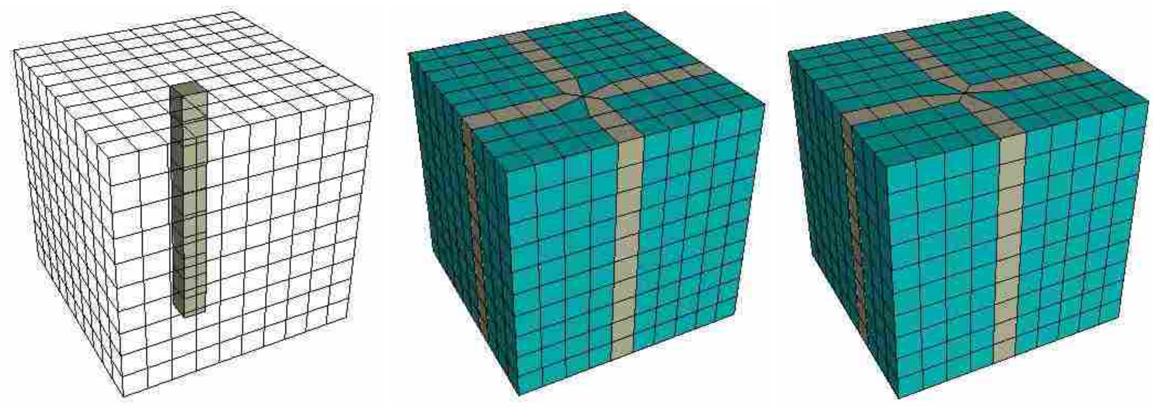


(a) Example of highlighted sheet

(b) Intersection of two sheets defines a column

Figure 1-1: Intersection of two sheets forms a column.

Sheet operations of importance to this work include column collapse, sheet extraction, and pillowing. Column collapse is the process of merging two diagonally opposite edges of a hex to form a single edge. Each column has two possible directions to collapse, each creating a unique sheet as shown in Figure 1-2. Sheet extraction is the process of removing a hexahedral sheet by collapsing the edges that form the sheet as shown in Figure 1-3. Pillowing may be considered the opposite of sheet extraction and is the process of inserting a sheet into a mesh [8]. A pillow may be applied to a contiguous set of hexahedra known as a “shrink” set. This set of hexahedra are reduced in size and pulled away from the rest of the mesh. A new sheet is then inserted into the ensuing gap as shown in Figure 1-4.

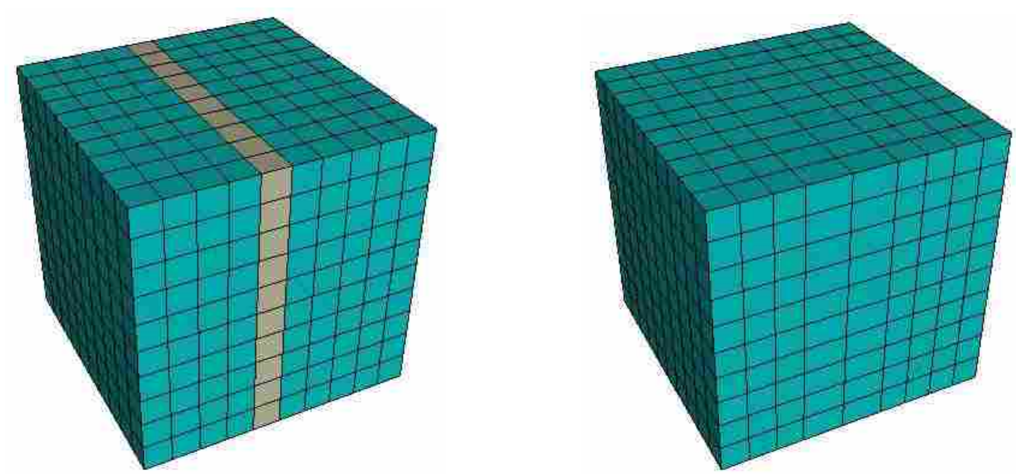


(a) Highlighted column

(b) Column collapsed one direction with effected sheets highlighted

(c) Column collapsed in the other direction with effected sheets highlighted

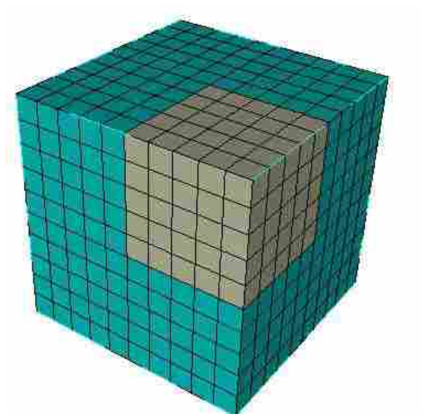
Figure 1-2: Example of column collapse in each direction.



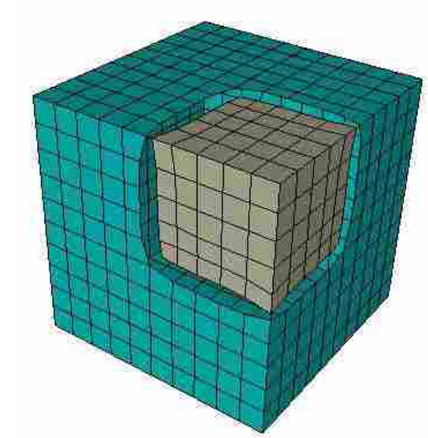
(a) Highlighted hexahedra define the sheet

(b) Final mesh after the sheet is removed

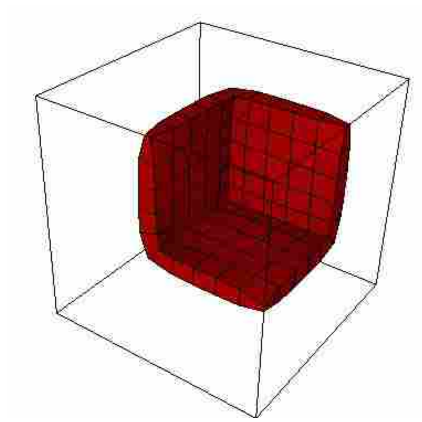
Figure 1-3: Example of sheet extraction.



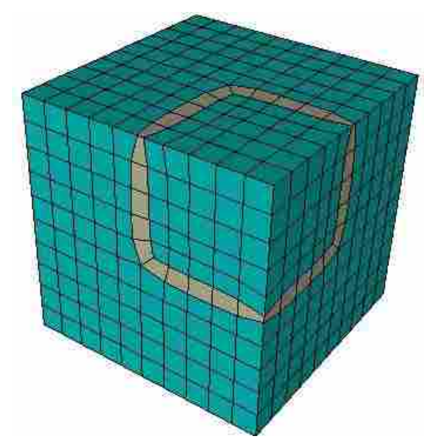
(a) Highlighted region will be pillowed



(b) Region is separated from surrounding mesh



(c) A hexahedron sheet is inserted into the remaining gap



(d) Resulting mesh with pillow highlighted

Figure 1-4: Pillowing procedure.

2. EDGE VALENCE BACKGROUND

2.1 Review of Edge Valence

As previously stated, the accuracy of the FEM is dependent on the quality of the mesh provided. Assessing the quality of a mesh before performing the FEM can be done using one or more of the many available metrics [9-11]. Several of the available metrics such as the scaled Jacobian, shape, and condition number rely on the Jacobian matrix of the element shown as Equation 2-1 with y_n = natural coordinate and x_n = actual coordinate. The natural coordinates are taken from the unit cube while the actual coordinates are taken from the actual element node locations. The Jacobian matrix, computed for each of the nodes of a hexahedral element, provides useful information such as element shape, volume, and orientation. The determinate of the Jacobian matrix is often referred to simply as the element Jacobian.

$$J(x_1,\ldots x_n) = \begin{matrix} \frac{\partial y_1}{\partial x_1} & . & . & . & \frac{\partial y_1}{\partial x_n} \\ . & . & & & . \\ . & & . & & . \\ . & & & . & . \\ \frac{\partial y_n}{\partial x_1} & . & . & . & \frac{\partial y_n}{\partial x_n} \end{matrix} \qquad (2\text{-}1)$$

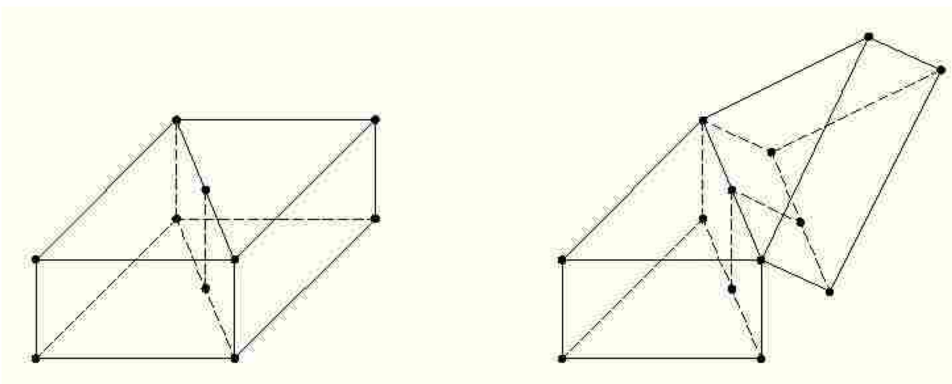
For hexahedral elements, the Jacobian matrix for each node can be divided by thelengths of the three edge vectors that intersect that particular node to produce the scaled

Jacobian matrix. The determinate of each of these scaled Jacobian matrices can then be calculated to produce the scaled Jacobian for each node. The minimum scaled Jacobian for each of the eight nodes is then taken as the minimum scaled Jacobian for that element. Scaled Jacobians have a range from 1 for perfect cubes to -1 for inverted cubes.

When adapting existing meshes, several candidate adaptations are often available. Ideally, a quality metric is used to determine which candidate is best. Woodbury [12] uses a shape metric based on the element Jacobian matrices to choose between different coarsening adaptations. Unfortunately, the Jacobian matrix is dependent on the global spatial positions of the nodes of an element. During adaptation, the precise locations of new nodes resulting from a potential adaptation are not known, making any metric based on the Jacobian matrix difficult to use. Thus, we seek a metric which will predict element quality without precise node locations and, if obtained, consistently result in good scaled Jacobians.

Edge valence is a relatively new quality metric developed by Staten [13] that has shown to be an accurate predictor of hexahedral mesh quality. Edge valence is defined as the number of quadrilateral faces connected to a single edge. Staten asserts that if the edge valence of all edges in a hexahedra mesh is 3, 4, or 5 then the scaled Jacobian of that element will be greater than zero and likely much higher, in the absence of over constraining geometric topology. For an edge on the interior of the mesh, if the edge valence is less than 3 the element will contain a doublet and consequently be inverted. Elements with an edge valence greater than 5 may have acceptable quality but if a doublet exists, the element will be inverted and admit only poor quality.

Two types of doublets may exist in elements with an edge valence greater than 5 or less than 3. Face doublets occur when two elements share two adjacent faces and edge doublets occur when two elements share two or more adjacent edges without sharing a face as shown in Figure 2-1. While the existence of doublets does impact mesh quality, the algorithm presented later in this work does not predict the creation of doublets. Rather, we avoid the introduction of doublets by only allowing operations that guarantee edge valences of 3, 4, or 5 within the volume of the mesh.



(a) Hexahedral face doublet (b) Hexahedral edge doublet

Figure 2-1: Hexahedral doublets.

2.2 Review of Localized Hexahedral Coarsening

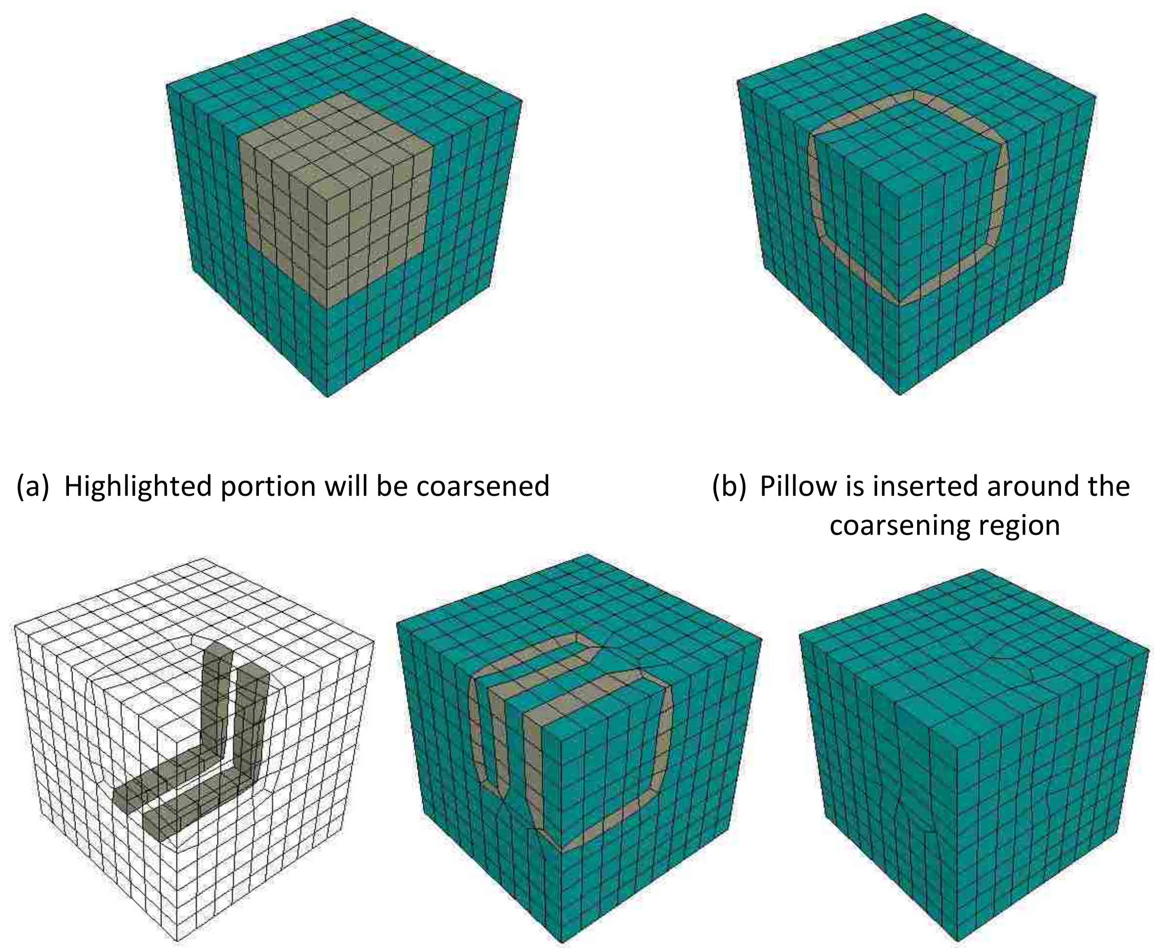
Several mesh adaptation algorithms exist for hexahedral meshes including coarsening [12], refinement [4], mesh matching [6], and grafting [5]. An existing mesh may be modified to adjust node density through refining and/or coarsening to increase the quality of elements of a mesh, to create a conformal mesh, or to more easily mesh a domain. The coarsening algorithm presented by Woodbury [12] uses pillowing, column collapse, and sheet extraction to achieve completely local coarsening of an existing mesh. A brief review of this process is presented below.

There are 3 main steps required for localized coarsening of hexahedral meshes according to Woodbury's algorithm [12]. Step 1 is to pillow the desired coarsening region. By inserting a pillow, columns are created that will be local to the desired coarsening region. Step 2 is to collapse columns within the pillow in such a manner as to create sheets local to the coarsening region. Step 3 is to extract the local sheets that were created in step 2. Figure 2-2 shows these coarsening steps on a simple structured mesh.

Several columns are often available for collapse in step 2. In addition, each column can be collapsed two unique directions, often providing many different collapse and extract combinations. The two criteria that are used to decide which columns to collapse are:

1. Level of coarsening desired for resulting mesh and,
2. Resulting mesh quality

Which columns are collapsed determines which sheets will be extracted; the number of elements within those sheets must not exceed the target number of elements to be removed. Determining the resulting mesh quality for a particular column collapse option can be difficult. Woodbury's coarsening algorithm decides which column collapses will result in a poor mesh quality by evaluating the quality of the sheets bordering the sheet that would be extracted as shown in Figure 2-3.



(a) Highlighted portion will be coarsened

(b) Pillow is inserted around the coarsening region

(c) Highlighted columns will be collapsed

(d) After columns are collapsed, local sheets are extracted

(e) Final coarsened mesh

Figure 2-2: Overview of coarsening process showing pillowing (b), column collapse (d), and sheet extraction (e).

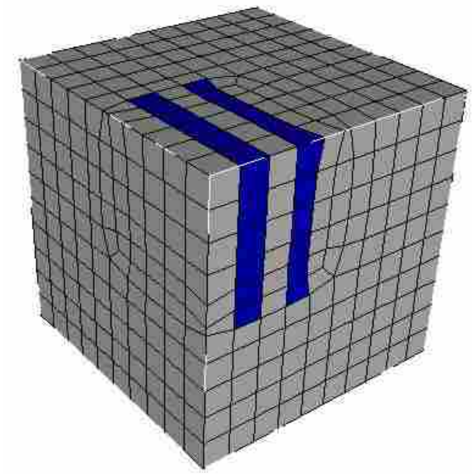


Figure 2-3: Shaded hexahedral represent bordering sheets for the sheet in between the two.

The quality of these bordering sheets is evaluated by calculating a shape metric, f_{shape} as presented by Knupp [14], for each element within that sheet. This metric is mathematically defined in Equation 2-2 and has a value of 1.0 for a perfect cube and 0 for a degenerate element. For this equation, the metric tensor is defined as $A^T_k A_k$ with A_k being the Jacobian matrix for the k^{th} node. The minimum element f_{shape} for each pair of bordering sheets is then taken as the quality of the sheet and compared to the qualities of the sheets bordering other potential extraction sheets. This method of predicting sheet quality has been effective and reliable for structured meshes but does not guarantee acceptable mesh quality because it uses the pre-adaptation location of nodes.

$$f_{shape} = \frac{24}{\sum_{k=0}^{7} (\lambda_{11}^{k} + \lambda_{22}^{k} + \lambda_{33}^{k}) / \alpha_{k}^{2/3}} \qquad \text{(2-2)}$$

where: λ_{ij}^{k} = the ijth component of the kth metric tensor

α_k = the determinate of the kth jacobian matrix

The difficulty in using Woodbury's approach to predict mesh quality stems from the fact that the mesh is being evaluated prior to any mesh manipulation. It is an oversimplification to assume that the mesh quality will not significantly change after a series of pillowing, collapsing columns, and extracting sheets. Any attempt to precisely predict f_{shape} through these coarsening steps, and smoothing, would be unrealistic due to the many possibilities of node positioning.

3. EDGE VALENCE PREDICTION ALGORITHM

A new algorithm that can be used during hexahedral coarsening is presented below. This algorithm accurately predicts the valence of each edge in a given mesh through the steps of column collapse and sheet extraction. The algorithm assumes the coarsening region has been selected, the pillow has been inserted, and a coarsening layout has been determined. For the purpose of this thesis, a coarsening layout is defined as a set of columns and sheets that could be, respectively, collapsed and extracted to produce a coarsened mesh.

A coarsening layout is created by evaluating every sheet within the coarsening region and determining which of these sheets will produce a sufficiently coarsened mesh. Woodbury's algorithm [12] will only generate one coarsening layout at any given time; for this reason, if a coarsening layout is rejected due to poor edge valence, a new layout must be created using a different combination of sheets. If the initial mesh in the coarsening region does not contain acceptable edge valences, the edge valence prediction algorithm would be skipped and Woodbury's original method of evaluating mesh quality, presented earlier, would be used. For this algorithm, unacceptable edge valences are those greater than 5 or less than 3 for an interior edge.

For this algorithm equation 3-1 is used to predict the edge valence for edges that will merge with another edge during column collapse and equation 3-2 is used for edges that will not merge with another edge during column collapse. Equation 3-3 is used to predict edge valence for edges during sheet extraction. These equations are presented below with examples.

$$PEVC = m_1 + m_2 - 2m_3 + \alpha \tag{3-1}$$

$$PEVC = m - 1 + \beta \tag{3-2}$$

where: *PEVC* = Predicted edge valence for column collapse

m_1 = Number of hexes attached to edge 1

m_2 = Number of hexes attached to edge 2

m_3 = Number of hexes common to edge 1, edge 2, and the column

m = Number of hexes attached to edge

α = 1 if at least 1 edge is on the mesh boundary

0 otherwise

B = 1 if edge is on mesh boundary

0 otherwise

$$PEVS = m_1 + m_2 - 2m_3 + \alpha \tag{3-3}$$

where: PEVS = Predicted edge valence for sheet extraction

m_1 = Number of hexes attached to edge 1

m_2 = Number of hexes attached to edge 2

m_3 = Number of hexes common to edge 1, edge 2, and the sheet

α = 0 if neither edge is on the mesh boundary

1 otherwise

To demonstrate these equations, a simple structured mesh will be used. Figure 3-1 shows a highlighted column with three labeled edges. For this example, the column will be collapsed such that edge #1 and edge #3 will be merged together and edge #2 will not merge with any other edge. Using equation 3-1 we can predict the edge valence of the resulting edge when edge #1 and edge #3 are merged as follows.

$$\begin{aligned} PEVC &= 2 + 2 - 2(1) + 1 \\ &= 3 \end{aligned} \tag{3-4}$$

The edge valence of edge #2 may be predicted through column collapse by using equation 3-2 as follows with the results verified in Figure 3-2.

$$\begin{aligned} PEVC &= 4 - 1 \\ &= 3 \end{aligned} \tag{3-5}$$

Continuing with the same mesh that was used to demonstrate the equations used for column collapse, we will now demonstrate equation 3-3 for sheet extraction. Figure 3-3 shows a highlighted sheet that will be extracted and two edges labeled edge #4 and edge #5 that will be merged in the extraction process. Using equation 3-3 the valence can be predicted for the resulting edge when edge #4 and edge #5 are merged as follows. Results may be verified from Figure 3-4.

$$\begin{aligned} \text{PEVS} &= 3 + 6 - 2(2) + 0 \\ &= 5 \end{aligned} \tag{3-6}$$

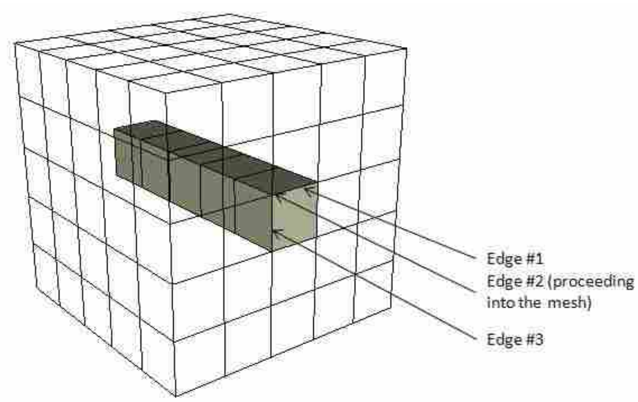


Figure 3-1: Highlighted column will be collapsed.

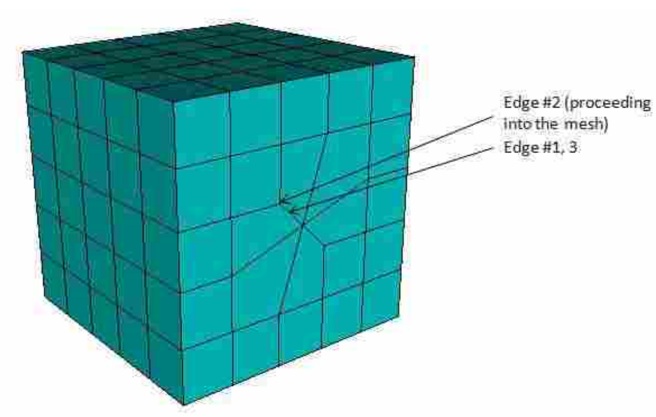


Figure 3-2: Resulting mesh after column collapse.

Using edge valence as the quality metric for accurately predicting mesh quality through sheet operations is achievable partly due to the simplicity of edge valence. Unlike the Jacobian, edge valence is a positive integer value that is based on mesh topology rather than node location. Therefore, most smoothing techniques will not alter edge valence and the discrete values are more easily calculated. Cleanup operations that alter node connectivity [15] may change the edge valence of a mesh but these operations only occur after the sheet operations have taken place and should only improve the mesh quality. These attributes make it possible to accurately predict edge valence through sheet operations.

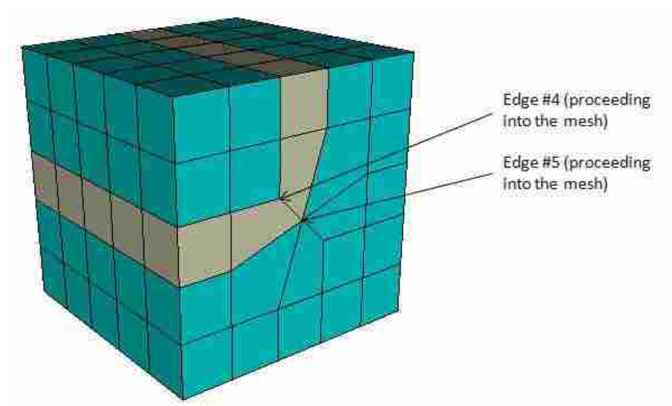


Figure 3-3: Highlighted sheet will be extracted.

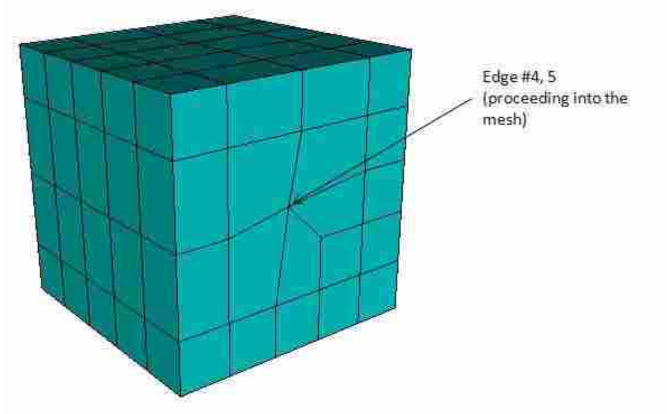


Figure 3-4: Resulting mesh after sheet extraction.

With the edge valence prediction equation presented, the following 6 steps now show the algorithm as applied to the coarsening process.

1. From the coarsening layout, find the hexahedral columns that will collapse and the collapse direction.
2. For each edge that will collapse find the opposite edge with which it will be merged.
3. Predict the valence for all the edges that are part of the hex columns that will be collapsed. If the edge will merge with another edge use Equation 3-1, otherwise use Equation 3-2 and save these new valences.
4. From the coarsening layout, find each of the hexahedral sheets that will be removed.
5. For each edge that will collapse find the opposite edge with which it will be merged.
6. Predict the valence for all the edges that will be collapsed using Equation 3-3 and the valences calculated in step 3 as needed.

The algorithm presented above was used to create the examples in the results section. A flowchart of this algorithm is presented in Figure 3-5.

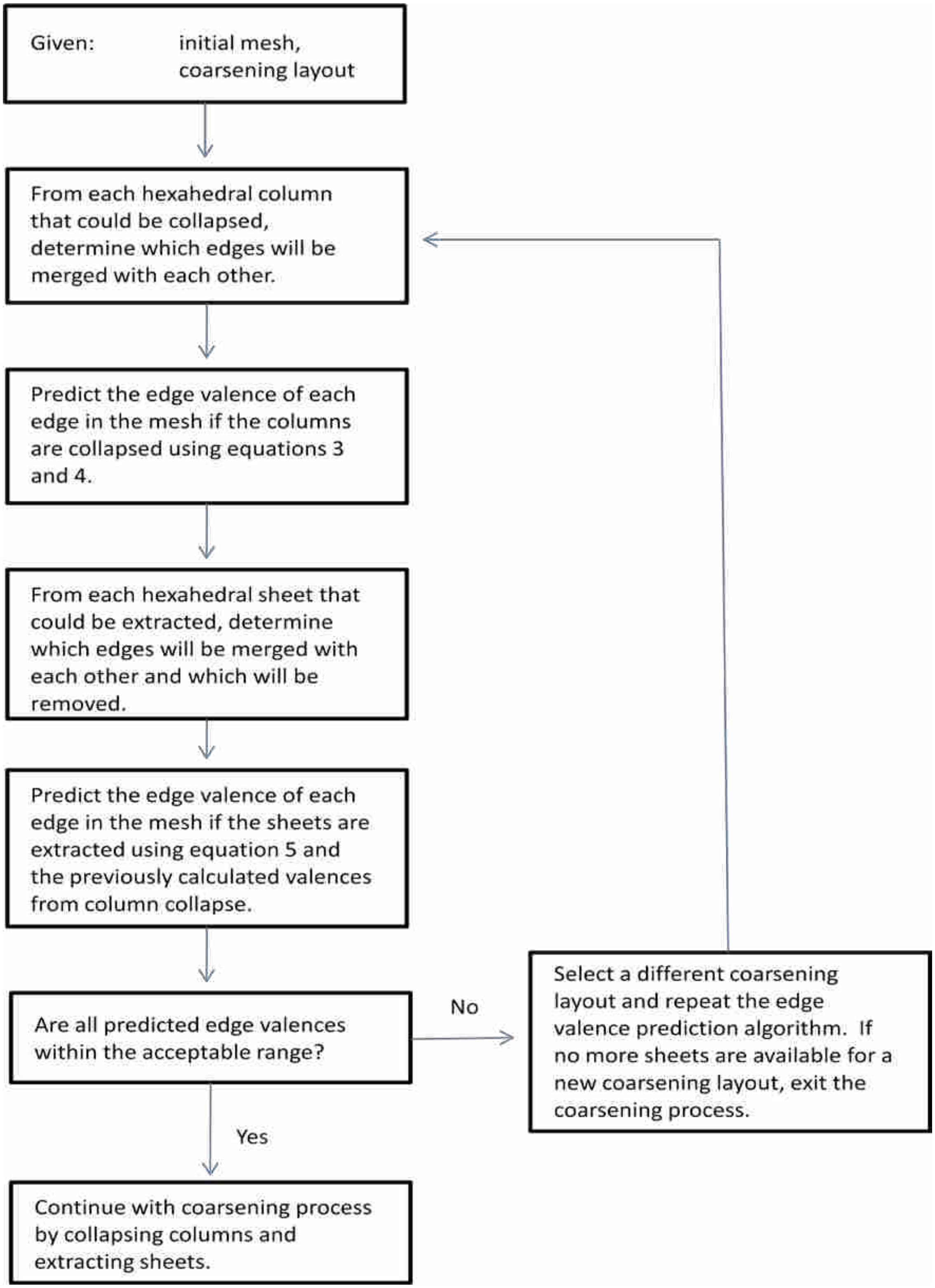
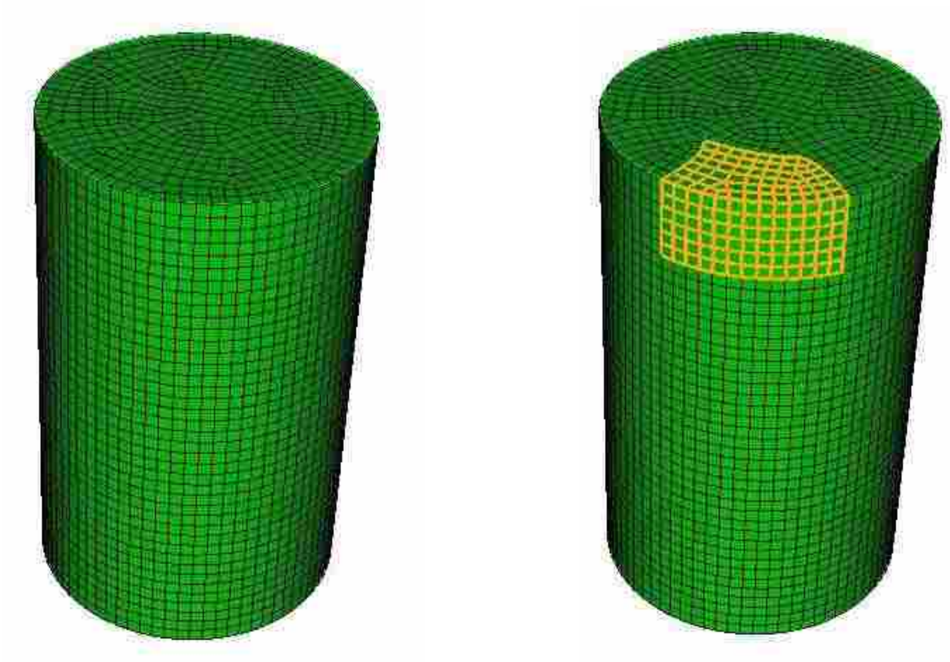


Figure 3-5: Flowchart of the edge valence prediction algorithm.

4. RESULTS

The following examples have been generated using the CUBIT geometry and mesh generation software [16]. The examples demonstrate the ability of the edge valence prediction algorithm to accurately predict edge valence through sheet operations as well as its ability to guide the coarsening process. The first example uses a simple cylinder with a swept mesh shown in Figure 4-1 (a). The area highlighted in Figure 4-1 (b) will be coarsened by 50% using Woodbury's coarsening algorithm. Figure 4-2 (a) shows the resulting mesh with the original algorithm and Figure 4-2 (b) shows the resulting mesh when edge valence prediction is used to guide the coarsening process. Resulting edge valence and scaled Jacobian for the two coarsened meshes are shown in Table 4-1. A histogram showing the distribution of scaled Jacobians among elements within the coarsening region is shown in Figure 4-3.

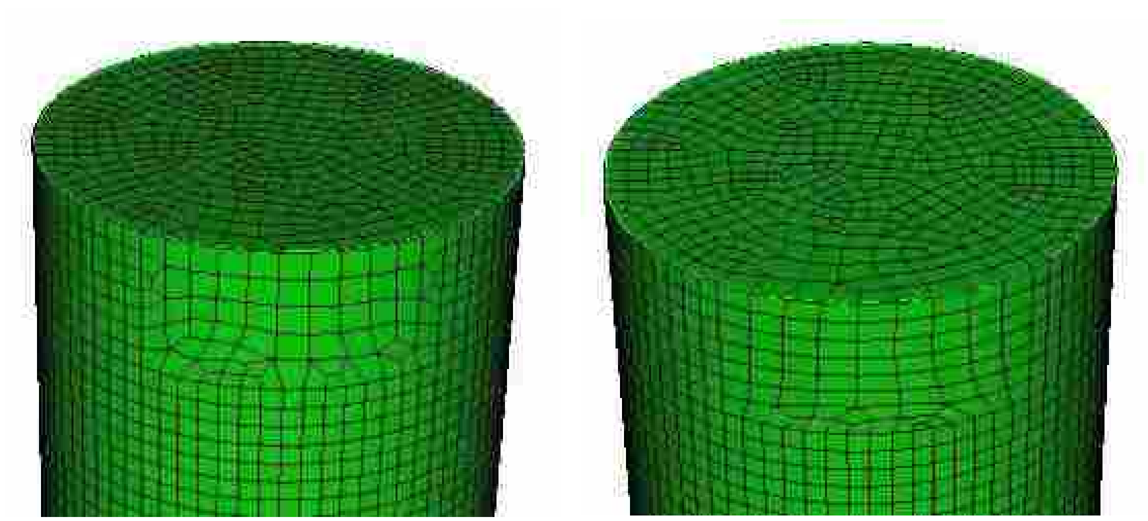
This next example uses another swept mesh, shown in Figure 4-4, which will be coarsened to 50% and 75% as shown in Figure 4-5 and Figure 4-6 respectively. Edge valence and Jacobian results are shown in Table 4-2 and Table 4-3. Again, histograms are provided in Figure 4-7 and Figure 4-8 for the 50% and 75% coarsening respectively. These histograms show that not only does the minimum scaled Jacobian increase but the overall average element scaled Jacobian also increases when edge valence prediction is used.



(a) Original mesh

(b) Highlighted portion will be coarsened 50%

Figure 4-1: Original mesh on cylinder with highlighted portion to be coarsened by 50%.



(a) 50% coarsening without edge valence prediction

(b) 50% coarsening with edge valence prediction

Figure 4-2: Results of coarsened cylinder without edge valence prediction (a), and with edge valence prediction (b).

Table 4-1: Edge Valence and Jacobian Results for 50% Coarsening of Cylinder

50% Coarsening Without Edge Valence Prediction			50% Coarsening With Edge Valence Prediction		
Actual Coarsening: 46.03%			Actual Coarsening: 40.74%		
Min. Scaled Jacobian: 0.3162			Min. Scaled Jacobian: 0.4206		
Ave. Scaled Jacobian in coarsening region: 0.8466			Ave. Scaled Jacobian in coarsening region: 0.9206		
Volume Edge Valence	# Edges		Volume Edge Valence	# Edges	
	Predicted	Actual		Predicted	Actual
3	-	1141	3	1107	1107
4	-	56040	4	56169	56169
5	-	979	5	953	953
6	-	5	6	0	0
7	-	0	7	0	0
Surface/Curve Edge Valence	# Edges		Surface/Curve Edge Valence	# Edges	
	Predicted	Actual		Predicted	Actual
2	-	148	2	146	146
3	-	7908	3	7902	7902
4	-	0	4	0	0

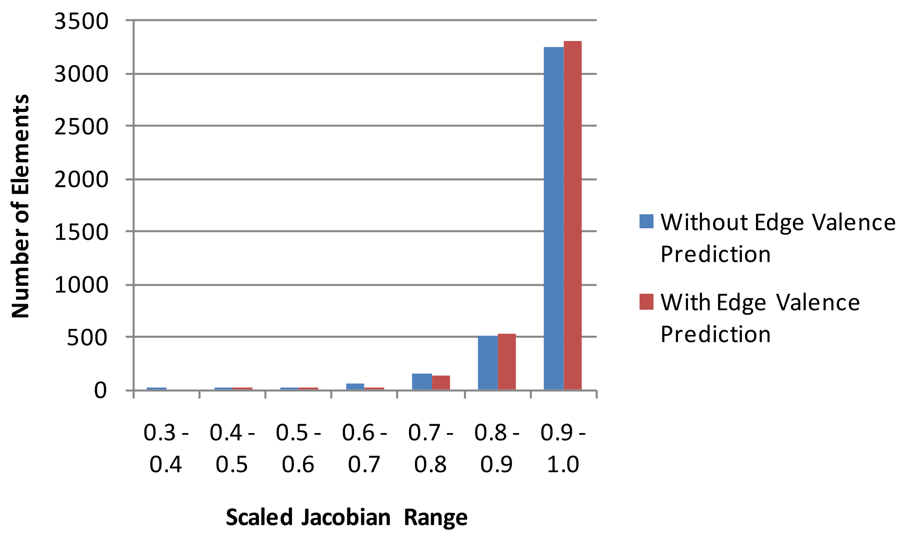
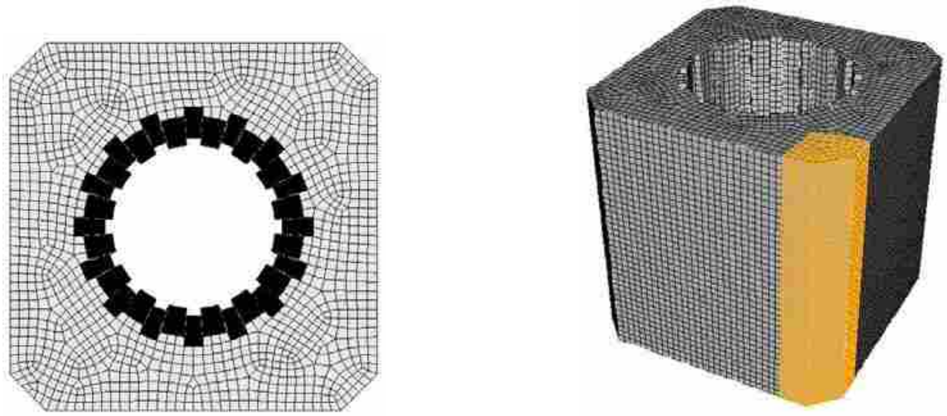


Figure 4-3: Histogram of Scaled Jacobian for 50% coarsening of cylinder.

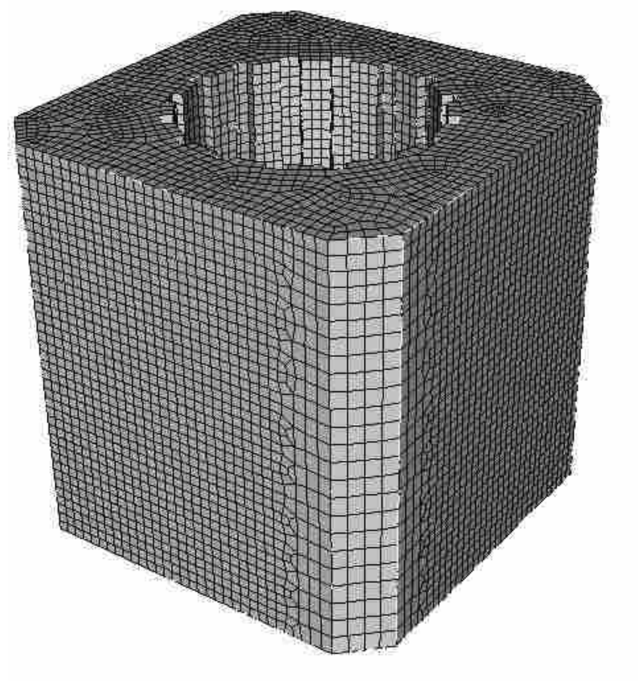


(a) Top view of original mesh (b) Highlighted portion will be coarsened

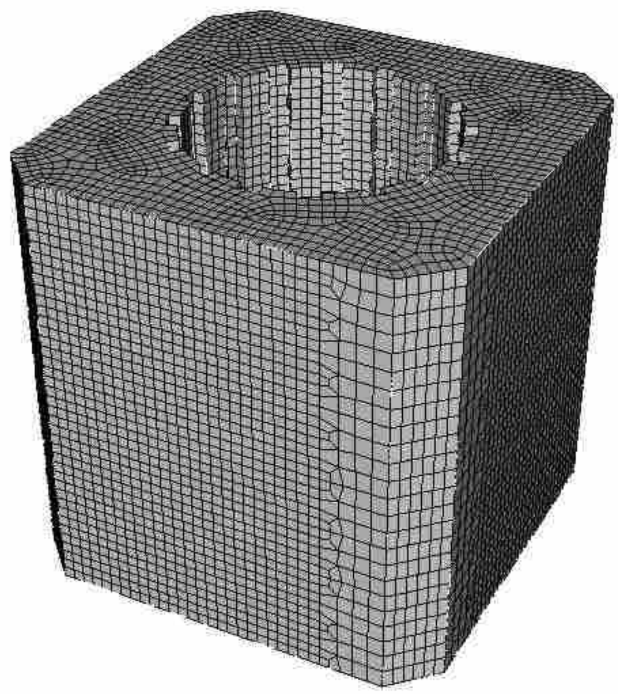
Figure 4-4: Original mesh on mechanical part with highlighted portion to be coarsened by 50% and 75%.

From this data we can make two observations. The first is that use of edge valence prediction has produced a coarsened mesh with a higher minimum scaled Jacobian. The second is that using edge valence prediction may allow for the mesh to be coarsened closer to the prescribed amount of coarsening; in this case, 71% coarsening was achieved instead of 66%.

This second observation will not always occur and is in reality an exception to what will most likely occur. More often, the use of edge valence prediction will result in a reduction in coarsening when compared to Woodbury's original algorithm because of greater scrutiny of each coarsening layout.

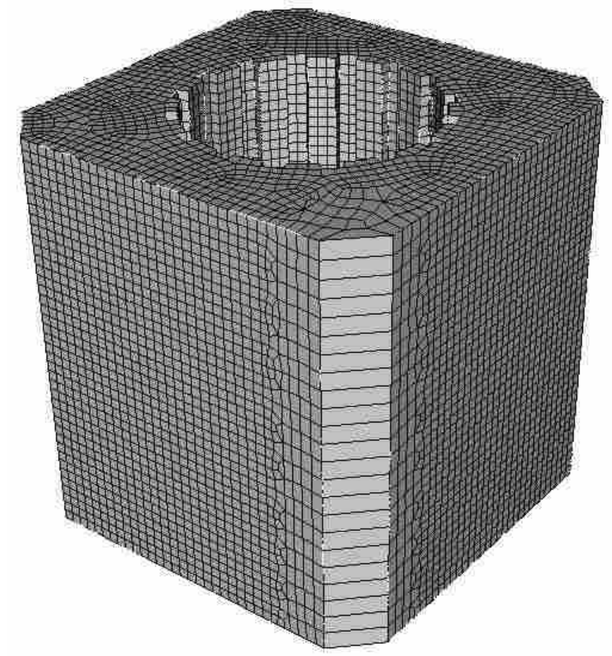


(a) Without edge valence prediction

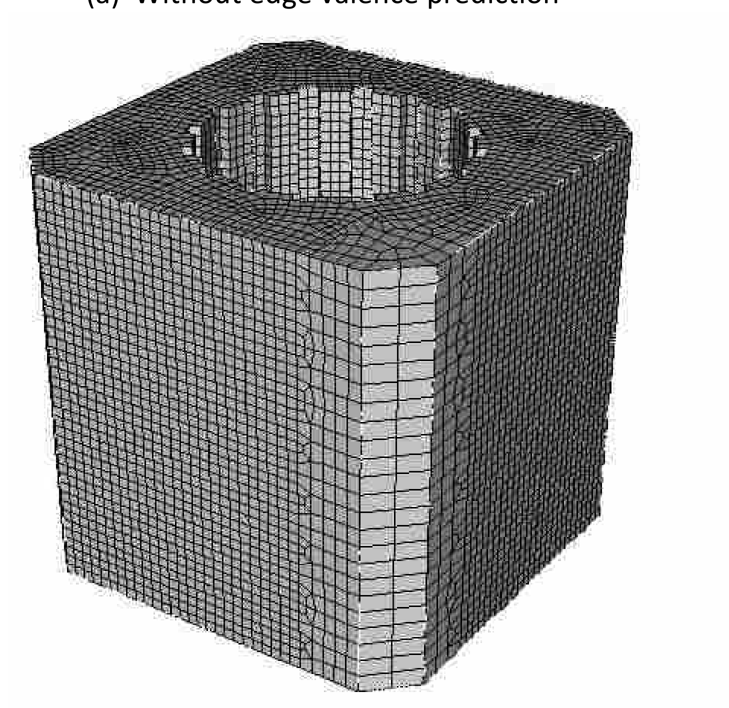


(b) With edge valence prediction

Figure 4-5: Results of 50% coarsened mechanical part without edge valence prediction (a), and with edge valence prediction (b).



(a) Without edge valence prediction



(b) With edge valence prediction

Figure 4-6: Results of 75% coarsened mechanical part without edge valence prediction (a), and with edge valence prediction (b).

Table 4-2: Edge Valence and Jacobian Results for 50% Coarsening of Mechanical Part

50% Coarsening Without Edge Valence Prediction		
Actual Coarsening: 50%		
Min. Scaled Jacobian: 0.4317		
Ave. Scaled Jacobian in coarsening region: 0.8717		
Volume Edge Valence	# Edges	
	Predicted	Actual
3	-	1566
4	-	118073
5	-	1518
6	-	40
7	-	0
Surface/Curve Edge Valence	# Edges	
	Predicted	Actual
2	-	1804
3	-	22008
4	-	1280

50% Coarsening With Edge Valence Prediction		
Actual Coarsening: 49.58%		
Min. Scaled Jacobian: 0.4366		
Ave. Scaled Jacobian in coarsening region: 0.8908		
Volume Edge Valence	# Edges	
	Predicted	Actual
3	1517	1517
4	118189	118189
5	1549	1549
6	0	0
7	0	0
Surface/Curve Edge Valence	# Edges	
	Predicted	Actual
2	1804	1804
3	21964	21964
4	1280	1280

Table 4-3: Edge Valence and Jacobian Results for 75% Coarsening of Mechanical Part

75% Coarsening Without Edge Valence Prediction		
Actual Coarsening: 65.88%		
Min. Scaled Jacobian: 0.3726		
Ave. Scaled Jacobian in coarsening region: 0.7999		
Volume Edge Valence	# Edges	
	Predicted	Actual
3	-	1644
4	-	116715
5	-	1528
6	-	39
7	-	24
Surface/Curve Edge Valence	# Edges	
	Predicted	Actual
2	-	1798
3	-	21798
4	-	1280

75% Coarsening With Edge Valence Prediction		
Actual Coarsening: 70.99%		
Min. Scaled Jacobian: 0.3958		
Ave. Scaled Jacobian in coarsening region: 0.8010		
Volume Edge Valence	# Edges	
	Predicted	Actual
3	1622	1622
4	116332	116332
5	1654	1654
6	0	0
7	0	0
Surface/Curve Edge Valence	# Edges	
	Predicted	Actual
2	1792	1792
3	21616	21616
4	1280	1280

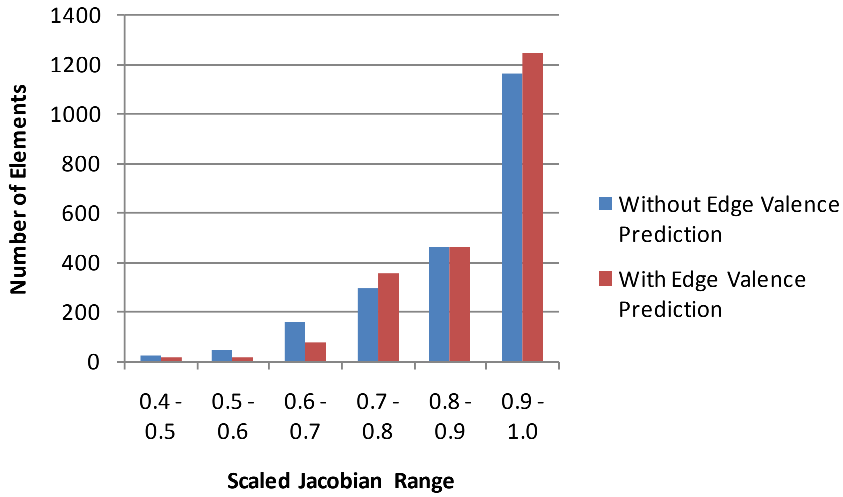


Figure 4-7: Histogram of scaled Jacobian for 50% coarsening of mechanical part.

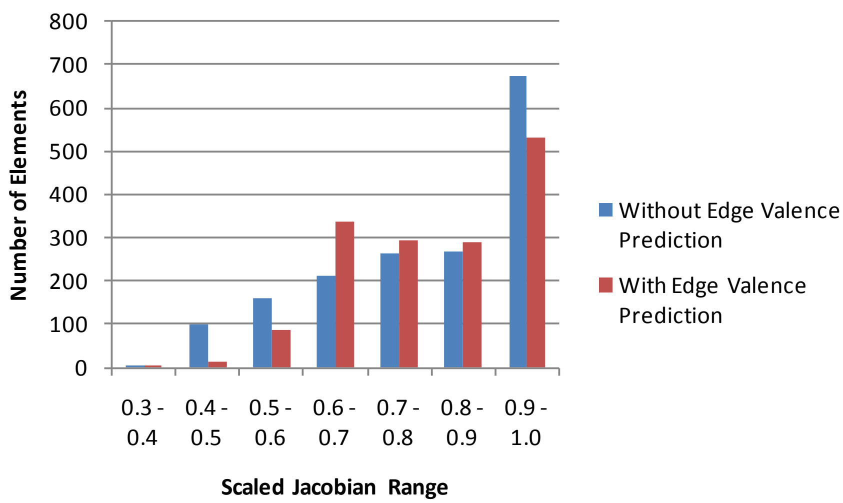
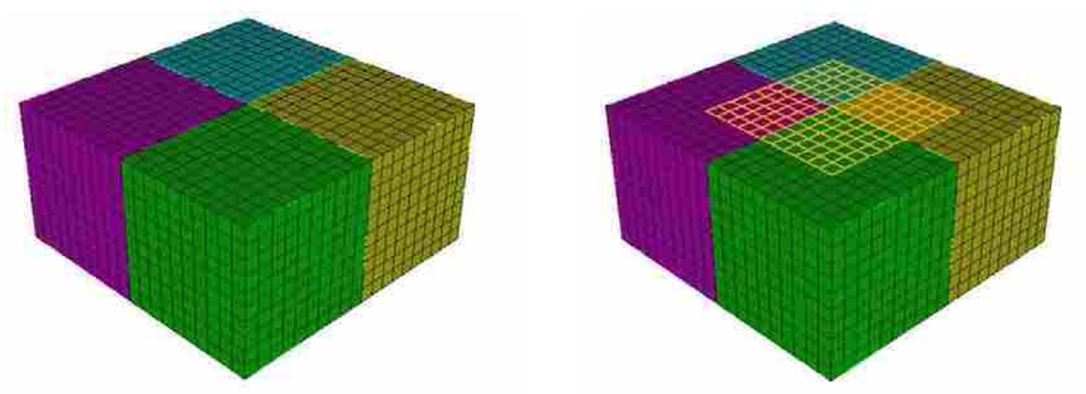


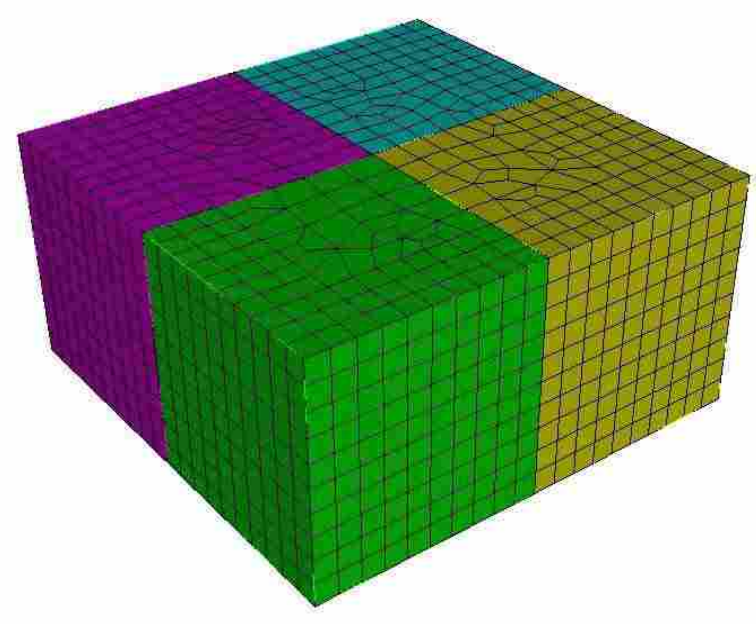
Figure 4-8: Histogram of scaled Jacobian for 75% coarsening of mechanical part.

The last example presented in Figure 4-9 shows how much the minimum scaled Jacobian can be improved using the edge valence prediction algorithm and the ability of the algorithm to handle merged surfaces. Figure 4-10 shows the two resulting meshes after 66% coarsening. Table 4-4 shows again how the edge valence prediction algorithm was able to accurately predict the edge valence of each edge in a mesh through the steps of column collapse and sheet extraction. Figure 4-11 shows a histogram of the scaled Jacobian for these two coarsening methods. These histograms show that using edge valence prediction helps to increase the overall scaled Jacobian in the coarsening region. These results also show the ability of the algorithm to successfully guide Woodbury's coarsening algorithm to produce a mesh with a higher minimum scaled Jacobian. The original algorithm produced a mesh with a minimum scaled Jacobian of 0.2087 which is borderline acceptable for some solvers whereas the application of edge valence prediction produced a mesh with a minimum scaled Jacobian of 0.4305.

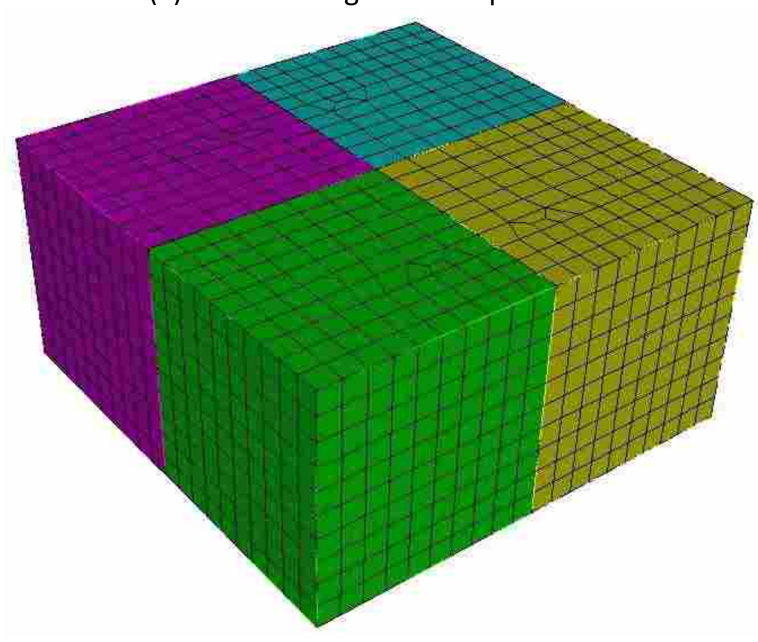


(a) Original Mesh (b) Highlighted portion to be coarsened

Figure 4-9: Original mesh on blocks with highlighted portion to be coarsened by 66%, highlighted portion extends down 5 layers into the mesh.



(a) Without edge valence prediction



(b) With edge valence prediction

Figure 4-10: Results of 66% coarsened block without edge valence prediction (a), and with edge valence prediction (b).

Table 4-4: Edge Valence and Jacobian Results for 66% Coarsening of Block

66% Coarsening Without Edge Valence Prediction			66% Coarsening With Edge Valence Prediction		
Actual Coarsening: 57.16%			Actual Coarsening: 46.28%		
Min. Scaled Jacobian: 0.2087			Min. Scaled Jacobian: 0.4305		
Ave. Scaled Jacobian in coarsening region: 0.7412			Ave. Scaled Jacobian in coarsening region: 0.8731		
Volume Edge Valence	# Edges		Volume Edge Valence	# Edges	
	Predicted	Actual		Predicted	Actual
3	-	294	3	172	172
4	-	8046	4	8492	8492
5	-	247	5	188	188
6	-	35	6	0	0
7	-	3	7	0	0
Surface/Curve Edge Valence	# Edges		Surface/Curve Edge Valence	# Edges	
	Predicted	Actual		Predicted	Actual
2	-	200	2	200	200
3	-	2912	3	2920	2920
4	-	616	4	626	626

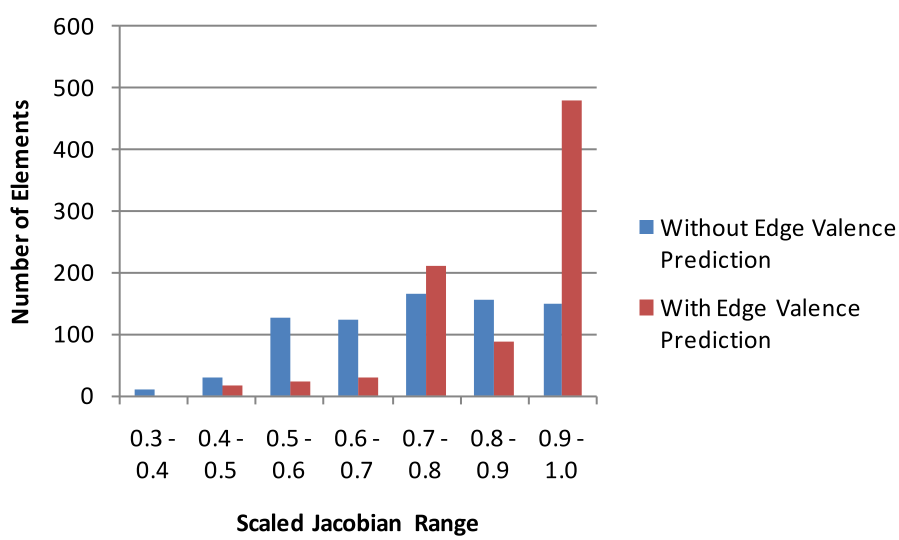


Figure 4-11: Histogram of scaled Jacobian for 66% coarsening of block.

5. CONCLUSIONS

This thesis presents an algorithm for predicting the edge valence of edges in an all-hexahedral mesh through the sheet operations of column collapse and sheet extraction. This prediction algorithm allows for mesh modifications to be analyzed without actually altering the mesh in any way. The operation that will maintain mesh quality can then be selected for execution. This is critical for mesh modification algorithms, which make incremental decisions during the modification process, because it allows for an objective quality metric to guide mesh modifications without actually having to carry out those modifications. This ability to guide mesh modifications without actually altering the mesh is the principle contribution of this thesis.

To demonstrate this edge valence prediction capability, the algorithm has been applied to the localized coarsening process presented by Woodbury [12] with examples shown. In the examples shown, edge valence prediction resulted in higher mesh quality as measured by the minimum and average scaled Jacobian. In two of the four examples, the coarsening process was stopped pre-maturely in order to maintain good element quality.

Future work with edge valence prediction may be the application of this algorithm to other all-hex topology modification techniques such as refining or mesh matching. It is expected that the application of this new algorithm would improve these processes because of its successful application to coarsening but, at this point it is unknown to what extent the algorithm may be applied. Future work may also be considered for adapting the edge valence prediction algorithm presented in this work to allow for its use on meshes with an initial poor quality. As stated, this algorithm will not evaluate meshes with unacceptable edge valences due to the possible presence and creation of doublets. However, the algorithms extension to poor quality meshes may be useful and possibly controlled by the development of doublet prediction algorithms.

The efficiency of this algorithm may also be improved, as applied to the coarsening process, by restructuring Woodbury's coarsening algorithm to make more use of edge valence prediction. This may be done by evaluating multiple coarsening layouts at a single time and executing the one that provides the best mesh quality, thus eliminating the current greedy algorithm. Another facet of Woodbury's original algorithm that may benefit from edge valence prediction is in deciding which direction to collapse the columns. Currently, there is no quality check for different collapse directions but the application of edge valence prediction may improve this process.

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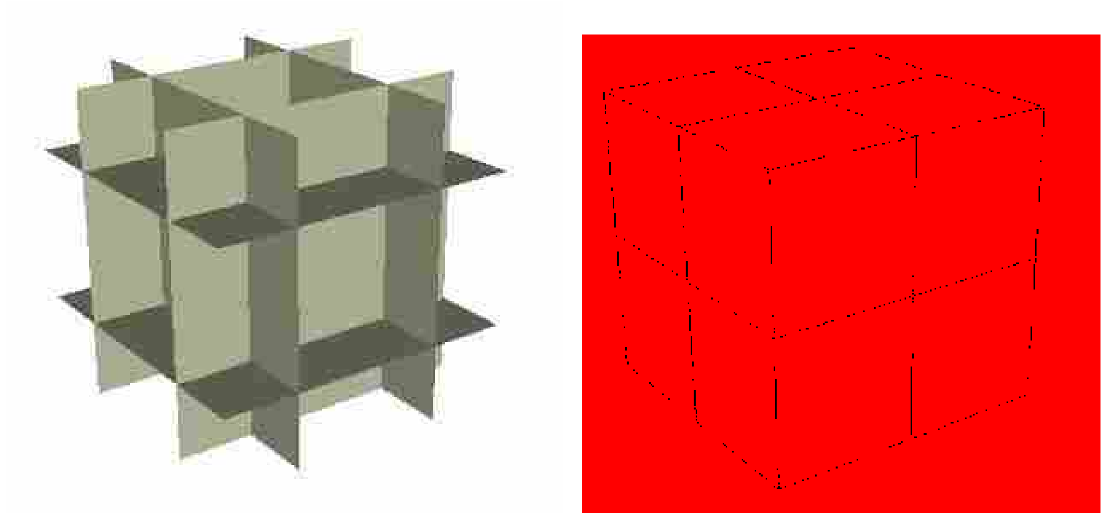
APPENDIX A: HEXAHEDRAL TOPOLOGY MODIFICATION OPERATIONS

Developing hexahedral topology modification techniques has proven to be a difficult matter due to the perpetuation of local changes throughout the mesh. Efforts to develop robust, and local, hexahedral modification techniques is an ongoing area of research with steady progress being made. The following is a short survey of current hexahedral modification techniques.

Every hexahedral mesh may be understood in terms of its dual or as a primal mesh as shown in Figure 0-1. The dual of a mesh may be seen as a set of intersecting surfaces that bisect hexahedral surfaces in each direction. The dual of a mesh may also be known as the spatial twist continuum [17]. Operations on a hexahedral mesh may be categorized into one of three main types: sheet operations, flipping operations [18], and atomic operations [15].

Sheet operations already discussed include column collapse, sheet extraction and pillowing. Another sheet operation not used in the edge valence prediction algorithm is dicing [19]. Dicing is used for refining a hexahedral mesh for the purpose of generating large meshes. Dicing is the process of sub-dividing each hexahedron in a mesh to create a refined mesh.



(a) Dual of a mesh (b) Primal of a mesh

Figure 0-1: Two methods for presenting a mesh are in terms of its dual (a) and primal (b).

Flipping operations are well known and commonly used for tetrahedral meshes and have been extended for hexahedral use. Flipping operations are an entirely local process that effect small pockets of hexahedron. Flipping has not been applied to hexahedron with the same success as when it was applied to tetrahedron but efforts are still being made to exploit their use [20]. For diagrams and further descriptions of the flipping process the reader may refer to already published literature [18, 20].

Atomic operations are defined as irreducible local operations that may be used to describe any higher order operation such as a sheet or flipping operation. These atomic operations include atomic pillow, face shrink, and face open-collapse. A fourth operation may be added to this list but understanding of its primal expression is limited and thus application towards hexahedral topology modifications is not possible [21].

The atomic pillow is performed on a single quadrilateral face that separates two hexahedral. The quad is pulled apart from the hexahedral and split into two separate quads. The gap now present between the two quads is filled with two hexahedral each sharing five faces. This process is depicted in two dimensions in Figure 0-2.

Face shrink involves two adjacent hexahedral that shares a single face. Vertices of the common face are shrunk towards the center of the face with a new hexahedron inserted into each of the four remaining voids. The face shrink operation is the same as “Inflate Hex Ring” operation introduced by Knupp and Mitchell [22].This process is depicted in Figure 0-3. Face open-collapse is the process of splitting and merging neighboring dual sheets. This is done by opening dual edges and reconnecting them in a different but still conformal manner. The face open-collapse is presented in sheet diagrams in already published literature [20] but the primal mesh is much more difficult and has been omitted from this survey.

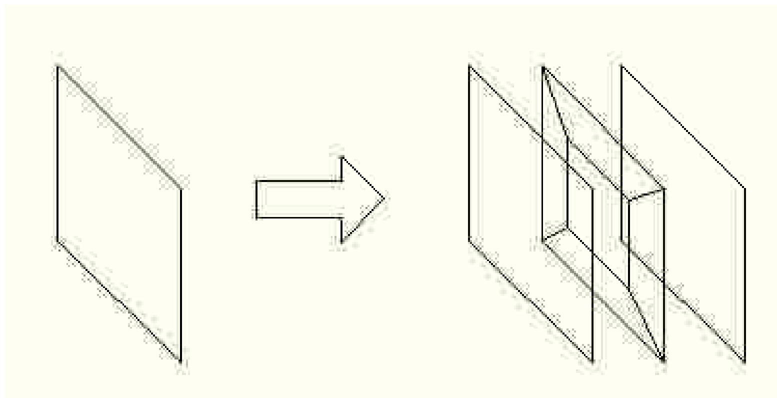


Figure 0-2: Atomic pillowing in the primal mesh.

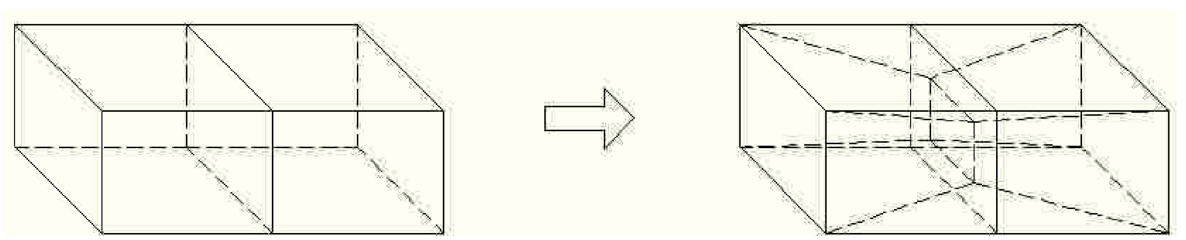


Figure 0-3: Face shrink in the primal mesh.