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Is there a relationship between curvature and inductance in the Josephson junction?

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ABSTRACT

A Josephson junction is a device made of two superconducting electrodes separated by a very thin layer of isolator or normal metal. This relatively simple device has found a variety of technical applications in the form of Superconducting Quantum Interference Devices (SQUIDs) and Single Electron Transistors (SETs). One can expect that in the near future the Josephson junction will find applications in digital electronics technology RSFQ (Rapid Single Flux Quantum) and in the more distant future in construction of quantum computers. Here we concentrate on the relation of the curvature of the Josephson junction with its inductance. We apply a simple Capacitively Shunted Junction (CSJ) model in order to find condition which guarantees consistency of this model with prediction based on the Maxwell and London equations with Landau-Ginzburg current of Cooper pairs. This condition can find direct experimental verification.

Introduction

A Josephson junction is a device that usually is made of two superconducting electrodes separated by very thin layer of non-superconducting material. The barrier which separates the electrodes must be very thin. If the layer is made of isolator (S-I-S junction) then its thickness is of order of several Angstroms. In case of nonsuperconducting metal (S-N-S junction) this thickness can be on the level of several microns. The junction can also be made as a constriction that weakens the superconductivity at the point of contact (S-s-S junction). Until a critical current is reached, a current of electron pairs can flow across the barrier without any resistance. First time, theoretical prediction that Cooper pairs can tunnel from one electrode to another was given by Brian Josephson in 1962 [1]. In his paper Josephson predicted relationships for the current and voltage across the weak link. This prediction was confirmed by Philip Anderson and John Rowell [2].

Depending on the number of dimensions the Josephson device can be considered as a two (large area Josephson junction), one (long Josephson junction) and even zero dimensional system (point Josephson junction). In particular, the junction can be considered as the long Josephson junction if its transverse dimension is smaller than the Josephson length. Josephson effect, similarly like superconductivity, is an example of a macroscopic quantum phenomenon. The leading dynamical variable that describes behavior of this system is a scalar field ϕ. This variable is a gauge invariant difference of phases of the macroscopic wave functions that describe superconducting electrodes. The behavior of this dynamical variable is determined by the sine-Gordon model. The solutions of this nonlinear model are studied for years [3,4].

The effective equation that describes dynamical processes in the junction can be obtained in many ways [5,6]. In one of simplified approaches the point junction is replaced by a circuit that contains the supercurrent (Josephson current) and some capacitive element (see Fig. 1). This model is described as Capacitively Shunted Junction – CSJ. The CSJ model presumes that the quasiparticle current in the junction is so small that can be neglected. The currents in the direction normal to the isolator layer are Josephson current of Cooper pairs and additionally displacement current due to capacitive effects.

The present paper aims in application of the CSJ model to description of the curvature effects in the long Josephson junction. We would like to indicate the condition which allows to recover the result obtained for the same system on the background of the Maxwell and London's equations with Landau-Ginzburg current.

Finally, let us underline that one of the promising areas of applications of the Josephson junction is the RSFQ (Rapid Single Flux Quantum) electronics [7]. The digital information in these instruments is carried by magnetic flux quanta identified with the kink solutions of the sine-Gordon model. An interesting role in this area can be played by the curved Josephson junctions. On the base of the modified sine-Gordon equation several geometries were

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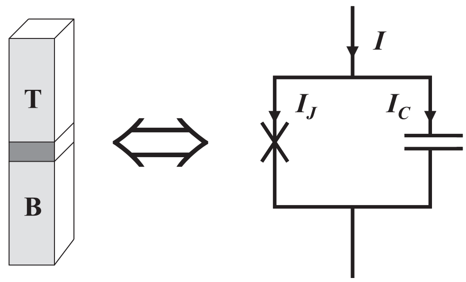

Fig. 1. In the CSJ model the point Josephson junction (left side of the figure) is equivalent to the circuit located on the right side of the picture. Here I_J and I_C denotes the Josephson and capacitive currents.

identified which can be used as electronic elements in RSFQ electronic devices [8].

The paper is organized as follows. For the sake of completeness, in the next Section we present Feynman approach for two-piece quantum system in order to obtain time changes of the phases of the macroscopic wave functions which describe behavior of the superconducting electrodes. Section "Curved junction in the CSJ model" contains construction of the field equation, in a curved system, on the base of the CSJ model. The last Section contains remarks.

Josephson relation

In the Feynman approach the Josephson junction can be viewed as quantum system that consists of two subsystems (superconducting electrodes) described by the many particle wave functions ψ_T and ψ_B. The indices T, B correspond to top and bottom electrode. The system of this type is described by the Schrödinger equation

$$\imath\hbar\frac{\partial}{\partial t}|\psi\rangle = \widehat{H}|\psi\rangle, \tag{1}$$

where the quantum state and the matrix elements of the hamiltonian are the following:

$$|\psi\rangle = \begin{bmatrix}\psi_T\\ \psi_B\end{bmatrix}, \qquad \begin{bmatrix}\langle\psi_T|\widehat{H}|\psi_T\rangle, & \langle\psi_T|\widehat{H}|\psi_B\rangle\\ \langle\psi_B|\widehat{H}|\psi_B\rangle, & \langle\psi_B|\widehat{H}|\psi_T\rangle\end{bmatrix} = \begin{bmatrix}E_T, & K\\ K, & E_B\end{bmatrix}. \tag{2}$$

The Schrödinger Eq. (1) can be transformed to the system of coupled equations

$$\imath\hbar\frac{\partial\psi_T}{\partial t} = E_T\psi_T + K\psi_B, \qquad \imath\hbar\frac{\partial\psi_B}{\partial t} = E_B\psi_B + K\psi_T, \tag{3}$$

where K describes interaction of the quantum subsystems. One can see that for $K = 0$ the subsystems are independent and are described by two independent Schrödinger equations. Next, we separate the modulus and the phase of the wave functions that describe the electrodes

$$\psi_T = |\psi_T|e^{\imath\varphi_T}, \qquad \psi_B = |\psi_B|e^{\imath\varphi_B}. \tag{4}$$

The real parts of Eq. (3) can be written in the form

$$-\hbar|\psi_T|\frac{\partial\varphi_T}{\partial t} = E_T|\psi_T| + K|\psi_B|\cos\phi,$$
$$-\hbar|\psi_B|\frac{\partial\varphi_B}{\partial t} = E_B|\psi_B| + K|\psi_T|\cos\phi. \tag{5}$$

where we denoted the phase difference as follows $\phi = \varphi_T - \varphi_B$. If we assume that modules of the wave functions are identical i.e. that the density of the Cooper pairs ρ in both electrodes are equal

$$|\psi_T|^2 = |\psi_B|^2 = \rho,$$

and also denote

$$E_T = qV_T, \qquad E_B = qV_B,$$

then Eq. (5) can be transformed as follows

$$\frac{\partial\varphi_i}{\partial t} = -\frac{q}{\hbar}V_i - \frac{K}{\hbar}\cos\phi. \tag{6}$$

In the above equation the index i denotes top and bottom electrode $i \in \{T, B\}$.

Curved junction in the CSJ model

In the literature the junction is studied in many ways [5]. In the analysis presented here we adopt the approach presented in papers [6]. The central part of the curved Josephson junction is depicted in Fig. 2. We consider the junction that has a form of a circle. The arc in the bottom electrode we parameterize by the parameter x. The arc in the top electrode is parameterized by the parameter y. The central curve of the isolator layer is parameterized by the parameter s. In the junction we chose the closed curve (with the corners **A**, **B**, **C**, **D**) and then add the phase differences between points **AB**, **BC**, **CD** and **DA**. Due to requirement of uniqueness of the wave function of the whole system, the sum of the phase differences in the considered loop is equal to multiplicity of 2π

$$[\varphi_T(y,t) - \varphi_T(y+\Delta y,t)] + [\varphi_T(y+\Delta y,t) - \varphi_B(x+\Delta x,t)]$$
$$+ [\varphi_B(x+\Delta x,t) - \varphi_B(x,t)] + [\varphi_B(x,t) - \varphi_T(y,t)] = 2\pi n. \tag{7}$$

Next, we introduce a new dynamical variable. Nontriviality of this variable follows from the fact that, from mathematical point of view, the system is not simply connected. Moreover, for appropriate choice of the electromagnetic vector potential the proposed variable coincides with the gauge invariant phase difference of the manyparticle wave functions of the superconducting electrodes. This variable is defined as the phase difference of the phases of the wave functions that describe the superconducting electrodes

$$\phi(s,t) \equiv \varphi_T(y,t) - \varphi_B(x,t). \tag{8}$$

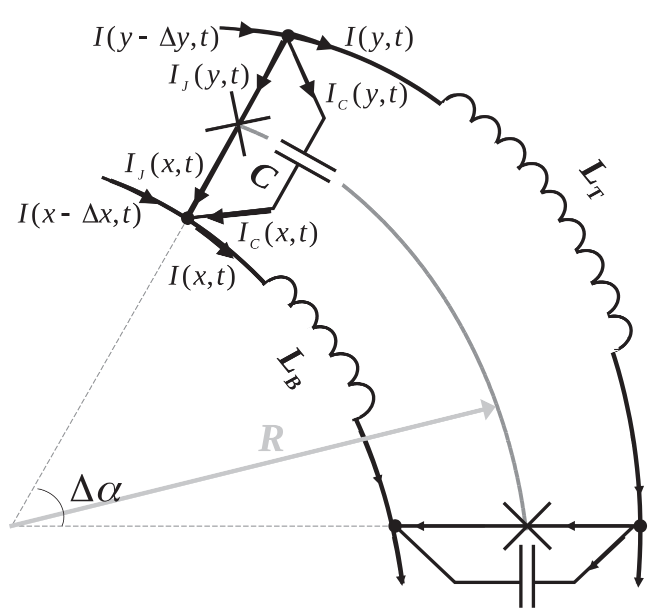

Fig. 2. The Josephson junction consists of the two superconducting electrodes (top and bottom). The electrodes are separated by the dielectric layer of thickness a. R_B is a radius of the circle located in the bottom electrode. This circle is parameterized by the parameter x. R_T is a radius of the the circle located in the top electrode. This arch is parameterized by the parameter y. R denotes the curvature radius of the central curve of the isolator which is parameterized by the parameter s. We consider long Josephson junction and therefore the transverse direction can be neglected.

In the neighboring cross section we have similar expression

$$\phi(s+\Delta s,t)=\varphi_T(y+\Delta y,t)-\varphi_B(x+\Delta x,t). \tag{9}$$

Eqs. (8) and (9) help us to transform the loop rule (7) to the form

$$\phi(s,t)-\phi(s+\Delta s,t)=-[\varphi_T(y+\Delta y,t)-\varphi_T(y,t)] + [\varphi_B(x+\Delta x,t)-\varphi_B(x,t)]+2\pi n. \tag{10}$$

We differentiate the last formula with respect to time

$$\partial_t\phi(s,t)-\partial_t\phi(s+\Delta s,t)=-[\partial_t\varphi_T(y+\Delta y,t)-\partial_t\varphi_T(y,t)] + [\partial_t\varphi_B(x+\Delta x,t)-\partial_t\varphi_B(x,t)]. \tag{11}$$

If we use the formula (6) then the time derivatives of the phases in Eq. (11) can be replaced by potentials

$$\partial_t\phi(s,t)-\partial_t\phi(s+\Delta s,t)=\frac{2\pi}{\Phi_0}[(V_T(y+\Delta y,t)-V_T(y,t)) -(V_B(x+\Delta x,t)-V_B(x,t))], \tag{12}$$

where we introduced the flux quantum Φ_0 i.e. $\frac{q}{\hbar}=\frac{2\pi}{\Phi_0}$. Next, we introduce a voltage which is the difference of electric potentials between points of each superconducting electrode

$$U_T=V_T(y+\Delta y,t)-V_T(y,t),$$
$$U_B=V_B(x+\Delta x,t)-V_B(x,t).$$

The formula (12) now simplifies as follows:

$$\partial_t\phi(s,t)-\partial_t\phi(s+\Delta s,t)=\frac{2\pi}{\Phi_0}(U_T-U_B). \tag{13}$$

The long Josephson junction in the framework of the CSJ model looks (see Fig. 3) like a net of inductors, capacitors and Josephson elements. For example, in ideal superconductors currents flow without ohmic resistance and capacitive effects and therefore along the superconducting electrodes we have only inductive effects hence

$$U_T=L_T\frac{dI_T(y,t)}{dt}, \qquad U_B=L_B\frac{dI_B(x,t)}{dt}.$$

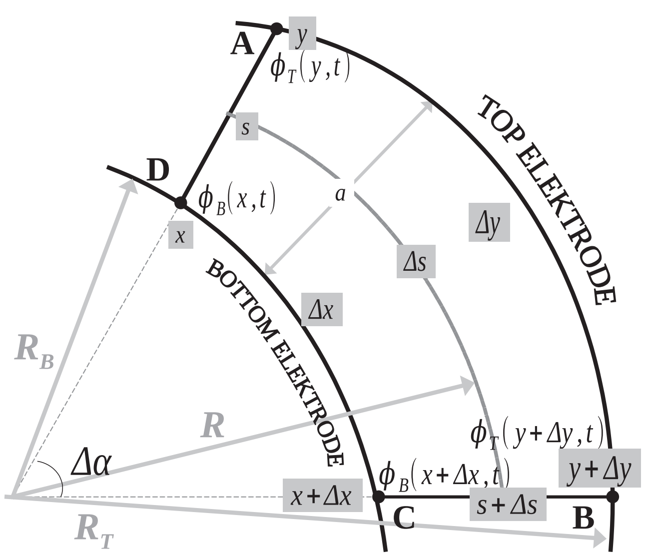

Fig. 3. In perfect superconductor the current along the superconductor has purely inductive nature. The inductances of the electrodes are denoted by L_T and L_B. The current in the normal direction to the isolator has in part capacitive character and in part it is supercurrent of Cooper pairs (Josephson current).

Now, Eq. (13) can be written as

$$\partial_t\phi(s,t)-\partial_t\phi(s+\Delta s,t)=\frac{2\pi}{\Phi_0}\left(L_T\frac{dI_T(y,t)}{dt}-L_B\frac{dI_B(x,t)}{dt}\right). \tag{14}$$

Let us notice that the time derivatives from the last equation can be easy eliminated by integration with respect to time

$$\phi(s,t)-\phi(s+\Delta s,t)=\frac{2\pi}{\Phi_0}[L_TI_T(y,t)-L_BI_B(x,t)], \tag{15}$$

where for simplicity we presumed zero integration constant. Moreover, the similar equation is satisfied also in the node shifted by $\Delta\alpha$

$$\phi(s-\Delta s,t)-\phi(s,t)=\frac{2\pi}{\Phi_0}[L_TI_T(y-\Delta y,t)-L_BI_B(x-\Delta x,t)]. \tag{16}$$

Subsequently, from Eq. (15) we subtract Eq. (16) yielding

$$-\phi(s-\Delta s,t)+2\phi(s,t)-\phi(s+\Delta s,t) =\frac{2\pi}{\Phi_0}[L_T(I_T(y,t)-I_T(y-\Delta y,t))-L_B(I_B(x,t)-I_B(x-\Delta x,t))]. \tag{17}$$

Kirchhoff's first law for node A has the form

$$I_T(y-\Delta y,t)=I_T(y,t)+I_J(s,t)+I_C(s,t),$$

similarly for node D, we have (see Figs. 2and 3)

$$I_B(x-\Delta x,t)=I_B(x,t)-I_J(s,t)-I_C(s,t).$$

The Kirchhoff's current rule allows for conversion of the currents parallel to the superconducting electrodes to the currents perpendicular to it. The Kirchhoff's current rule allows for the following transformation of the Eq. (17)

$$-\phi(s-\Delta s,t)+2\phi(s,t)-\phi(s+\Delta s,t) =-\frac{2\pi}{\Phi_0}\left[L_T\big(I_J(s,t)+I_C(s,t)\big)+L_B\big(I_J(s,t)+I_C(s,t)\big)\right]. \tag{18}$$

In the next step we expand the gauge invariant phase difference ϕ around a point s, with accuracy to the second order

$$\phi(s\pm\Delta s,t)=\phi(s,t)\pm\phi_s(s,t)\Delta s+\frac{1}{2}\phi_{ss}(s,t)\Delta s^2+\dots\ .$$

Above expansion reduces the left side of the Eq. (18)

$$\partial_s^2\phi(s,t)\Delta s^2=\frac{2\pi}{\Phi_0}(L_T+L_B)\left[I_J(s,t)+I_C(s,t)\right]. \tag{19}$$

Now we introduce the total inductance $L=L_T+L_B$, and then replace this quantity by inductance per unit length $L=l\Delta s$. In addition, the current flowing in the direction normal to the insulator layer is replaced by the current density. Since we are dealing with a long Josephson junction, the current density flowing through the dielectric layer is not calculated per unit area but per unit length i.e. $I_C=\Delta sJ_C$ and $I_J=\Delta sJ_J$. We insert these current densities into the Eq. (19)

$$\partial_s^2\phi(s,t)=\frac{2\pi}{\Phi_0}\,l\big[J_J(s,t)+J_C(s,t)\big], \tag{20}$$

where the factor $l=l(R)$ is some function of the curvature radius R. The dependence of the inductance per unit length on the geometry of the junction can be extracted as follows $l=l_0\mathcal{G}(R)$, where $\mathcal{G}$ is some dimensionless function of curvature. Our aim is to write the Eq. (20) as the equation for the ϕ variable. In order to realize this purpose, we use the first Josephson relation [1]

$$J_J=J_0\sin\phi. \tag{21}$$

Moreover, we use the fact that for a capacitive element we have

$$I_C = C\frac{dU}{dt} \quad \text{or} \quad J_C = c\frac{dU}{dt}, \tag{22}$$

where c is a capacity per unit length of the junction. Additionally, we insert the second Josephson relation [1]

$$U = \frac{\Phi_0}{2\pi}\,\partial_t\phi, \tag{23}$$

to the formula (22) and we obtain

$$J_C = c\,\frac{\Phi_0}{2\pi}\,\partial_t^2\phi. \tag{24}$$

Now we are ready to write the Eq. (20) in the form

$$\partial_s^2\phi(s,t) = \frac{2\pi}{\Phi_0}\,l_0\mathcal{G}\left[J_0\sin\phi + c\,\frac{\Phi_0}{2\pi}\,\partial_t^2\phi\right]. \tag{25}$$

One can easy recognize that the dynamics of the gauge invariant phase difference ϕ is described by the sine-Gordon model

$$\frac{\Phi_0}{2\pi}\frac{c}{J_0}\partial_t^2\phi - \frac{\Phi_0}{2\pi}\frac{1}{l_0J_0}\frac{1}{\mathcal{G}}\partial_s^2\phi + \sin\phi = 0. \tag{26}$$

The standard form of this equation is obtained after rescaling the time and space variables

$$t \to t' = \sqrt{\frac{2\pi J_0}{\Phi_0 c}}t,$$

$$s \to s' = \sqrt{\frac{2\pi l_0 J_0}{\Phi_0}}s.$$

After this operation the Eq. (26) can be transformed to the form

$$\partial_{t'}^2\phi - \widetilde{\mathcal{F}}\partial_{s'}^2\phi + \sin\phi = 0, \tag{27}$$

where the factor $\widetilde{\mathcal{F}} = 1/\mathcal{G}(R)$ describes curvature effects in the junction.

Remarks

In the previous section we obtained, on the background of the CSJ model, the Eq. (27) that describes the dynamics of the fluxon in the curved junction with constant curvature. The curvature effects in this equation are included in the inductance per unit length. Let us notice that the form of this equation is in full analogy with equation obtained (for constant curvatures) on the background of the Londons and Maxwell equations with Landau - Ginzburg current of Cooper pairs [9].

$$\partial_{t'}^2\phi - \mathcal{F}\partial_{s'}^2\phi + \sin\phi = 0, \tag{28}$$

where

$$\mathcal{F} = \frac{1}{aK}\ln\left[\frac{2+aK}{2-aK}\right], \tag{29}$$

here K is curvature of the junction. The geometrical formalism that is used in description of the system considered in this paper has found applications in many branches of science [10]. The factor $\mathcal{F}$ in the above equation affects the speed of the fluxon that propagates along the junction [11]. In case of constant curvature the function $\mathcal{F}$ has the form

$$\mathcal{F} = \frac{R}{a}\ln\left[\frac{2+\frac{a}{R}}{2-\frac{a}{R}}\right]. \tag{30}$$

Let us notice that only if we identify functions $\widetilde{\mathcal{F}} = \mathcal{F}$ then both approaches lead to the same result. On the other hand, if the CSJ model is appropriate for description of curvature effects in the Josephson junction then this identification provides the curvature dependence of the inductance per unit length

$$l \approx l_0\frac{a}{R}\ln^{-1}\left[\frac{2+\frac{a}{R}}{2-\frac{a}{R}}\right]. \tag{31}$$

The curvature dependence of this quantity, in the Josephson junction, can find direct experimental verification and therefore the applicability of the CSJ model to description of the curvature effects in the Josephson junction can be directly verified.

Acknowledgments

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References

[1] Josephson BD. Phys Lett 1962;1:251.
[2] Anderson PW, Rowell JM. Phys Rev Lett 1963;10:230.
[3] Babelon O, Bernard D, Talon M. Introduction to classical integrable systems. Press: Cambridge Univ; 2003.
[4] Ablowitz MJ, Clarkson PA. Solitons. nonlinear evolution equations and inverse scattering. Cambridge Univ Press; 1999;
Ferreira LA, Piette B, Zakrzewski WJ. Phys Rev E 2008;77:036613;
Ferreira LA, Piette B, Zakrzewski WJ. J Phys C 2008;128:012027;
Toda M. Ann Global Anal Geom 2005;27:257.
[5] Sakai AV, Ustinov S, et al. Phys Rev B 1994;50:12905;
Sakai S, Bodin P, Pedersen NF. J Appl Phys 1993;73:2411;
Wallraff A, Goldobin E, Ustinov AV. J. Appl. Phys. 1996;80:6523;
Swihart J. J. Appl. Phys. 1961;32:451;
Josephson BD. Adv. Phys. 1965;14:419;
Barone A, Paterno G. Physics and applications of the josephson effect. New York: Wiley; 1982.
[6] Grnbech-Jensen N, Samuelsen MR, et al. Phys Rev B 1990;42:3976;
Parmentier RD. Fluxons in long Josephson junctions. In: Lonngren K, Scott A, editors. Solitons in Action. New York: Academic Press; 1978;
Pederson NF. Solitons in Josephson transmission lines. In: Trullinger SE et al., editors. Solitons. Amsterdam: Elsevier Science Publishers; 1986.
[7] Likharev KK, Mukhanov OA, Semenov VK. Resistive Single Flux Quantum Logic for the Josephson-Junction Technology, SQUID'85 1103 (1985).
[8] Dobrowolski T. Geom Integrability Quantization 2016;17:182;
Dobrowolski T. Eur Phys J B 2013;86:346.
[9] Dobrowolski T. Ann Phys (N Y) 2012;327:1336;
Jarmoliński A, Dobrowolski T. Physica B Condens Matter 2017;514:24;
Dobrowolski T. J Geom Symmetry Phys 2014;34:13.
[10] Marinov PI, Mladenov IM. J Geom Symmetry Phys 2012;27:93;
Arodź H. Nucl Phys B 1998;509:273;
Toda M, Athukoralage B. AIP Conf Proc 2013;1558:883;
Sławianowski JJ, Kovalchuk V, Gołubowska B, Martens A, Rożko EE. Acta Phys Pol B 2010;41:165;
Sławianowski JJ, Kovalchuk V, Martens A, Goubowska B, Rozko EE. Math Method Appl Sci 2011;34:1512;
Ogawa N. Phys Rev E 2010;81:061113.
[11] Dobrowolski T. Phys Rev E 2009;79:046601;
Dobrowolski T. Phys Lett A 2009;373:3867;
Dobrowolski T. DCDS-S 2011;4:1095.